On the Cayley graphs of upper triangular matrix rings

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Abstract. Let R be a commutative ring with nonzero identity. In this paper, we define and study the Cayley graph $\vec{\Gamma'}_{T_n(R)}$ of upper triangular matrix rings, where n is a natural number. We obtain some graph theoretical properties of $\vec{\Gamma'}_{T_n(R)}$ including its diameter, planarity and girth. Then, we study the Cayley graph $\vec{\Gamma'}_{T_2(\mathbb{F})}$, where $\mathbb{F}$ is a field.Let R be a commutative ring with nonzero identity. In this paper, we define and study the Cayley graph $\vec{\Gamma'}_{T_n(R)}$ of upper triangular matrix rings, where n is a natural number. We obtain some graph theoretical properties of $\vec{\Gamma'}_{T_n(R)}$ including its diameter, planarity and girth. Then, we study the Cayley graph $\vec{\Gamma'}_{T_2(\mathbb{F})}$, where $\mathbb{F}$ is a field.

1. Introduction

The investigation of graphs related to various algebraic structures is a very large and growing area of research. Many fundamental papers devoted to graphs assigned to a ring have appeared recently, see for example [1], [2], [5], [6] and [8]. Among all types of graphs related to various algebraic structures, Cayley graphs have attracted serious attention in the literature, since they have many useful applications, see [13], [14], [15] and [16].

Let R be a commutative ring with $1 \neq 0$ and S be a subset of R. The Cayley graph $\mathrm{Cay}(R,S)$ of R relative to S is defined as a digraph with vertex set R and edge set $E(R,S)$ consisting of those pairs (x,y) such that $y = sx$, for some $s \in S$. By the ordered pair (x,y), we mean that $x \to y$. Also, let $T_n(R)$ denote the $n \times n$ upper triangular matrix ring over R and $Z(R)$ denote the set of zero divisors of R. When there is no confusion, we write T instead of $T_n(R)$.

In this paper, we associate a digraph to the upper triangular matrix rings. Let $J = \{A \in T \mid \det(A) \in Z(R)\}$ and $J^* = J \setminus \{0\}$. The digraph on the upper triangular matrix ring R, denoted by $\vec{\Gamma'}_T$, is a digraph whose vertex set is the set J^* and, for every two distinct vertices A and B, there is an arc from A to B whenever there exists $C \in T^*$ such that $A = BC$. In fact the digraph $\vec{\Gamma'}_T$ is the Cayley graph $\mathrm{Cay}(J^*, T^*)$, where $T^* = T \setminus \{0\}$.

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We define and study the graph $\vec{\Gamma}'_T$. In Sections 2 and 3, we investigate some basic properties of the graph $\vec{\Gamma}'_T$ such as connectivity, diameter, girth and planarity. Also, in Section 4, we study the graph $\vec{\Gamma}'_{T_2(\mathbb{F})}$, where $\mathbb{F}$ is a finite field with $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$.

We will use the standard terminology in graph theory from [10].

A *simple graph* is a pair $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ are the sets of vertices and edges of G, respectively. In a graph G, the *distance* between two distinct vertices a and b, denoted by $d_G(a, b)$, is the length of the shortest path connecting a and b, if such a path exists, otherwise, we set $\mathrm{d}_G(a, b) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}_G(a, b) \mid a \text{ and } b \text{ are distinct vertices of } G\}$. For two distinct vertices a and b in G, $a - b$ means that a and b are adjacent. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if each pair of distinct vertices is joined by an edge. For a positive integer n, we use K_n to denote the complete graph with n vertices. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of the shortest cycle in G, if G contains a cycle; otherwise, $\mathrm{gr}(G) := \infty$. A graph is called *planar* if it can be drawn in the plane without any edge crossing. The Kuratowski Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [10, p. 153]). A simple graph is an *outer planar* if it can be drawn in the plane without crossings in such a way that all of the vertices to the unbounded face of the drawing. Also, the union of the graphs G_1 and G_2, which is denoted by $G_1 \cup G_2$, where G_1 and G_2 are two vertex-disjoint graphs, is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. We say that a digraph X is *connected* if the undirected underlying simple graph obtained by replacing all directed edges of X with undirected edges is a connected graph. Also, for distinct vertices x and y in X, we use the notation $x \to y$ to show that there is an arc from x to y.

2. Girth and diameter

We begin this section with the following result.

Theorem 2.1. (cf. [17, Theorem 2.1]) *Suppose that R is a commutative ring with identity $1 \neq 0$, and suppose that $Q(R)$ is the total quotient ring of R. Then $\vec{\Gamma}(T_n(R)) \cong \vec{\Gamma}(T_n(Q(R)))$.*

By Theorem 2.1, we may assume that throughout this paper every element of R is either a unit or a zero-divisor.

Lemma 2.2. (cf. [18, Lemma 2.2]) *Let $A = [a_{ij}] \in T$. Then $\det(A)$ is a zero-divisor in R if and only if a_{jj} is a zero-divisor in R for some $j \in \{1, 2, \ldots, n\}$.*

Lemma 2.3. (cf. [18, Lemma 2.4]) *Let $A \in T$. Then*

$$A \in Z_L(T) \Longleftrightarrow \det(A) \in Z(R) \Longleftrightarrow A \in Z_R(T).$$

Lemma 2.4. *Let* $Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in V(\vec{\Gamma'_T})$. *Suppose that* A *is a vertex such that* a_{11} *is unit. Then* $Y \longrightarrow A$.

Proof. Suppose that $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is an arbitrary upper triangular matrix such that a_{11} is unit. Now consider $C = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, where $x_{1i} = a_{11}^{-1} y_{1i}$, for $i = 1, 2, \cdots, n$. Hence clearly $AC = Y$. Therefore the result holds. □

Suppose that E_{ij} denote the matrix with 1 in the (i, j)-position and zero elsewhere.

Lemma 2.5. *Let* A *be a vertex in* $\vec{\Gamma'_T}$ *such that* a_{ii} *is a unit element for some* $1 \leqslant i \leqslant n$. *Then* $E_{ii} \longrightarrow A$.

Proof. Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \in V(\vec{\Gamma'_T})$ be such that a_{ii} is unit, for some $1 \leqslant i \leqslant n$. Consider the matrix

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & a_{ii}^{-1} & \vdots & 0 \\ 0 & 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then we have $AC = E_{ii}$, which means that $E_{ii} \to A$. □

Lemma 2.6. (cf. [18, Proposition 3.1]) *Let* R *be a finite ring with* $|R| = k$ *and* $|Z(R)| = d$. *Then*

$$|V(\vec{\Gamma'_T})| = k^{\frac{n(n-1)}{2}}[k^n - (k-d)^n] - 1.$$

In the following example, we see that $\vec{\Gamma'_T}$ is not connected in general.

Example 2.7. Suppose that $T_2(\mathbb{Z}_2)$ is the set of upper triangular matrices 2×2 on $\mathbb{Z}_2$. Then, by Lemma 2.6, $|V(\overrightarrow{\Gamma'_{T_2(\mathbb{Z}_2)}})| = 5$ and this five vertices are,

$$A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ C=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ D=\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\ E=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence we have,

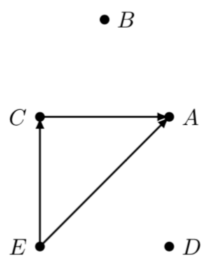

Figure 1:

Now, in the following two propositions, we study the connectness of some induced subgraphs of $\overrightarrow{\Gamma'_T}$.

Proposition 2.8. *The induced subgraph X of $\overrightarrow{\Gamma'_T}$ consists of all vertices that have at least a unit element on the principal diagonal, is connected with diameter less than or equal to two.*

Proof. Let A and B be two arbitrary vertices in X. Without less of generality, we may assume that a_{11} and b_{11} are unit elements of A and B, respectively. Then, by Lemma 2.4, $Y \to A$ and $Y \to B$, where $Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ is a vertex in $\overrightarrow{\Gamma'_T}$ and y_{11} is unit. So we have the path $A \leftarrow Y \to B$ in X. Hence the result holds. $\square$

Proposition 2.9. *Let $\mathbb{F}$ be a finite field. Then the induced subgraph X' of $\overrightarrow{\Gamma'_T}(\mathbb{F})$ consists of all vertices that all elements on the principal diagonal are zero-divisor, is connected with diameter less than or equal to two.*

Proof. Let A and B be two arbitrary vertices in X'. Suppose that

$$Y' = \begin{bmatrix} 0 & y'_{12} & \cdots & y'_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

is a vertex in X'. Then clearly, $A \to Y'$ and $B \to Y'$. Hence we have $A \to Y' \leftarrow B$ in X'. So the result holds. □

Now, in the following theorem, we determine a complete subgraph of $\vec{\Gamma'_T}$.

Theorem 2.10. *Let $\mathbb{F}$ be a finite field and*

$$\Gamma = \{A \in T_n(F) \mid a_{ii} \in F^*, \text{ for } 1 \leqslant i \leqslant n-1 \text{ and } a_{nn} = 0\}.$$

Then the induced subgraph of $\vec{\Gamma'_T}$ with vertex set Γ is complete.

Proof. We prove this result for $n = 3$. Suppose that $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & 0 \end{bmatrix}$ are two arbitrary vertices in Γ. So we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix},$$

where

$$x_{11} = b_{11}^{-1} a_{11}, \quad x_{12} = b_{11}^{-1}(a_{12} - b_{12}(b_{22}^{-1} a_{22})), \quad x_{13} = b_{11}^{-1}(a_{13} - b_{12}(b_{22}^{-1} a_{23})),$$

$$x_{22} = b_{22}^{-1} a_{22}, \quad x_{23} = b_{22}^{-1} a_{23} \quad \text{and} \quad x_{33} = 0.$$

Hence $A = BC$, where $C = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix}$. Now, for $n \geqslant 4$, one can easily check that the result also holds. □

In the next theorem, we show that $\mathrm{gr}(\vec{\Gamma'_T}) = 3$.

Theorem 2.11. *In the graph $\vec{\Gamma'_T}$, we have $\mathrm{gr}(\vec{\Gamma'_T}) = 3$.*

Proof. If $n = 2$, then consider the vertices

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

We have the cycle $A \longrightarrow B \longrightarrow D \longrightarrow A$ in $\vec{\Gamma'_T}$. Now, if $n \geqslant 3$, then by considering the vertices

$$A = E_{11} + E_{12}, \quad B = E_{11}, \quad D = \textstyle\sum_{j=1}^{3} E_{1j},$$

we obtain the cycle $A \longrightarrow B \longrightarrow D \longrightarrow A$, and so the result holds. □

3. Planarity of $\vec{\Gamma}'_T$

In this section, we study the planarity and outer planarity properties of the graph $\vec{\Gamma}'_T$.

Lemma 3.12. *Let E_{ij} and E_{ik} be two vertices in graph $\vec{\Gamma}'_T$ such that $j > k$. Then $E_{ij} \longrightarrow E_{ik}$.*

Proof. Suppose that E_{ij} and E_{ik} are two vertices in the graph $\vec{\Gamma}'_T$ and $j > k$. Then we have

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & a_{ij} & \vdots & 0 \\ 0 & 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & a_{ik} & \vdots & 0 \\ 0 & 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & a_{kj} & \vdots & 0 \\ 0 & 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

such that $a_{ik} = a_{kj} = a_{ij} = 1$, which means that $E_{ij} \longrightarrow E_{ik}$. □

Theorem 3.13. *The graph $\vec{\Gamma}'_{T_n(R)}$ is planar if and only if $n = 2$ and $R = \mathbb{Z}_2$.*

Proof. First assume that $\vec{\Gamma}'_{T_n(R)}$ is planar. If $n = 3$, then the set of vertices

$$\{A = \Sigma_{j=1}^3 E_{1j}, \quad B = E_{11} + E_{12}, \quad C = E_{11}, \quad D = A + E_{23}, \quad E = D + E_{22}\},$$

forms a complete graph K_5, which means that $\vec{\Gamma}'_{T_n(R)}$ is not planar and this is impossible. If $n = 4$, then the vertex set

$$\{A = \Sigma_{j=1}^4 E_{1j}, \quad B = \Sigma_{j=1}^3 E_{1j}, \quad C = E_{11} + E_{22}, \quad D = E_{11}, \quad E = A + E_{24}\},$$

forms a complete graph K_5, which is again impossible. If $n \geqslant 5$, then, by Lemma 2.10, we have $E_{ij} \longrightarrow E_{ik}$ and $j > k$. So, we obtain a subgraph isomorphic to K_5 in $\vec{\Gamma}'_{T_n(R)}$ as it is pictured in Figure 2. Hence $\vec{\Gamma}'_{T_n(R)}$ is not planar and this is a contradiction.

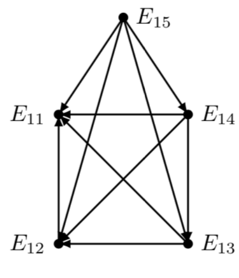

Figure 2:

Now, assume that $n = 2$. If $|U(R)| \geqslant 2$, then the vertices of the set

$$\left\{A=\begin{bmatrix}1&1\\0&0\end{bmatrix},\ B=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ C=\begin{bmatrix}a&a\\0&0\end{bmatrix},\ D=\begin{bmatrix}1&a\\0&0\end{bmatrix},\ E=\begin{bmatrix}a&1\\0&0\end{bmatrix}\right\}$$

forms the graph K_5, which is impossible. If $|U(R)| = 1$ and $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, then the vertices

$$A = \begin{bmatrix}(1,1)&(0,0)\\(0,0)&(0,0)\end{bmatrix},\quad B = \begin{bmatrix}(0,0)&(0,0)\\(0,0)&(1,1)\end{bmatrix},\quad C = \begin{bmatrix}(1,1)&(1,0)\\(0,0)&(0,0)\end{bmatrix},$$

$$D = \begin{bmatrix}(1,1)&(1,1)\\(0,0)&(0,0)\end{bmatrix},\quad E = \begin{bmatrix}(1,1)&(1,0)\\(0,0)&(0,1)\end{bmatrix},$$

forms a complete graph K_5, which is impossible. If $R = \mathbb{Z}_2$, then $\vec{\Gamma}'_{T_2(\mathbb{Z}_2)}$ is pictured in Figure 1, which is planar. Therefor if $\vec{\Gamma}'_{T_n(R)}$ is planar, then we have $n = 2$ and $R = \mathbb{Z}_2$.

The converse statement is obvious. □

Corollary 3.14. *The graph $\vec{\Gamma}'_{T_n(R)}$ is outer planar if and only if it is planar.*

4. The graph of $\vec{\Gamma}'_{T_2(\mathbb{F})}$

In this section, we suppose that $T_2(\mathbb{F})$ is the set of 2×2 matrices over an arbitrary finite field. We study the graph $\vec{\Gamma}'_{T_2(\mathbb{F})}$. We begin by drawing the graph $\vec{\Gamma}'_{T_2(\mathbb{Z}_3)}$. This simple example provides us with a template for the structure of this graph.

Let $\mathbb{F}$ be a finite field and $U = U(\mathbb{F})$. We first divide $T_2(\mathbb{F})$ into the following disjoint subsets:

$$T^{(0)} = \begin{bmatrix}0&U\\0&0\end{bmatrix},\quad T^{(1)} = \begin{bmatrix}U&0\\0&0\end{bmatrix},\quad T^{(2)} = \begin{bmatrix}0&0\\0&U\end{bmatrix},$$

$$T^{(3)} = \begin{bmatrix}U&U\\0&0\end{bmatrix},\quad T^{(4)} = \begin{bmatrix}0&U\\0&U\end{bmatrix}.$$

That is, $V(\vec{\Gamma}'_{T_2(\mathbb{F})}) = T^{(0)}\cup T^{(1)}\cup T^{(2)}\cup T^{(3)}\cup T^{(4)}$ is the disjoint union of the sets $T^{(i)}$ and $\vec{\Gamma}'_{T^{(i)}}$ is the induced subgraph of $\vec{\Gamma}'_{T_2(\mathbb{F})}$ with vertex set $T^{(i)}$.

Proposition 4.15. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = m$. Then:*

(*i*) *The graph $\vec{\Gamma}'_{T^{(i)}}$ is isomorphic to K_{m-1}, for $i = 0, 1, 2$.*

(*ii*) *The graph $\vec{\Gamma}'_{T^{(i)}}$ is isomorphic to $K_{(m-1)^2}$, for $i = 3, 4$.*

Proof. (*i*). Suppose that A and B are two arbitrary vertices in $\vec{\Gamma}'_{T^{(0)}}$. Then for vertices

$$A = \begin{bmatrix}0&x\\0&0\end{bmatrix},\quad B = \begin{bmatrix}0&y\\0&0\end{bmatrix},$$

where $x, y \in U$, we have,

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & y^{-1}x \end{bmatrix},$$

which implies that $A \to B$. Since $|U| = m-1$, we have $\vec{\Gamma'}_{T^{(0)}}$ is isomorphic to K_{m-1}. For $i = 1, 2$, the result follows similarly.

(ii) Suppose that A and B are two arbitrary vertices in $\vec{\Gamma'}_{T^{(3)}}$. Then for

$$A = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix},$$

where $x, y, z, w \in U$, we have

$$\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z^{-1}x & z^{-1}y \\ 0 & 0 \end{bmatrix}.$$

So $A \to B$, which implies that $\vec{\Gamma'}_{T^{(3)}}$ is isomorphic to $K_{(m-1)^2}$. One can easily see that $\vec{\Gamma'}_{T^{(4)}}$ is also isomorphic to $K_{(m-1)^2}$. Hence the result holds. □

Remark 4.16. For $i, j = 0, 1, 2, 3, 4$, we denote by $E(i, j)$ the set of all the directed edges from vertices in $\vec{\Gamma'}_{T^{(i)}}$ to vertices in $\vec{\Gamma'}_{T^{(j)}}$.

Note that, every directed edge from V_1 to V_2 can be represented by the ordered pair (V_1, V_2). With this representation, $E(i, j) \subseteq T^{(i)} \times T^{(j)}$, and the equality occurs when there is an edge from every vertex in $T^{(i)}$ to every vertex in $T^{(j)}$.

Proposition 4.17. *The following statements hold:*

(i) $E(i, 2) = E(j, 4) = E(2, 3) = \emptyset$, *for* $i = 0, 1$ and $j = 0, 1, 2, 3$.

(ii) $E(0, 1) = T^{(0)} \times T^{(1)}$, $E(0, 3) = T^{(0)} \times T^{(3)}$, $E(1, 3) = T^{(1)} \times T^{(3)}$.

Proof. (i). Suppose that A and B are two arbitrary vertices in $\vec{\Gamma'}_{T^{(0)}}$ and $\vec{\Gamma'}_{T^{(2)}}$, respectively. Then we consider the vertices

$$A = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix},$$

where $x, y \in U$. One can easily check that $A \nrightarrow b$ and $B \nrightarrow A$. So $E(0, 2) = \emptyset$. For other situations the result follows easily.

(ii). Suppose that A and B are two arbitrary vertices in $\vec{\Gamma'}_{T^{(0)}}$ and $\vec{\Gamma'}_{T^{(1)}}$, respectively. Then for vertices

$$A = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix},$$

where $x, y \in U$, we have $A \to B$ So $E(0, 1) = T^{(0)} \times T^{(1)}$. For $E(0, 3)$ and $E(1, 3)$ the result follows similarly. □

In the following example, we study the Cayley graph $\vec{\Gamma}'_{T_2(\mathbb{Z}_3)}$.

Example 4.18. The vertex set of $T_2(\mathbb{Z}_3)$ are

$$M_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, M_5 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, M_6 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, M_7 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$

$$M_8 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, M_9 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, M_{10} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, M_{11} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, M_{13} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

Now, we have $T^{(0)} = \{M_0, M_1\}$, $T^{(1)} = \{M_2, M_3\}$, $T^{(3)} = \{M_6, M_7, M_8, M_9\}$, $T^{(2)} = \{M_4, M_5\}$ and $T^{(4)} = \{M_{10}, M_{11}, M_{12}, M_{13}\}$. The graph $\vec{\Gamma}'_{T_2(\mathbb{Z}_3)}$ is pictured in Figure 3.

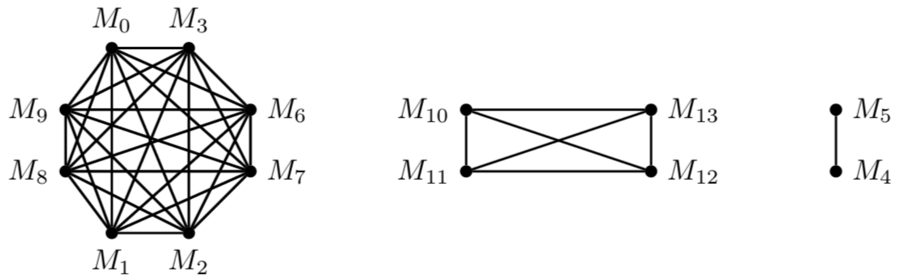

Figure 3: $\vec{\Gamma}'_{T_2(\mathbb{Z}_3)} \cong K_8 \cup K_4 \cup K_2$

Proposition 4.19. *If p is a prime number, then the graph $\vec{\Gamma}'_{T_2(\mathbb{Z}_p)}$ is isomorphic to the graph $K_{p^2-1} \cup K_{(p-1)^2} \cup K_{p-1}$.*

Proof. We know that $V(\vec{\Gamma}'_{T_2(\mathbb{Z}_p)}) = T^{(0)} \cup T^{(1)} \cup T^{(2)} \cup T^{(3)} \cup T^{(4)}$. Since $|T^{(0)}| = |T^{(1)}| = p-1$ and $|T^{(3)}| = (p-1)^2$, by Proposition 4.17 (ii), the vertex set $\{T^{(0)}, T^{(1)}, T^{(3)}\}$ forms a complete subgraph K_{p^2-1} in $\vec{\Gamma}'_{T_2(\mathbb{Z}_p)}$. Also, we have $|T^{(4)}| = (p-1)^2$ and $|T^{(2)}| = p-1$. So, by Proposition 4.15, $\vec{\Gamma}'_{T^{(4)}} \cong K_{(p-1)^2}$ and $\vec{\Gamma}'_{T^{(2)}} \cong K_{p-1}$. Now, by Proposition 4.17 (i), the result holds. □

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