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Generalized kernels of ordered semigroups

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Abstract. We investigate three kinds of generalized kernels of an ordered semigroup. Those that are the intersection of all prime ideals, the intersection of all maximal ideals, and the intersection of all completely prime ideals. We obtain a structure of an ordered semigroup for which the intersection of all prime ideals coincides with the intersection of all maximal ideals.

1. Preliminaries

Various kernels or radicals of a semigroup without order have been introduced and studied by many authors. For examples, R. Fulp [3], J. Luh [6] and S. Schwarz [9]. What is more, three generalized kernels of a semigroup without order, namely, the intersection of all prime ideals of S, denoted by Q^*, the intersection of all maximal ideals, denoted by M^*, and the intersection of all completely prime ideals, denoted by P^*, were introduced and studied by M. Satyanarayana in [8]. In the present paper followed [8] we investigate three kinds of generalized kernels of an ordered semigroup. Analogously, the intersection of all prime ideals, denoted by Q^*, the intersection of all maximal ideals, denoted by M^*, and the intersection of all completely prime ideals, denoted by P^*. We obtain a structure of an ordered semigroup for which M^* and Q^* coincide.

Now, let us recall some certain definitions and results used throughout the paper. A semigroup $(S, \cdot)$ together with a partial order $\leqslant$ that is *compatible* with the semigroup operation, meaning that, for any x, y, z in S,

$$x \leqslant y \ \text{ implies } \ zx \leqslant zy \ \text{ and } \ xz \leqslant yz$$

is called a *partially ordered semigroup* (or simply an *ordered semigroup*) (cf. [1], [2]). Under the trivial relation, $x \leqslant y$ if and only if $x = y$, it is observed that every semigroup is an ordered semigroup.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. For a non-empty subset A of S we define

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

In particular, we write Ax for $A\{x\}$, and similarly for xA. It is observed that the following hold (see [5]):

(1) $A \subseteq (A]$ (hence, $S = (S]$);

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(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A](B] \subseteq (AB]$;

(4) $(A \cup B] = (A] \cup (B]$;

(5) $((A]] = (A]$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A non-empty subset A of S is called a *left ideal* (respectively, a *right ideal*) of S if it satisfies the following conditions:

(i) $SA \subseteq A$ (respectively, $AS \subseteq A$);

(ii) $A = (A]$, that is, for any x in A and y in S, $y \leqslant x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal*, or simply an *ideal* of S. It is known that the union or intersection of any two ideals of S is an ideal of S.

For an element a of an ordered semigroup $(S, \cdot, \leqslant)$, the *principal ideal* generated by a of S will be denoted by $I(a)$. And it is of the form

$$I(a) = (a \cup Sa \cup aS \cup SaS].$$

A left ideal A of S is said to be *proper* if $A \subset S$. The symbol $\subset$ stands for proper subset of sets. A *proper right ideal* and a *proper ideal* of S are defined similarly. If S does not contain proper ideals, then we call S *simple*. A proper ideal A of S is said to be *maximal* if for any ideal B of S, $A \subset B \subseteq S$ implies $B = S$.

In an ordered semigroup S the Green relation $\mathcal{J}$ is defined by $a\mathcal{J}b$ if and only if $I(a) = I(b)$.

An ideal A of S is said to be

– *prime* if for any ideals B, C of S, $BC \subseteq A$ implies $B \subseteq A$ or $C \subseteq A$,

– *completely prime* if for any a, b in S, $ab \in A$ implies $a \in A$ or $b \in A$,

– *completely semiprime* if $a^n \in A$ for any positive integer n implies $a \in A$.

The intersection of all ideals of S, if it is non-empty, is called a *kernel* of S, and it will be denoted by K. The intersection of all prime ideals of S will be denoted by Q^*. The intersection of all maximal ideals of S will be denoted by M^*. And the intersection of all completely prime ideals of S will be denoted by P^*.

2. Results

We begin this section with the following definition.

Definition 2.1. An element x of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *semisimple element* in S if $x \in (xSxS]$ or $x \in (SxSxS]$. And S is said to be *semisimple* if every element of S is semisimple.

Example 2.2. Consider an ordered semigroup $(S, \cdot, \leqslant)$ with $S = \{a, b, c, d, e\}$ and

$\cdot$	a	b	c	d	e
a	d	b	b	d	e
b	b	b	b	b	e
c	b	b	c	b	e
d	d	b	b	d	e
e	b	b	e	b	e

$\leqslant = \{(a,a),(a,b),(a,c),(a,e),(b,b),(b,c),(b,e),(c,c),(d,b),(d,c),(d,d),(d,e),(e,e)\}$
We have that the ordered semigroup is semisimple.

An element a of an ordered semigroup $(S, \cdot, \leqslant)$ is said to be *left regular* (respectively, *right regular*, *regular*, *intra-regular*) if there exist x, y in S such that $a \leqslant xa^2$ (respectively, $a \leqslant a^2x$, $a \leqslant axa$, $a \leqslant xa^2y$). It is observed that left regular elements, right regular elements, regular elements, and intra-regular elements are all semisimple.

First, we have the following lemma.

Lemma 2.3. *Let A be an ideal of an ordered semigroup $(S, \cdot, \leqslant)$, and let $x \in S \backslash A$. If x is semisimple, then there exists a prime ideal Q containing A of S such that $x \notin Q$.*

Proof. Assume that x is semisimple. Let $\mathcal{T}$ be the family of all ideals not containing x of S. Then $A \in \mathcal{T}$, and hence $\mathcal{T}$ is non-empty. By Zorn's Lemma, there exits a maximal element Q in $\mathcal{T}$. We assert that Q is a prime ideal of S. Let A and B be ideals of S such that $AB \subseteq Q$. Suppose that $A \nsubseteq Q$ and $B \nsubseteq Q$. By the maximality of Q, $x \in A \cup Q$ and $x \in B \cup Q$. Then $x \in A$ and $x \in B$. If $x \in (xSxS]$, Then

$$x \in (xSxS] \subseteq (ASBS] \subseteq (AB] \subseteq Q.$$

This is a contradiction. Similarly, $x \in (SxSxS]$ implies $x \in Q$. This is impossible. Thus the assertion holds. □

Lemma 2.4. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup, and $H = \{x, x^2, x^3, ...\}$ a cyclic subsemigroup of S. If $H \cap A = \emptyset$ for some ideal A of S, then there exists a prime ideal Q of S such that $H \cap Q = \emptyset$.*

Proof. Let $\mathcal{T}$ be the collection of all ideals not meet H. Then $A \in \mathcal{T}$, and hence $\mathcal{T}$ is non-empty. By Zorn's Lemma, there exits a maximal element Q in $\mathcal{T}$. We assert that Q is a prime ideal of S. Let B and C be ideals of S such that $BC \subseteq Q$. Suppose that $B \nsubseteq Q$ and $C \nsubseteq Q$. By the maximality of Q, $x^n \in A \cup Q$ and $x^m \in B \cup Q$ for some positive integers n, m. Hence $x^n \in B$ and $x^m \in C$. Then $x^{n+m} \in (BC] \subseteq Q$. This is a contradiction. Thus the assertion holds. □

Proposition 2.5. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup, and let $x \in Q^*$. If A is any proper ideal of S, then $x^n \in A$ for some positive integer n.*

Proof. Let $x \in Q^*$. If $x^n \notin A$ for all positive integer n. By Lemma 2.4, then there exists a prime ideal Q of S such that $x^n \notin Q$ for all positive integer n. Thus $x \notin Q^*$. This is a contradiction. $\square$

An ordered semigroup $(S, \cdot, \leqslant)$ is called *archimedean* if for any a, b in S there exits a positive integer n such that $a^n \in (SbS]$. We have the following

Theorem 2.6. *For an ordered semigroup $(S, \cdot, \leqslant)$, Q^* is an archimedean subsemigroup of S.*

Proof. Let $x, y \in Q^*$. By Proposition 2.5, $x^n \in I(y)$ for some positive integer n. Thus $x^n \in (y]$ or $x^n \in (yS]$ or $x^n \in (Sy]$ or $x^n \in (SyS]$. And each of the cases implies $x^{n+2} \in (Q^* y Q^*]$. Hence Q^* is an archimedean subsemigroup of S. $\square$

Consequently,

Corollary 2.7. *For an ordered semigroup $(S, \cdot, \leqslant)$, K is an archimedean subsemigroup of S.*

Theorem 2.8. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If S is semisimple, then $Q^* = K$.*

Proof. Let $x \in Q^*$. If $x \notin A$ for some ideal A of S, then by Lemma 2.3 we have $x \notin Q^*$. This is a contradiction. $\square$

Theorem 2.9. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup, and let $x \notin M^*$.*

(1) *If $x \in (S^2]$, then x is semisimple.*

(2) *If $x \notin (S^2]$, then either x^2 is semisimple or $x^2 \in M^*$.*

Proof. (1). Let $x \in (S^2] \setminus M^*$. Then there exits a maximal ideal M of S such that $x \notin M$. Then $S = M \cup I(x)$. We have

$$(SM \cup SI(x)] \subseteq (M \cup S(x \cup xS \cup Sx \cup SxS]] \subseteq M \cup (Sx \cup SxS].$$

Then $x \in (Sx]$ or $x \in (SxS]$. If $x \in (SxS]$, then $x \leqslant a_1 x a_2$ for some $a_1, a_2 \in S$. Since $x \notin M$, we have $a_1, a_2 \notin M$. And $a_1, a_2 \in I(x)$. This implies that $x \in (xSxS]$ or $x \in (SxSxS]$. Thus x is semisimple. Similarly, in the second case, if $x \in (Sx]$, then x is semisimple.

(2). If $x \notin (S^2]$ and $x^2 \notin M^*$, then $x^2 \in (S^2]$; hence x^2 is semisimple. $\square$

Consequently,

Corollary 2.10. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If $(S^2] = S$, then $S \setminus M^*$ is semisimple.*

Theorem 2.11. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If $(S^2] = S$ and $M^* \subseteq Q^*$, then $M^* = Q^*$.*

Proof. We first prove that every maximal ideal of S is prime. Let M be a maximal ideal of S. We set

$$P = S \setminus M.$$

We have

$$S = (S^2] = ((M \cup P)^2] = (M^2 \cup MP \cup PM \cup P^2] \subseteq (M \cup P^2].$$

This implies that $P \subseteq (P^2]$. Let A and B be ideals of S such that $AB \subseteq M$. Suppose that $A \nsubseteq M$ and $B \nsubseteq M$. By the maximality of M, $S = M \cup A$ and $S = M \cup B$. Then $P \subseteq A$ and $P \subseteq B$. It follows that

$$P \subseteq (P^2] \subseteq (AB] \subseteq (M] = M.$$

This is a contradiction. Hence M is prime, and $Q^* \subseteq M^*$. Thus $M^* = Q^*$. □

Lemma 2.12. *Let M^* of an ordered semigroup $(S, \cdot, \leqslant)$ be an archimedean subsemigroup of S. Then $M^* \subseteq P^*$.*

Proof. Let P be a completely prime ideal of S. We prove that $M^* \cap P = M^*$. Suppose that $M^* \nsubseteq M^* \cap P$. Then there exists $x \in M^*$ and $x \notin P$. Since P is completely prime, $x^n \notin P$ for all positive integer n. Let $a \in M^* \cap P$. Since M^* is archimedean, $x^r \in (M^* a M^*]$ for some positive integer r. Consider

$$(M^* a M^*] \subseteq (SaS] \subseteq I(a) \subseteq P.$$

Then $x^r \in P$. This is a contradiction. Thus $M^* \cap P = M^*$, and $M^* \subseteq P^*$. □

Proposition 2.13. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then a maximal ideal of S is either trivial or prime.*

Proof. Let M be a nontrivial maximal ideal of S. For each $a, b \in S \setminus M$, $I(a) \cup M = S$ and $I(b) \cup M = S$. Then $b \in I(a)$ and $a \in I(b)$. This implies that $I(a) = I(b)$. Thus $S \setminus M$ is a $\mathcal{J}$-class disjoint from M. If $a, b \notin M$, then $ab \notin M$. Therefore, M is prime. □

Theorem 2.14. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then $Q^* \subseteq M^*$ if and only if $Q^* \subseteq (S^2]$.*

Proof. Let $x \in Q^*$. If $x \notin (S^2]$, then $S \setminus (x]$ is a maximal ideal of S by [10]. This implies that

$$x \in M^* \subseteq S \setminus (x].$$

This is a contradiction. Hence $Q^* \subseteq (S^2]$. Conversely, assume that $Q^* \subseteq (S^2]$. Let $x \in Q^*$. If $x \notin M^*$, then there exists a maximal ideal M of S and $x \notin M$. Since $x \in Q^*$, we have M is not prime. Take $M = S \setminus (y]$ for some $y \notin (S^2]$ by Proposition 2.13. Since $x \notin M$, $y \in I(x) \cup M$. This implies that $y \in I(x)$. Hence

$$y \in I(x) \subseteq Q^* \subseteq (S^2].$$

This is a contradiction. □

Theorem 2.15. *For an ordered semigroup $(S, \cdot, \leqslant)$, if $Q^* \subseteq M^*$, then:*

(1) *M^* is completely semiprime;*

(2) *$S \setminus M^*$ is semisimple;*

(3) *$S = (S^2]$.*

Proof. (1). Assume that M^* is completely semiprime. Let $x \in Q^* \setminus M^*$. If $x \in (S^2]$, then x is semisimple by Theorem 2.9. Thus there exists a prime ideal Q of S such that $x \notin Q$ by Lemma 2.3. This is a contradiction. Hence $x \in M^*$. If $x \notin (S^2]$, then either x^2 is semisimple or $x^2 \in M^*$. If x^2 is semisimple by Theorem 2.9. Thus there exits a prime ideal Q such that $x^2 \notin Q$ by Lemma2.3. This is a contradiction. Hence $x^2 \in M^*$. Since M^* is completely semiprime, $x \in M^*$.

For proving (2) and (3), it can be considered similarly. □

Theorem 2.16. *If M^* is an archimedean of an ordered semigroup $(S, \cdot, \leqslant)$ and prime ideals are completely prime, then $M^* \subseteq Q^*$.*

Proof. This follows by the hypothesis and Lemma 2.12. □

Now we show that Q^* is the maximal archimedean ideal of S if Q^* is a completely semiprime ideal .

Theorem 2.17. *Let Q^* be a completely semiprime ideal of an ordered semigroup $(S, \cdot, \leqslant)$. Then an ideal A of S is an archimedean if and only if $A \subseteq Q^*$.*

Proof. Assume that A is an archimedean. Since Q^* is non-empty, there exists $y \in A \cap Q^*$. Let $x \in A$; then there exists a positive integer n such that

$$x^n \in (AyA] \subseteq (AQ^*A] \subseteq Q^*.$$

Since Q^* is completely semiprime, $x \in Q^*$. The converse statement is obvious. □

From above theorem is an open problem whether this result is hold in general.

Corollary 2.18. *If Q^* is a completely semiprime ideal of an ordered semigroup $(S, \cdot, \leqslant)$, then the only archimedean ideal A of S such that $S \setminus A$ is semisimple is Q^* itself.*

Proof. Let A be an archimedean ideal of S such that $S \setminus A$ is semisimple. We have $A \subseteq Q^*$ by Theorem 2.17. Let $x \in Q^* \setminus A$. Then x is semisimple. Thus there exits a prime ideal Q containing A of S such that $x \notin Q$ by Lemma 2.3. This is a contradiction. Hence $Q^* \subseteq A$. □

Theorem 2.19. *Let Q^* be a non-empty completely semiprime ideal of an ordered semigroup $(S, \cdot, \leqslant)$. Then the following hold:*

(1) *$M^* = Q^*$ if and only if M^* is completely semiprime and M^* is an archimedean subsemigroup of S.*

(2) *Let $S = (S^2]$. Then $M^* = Q^*$ if and only if M^* is an archimedean subsemigroup of S.*

(3) *$S = (S^2]$ with $M^* = Q^*$ if and only if M^* is an archimedean subsemigroup of S such that $S \setminus M^*$ is semisimple.*

Proof. (1). This follows by the hypothesis, Theorem 2.6, Theorem 2.15 and Theorem 2.16.

(2). By Theorem 2.6, M^* is an archimedean subsemigroup of S. Conversely, since $S = (S^2]$, it follows by Theorem 2.15 that $Q^* \subseteq M^*$. And by Theorem 2.17, we have that $M^* \subseteq Q^*$. Hence $M^* = Q^*$.

(3). By Theorem 2.6 and Corollary 2.10, we have M^* is an archimedean subsemigroup of S such that $S \setminus M^*$ is semisimple. Conversely, assume that M^* is an archimedean subsemigroup of S such that $S \setminus M^*$ is semisimple. Since M^* is an archimedean subsemigroup, $M^* \subseteq Q^*$ by Theorem 2.17. Let $x \in Q^* \setminus M^*$. Thus x is semisimple. By Lemma 2.3, we have $x \notin Q^*$. This is a contradiction. Hence $M^* = Q^*$. Finally, we prove that $S = (S^2]$. Let $x \in S$. There are two cases to consider. If $x \notin M^*$, then x is semisimple, and thus $x \in (S^2]$. If $x \in M^*$ and $x \notin (S^2]$, then $S \setminus (x]$ is a maximal ideal of S. Thus

$$x \in M^* \subseteq S \setminus (x].$$

This is a contradiction. Hence $S = (S^2]$. □

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