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Semigroup action on groupoid ordered under co-order

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Abstract. In this paper we prove that if S is a commutative semigroup with apartness acting on an ordered groupoid G under co-order $\nleqslant_G$, then there exists a commutative semigroup $\mathfrak{S}$, constructed by S, acting on the ordered groupoid $(G \times S)/q$ under $\nleqslant_q$, where $(x,a)q \nleqslant_q (y,b)q \Longleftrightarrow bx \nleqslant_G ay$ and $q = \rho \cup \rho^{-1}$, $((x,a),(y,b)) \in \rho \Longleftrightarrow bx \nleqslant_G ay$ for any $(x,a),(y,b) \in G \times S$.

1. Introduction

Actions of commutative semigroups on groupoids have been considered by Tamura and Burnell in [12], based on the following definition: If G is a groupoid and Γ a commutative semigroup, the pair (G,Γ) is called a groupoid G with Γ if there is a mapping of $\Gamma \times G$ into G, $(\alpha,x) \longmapsto \alpha x$ such that $\alpha(x+y) = \alpha x + \alpha y$, $(\alpha\beta)x = \alpha(\beta x) = (\beta\alpha)x$ and $\alpha x = \alpha y$ implies $x = y$ for every $\alpha,\beta \in \Gamma$ and $x,y \in G$. In paper [12], they proved that each (G,Γ) can be embedded in a groupoid $\overline{G}$ with $\overline{\Gamma}$, and studied groupoids G with Γ, in general. In paper [7] Kehayopulu and Tsingelis studies actions of commutative semigroups on ordered groupoids. In the present paper we study actions of commutative semigroups with apartness on groupoids ordered under co-order. Background for this research is the Bishop's constructive mathematics - a mathematics with the Intuitionistic logic.

It is quite natural to ask the question: What is the difference between the results offer by Kehayopulu and Tsingelis in [7] and the results expressed in this text? First, we use the Intuitionist logic instead of the Classical logic. Since the principle of exclusion of the third is not valid in the Intuitionist logic, each set X is regarded as a relational system $(X,=,\neq)$. As a consequence of this, in the constructive algebra we have a significantly larger collection of substructures than in classical algebra. It is not uncommon for classically equivalent formulas to be non equivalent in constructive algebra. Therefore, in the constructive algebra, when determining algebraic concepts, the equivalency of formulas containing an equality and formulas containing a diversity/apartness must be taken into account. For example, in this text we use co-quasiorder and co-order relations on algebraic structures compatible with their internal operations instead of order and quasi-order relations.

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2. Preliminaries

This investigation is in Bishop's constructive mathematics in a sense of papers [5, 6, 8, 9, 10, 11] and books [1, 2, 3, 4] and [13] (Chapter 8: Algebra). Let $(S, =, \neq)$ be a constructive set (i.e., it is a relational system with the relation "$\neq$"). The *diversity relation* "$\neq$" is a binary relation on S, which satisfies the following properties:

$$\neg(x \neq x), \quad x \neq y \Longrightarrow y \neq x, \quad x \neq y \wedge y = z \Longrightarrow x \neq z \ .$$

If it satisfies the following condition

$$(\forall x, z \in S)(x \neq z \Longrightarrow (\forall y \in S)(x \neq y \vee y \neq z)),$$

then it is called *apartness* (A. Heyting). For subset X of S, we say that it is a strongly extensional subset of S if and only if $(\forall x \in X)(\forall y \in S)(x \neq y \vee y \in S)$. Following Bridges and Vita's (see for example [4]) definition for subsets X and Y of S, we say that set X is set-set apartness from Y, and it is denoted by $X \bowtie Y$, if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$. We set $x \bowtie Y$, instead of $\{x\} \bowtie Y$, and, of course, $x \neq y$ instead of $\{x\} \bowtie \{y\}$. With $X^c = \{x \in S : x \bowtie X\}$ we denote apartness complement of X. In the Cartesian product of sets X and Y we have

$$(\forall x, x' \in X)(\forall y, y' \in Y)((x, y) = (x', y') \Longleftrightarrow (x = x' \ \wedge \ y = y'))$$

$$(\forall x, x' \in X)(\forall y, y' \in Y)((x, y) \neq (x', y') \Longleftrightarrow (x \neq x' \ \vee \ y \neq y')).$$

For a function $f : (S, =, \neq) \longrightarrow (T, =, \neq)$ we say that it is a *strongly extensional* if and only if

$$(\forall a, b \in S)(f(a) \neq f(b) \Longrightarrow a \neq b);$$

a function f is an *embedding* if it holds

$$(\forall a, b \in S)(a \neq b \Longrightarrow f(a) \neq f(b)).$$

Let $(S, =, \neq)$ be a set with apartness. A total and strongly extensional function $w : S \times S \longrightarrow$ is an internal binary operation in S and $((S, =, \neq), w)$ is a groupoid. Let us note that, in that case, the operation w satisfies the following implications

$$(\forall x, y, x', y' \in S)((x, x') = (y, y') \Longrightarrow w(x, y) = w(x', y')),$$

$$(\forall x, y, x', y' \in S)(w(x, y) \neq w(x', y') \Longrightarrow (x \neq x' \ \vee \ y \neq y')).$$

A relation $\nleq$ on a nonempty set $G = (G, =, \neq)$ (groupoid $(G, =, \neq, +)$) is called a *co-quasiorder* on G (see [9, 11]) if it is consistent and co-transitive (and compatible with the groupoid operation on G ([5, 6]) in the following sense

$$(\forall x, y, z \in G)((x + y \nleq x + z \vee y + x \nleq z + x) \Longrightarrow y \nleq z).$$

A co-quasiorder $\nleq$ on a set G (a groupoid G) is a *co-order* ([9, 10]) if holds $\nleq \cup \nleq^{-1} = \neq$ (linearity). The concept of co-quasiorder plays an essential role in studying the structure of ordered sets and semigroups. This is because if $\nleq$ is a co-quasiorder on G, then the relation $q = \nleq \cup \nleq^{-1}$ is an co-equality relation on G ([8]) (co-congruence on groupoid G ([10])) and the the family $G/(q^c, q) = \{q^c x : x \in G\}$ is an ordered set (ordered groupoid)([9]).

For convenience, we give the following well known facts: An ordered groupoid under a co-order, is a groupoid at the same time an ordered set under a co-order and the operation being compatible with the ordering. Now let $(G, +, \nleq_G)$ and $(H, +, \nleq_H)$ be ordered groupoids and f a strongly extensional mapping of G into H. The mapping f is called *isotone* if $x \nleq_G y$ implies $f(x) \nleq_H f(y)$ and *reverse isotone* if $x, y \in G$, $f(x) \nleq_H f(y)$ implies $x \nleq_G y$. Let us note that each reverse isotone mapping between two ordered set under co-orders is a strongly extensive. The strongly extensional mapping f is called a *homomorphism* if it satisfies the property $f(x + y) = f(x) + f(y)$ for all $x, y \in G$. An isotone and reverse isotone injective, embedding and surjective homomorphism f is called an *isomorphism*. We say that G is *embedded* in H if G is isomorphic to a subgroupoid of G, i.e., if there exists a mapping $f : G \longrightarrow H$ which is isotone and reverse isotone injective and embedding homomorphism. For example, if q is a co-equality relation on a set G (a co-congruence relation on a groupoid G), then there exists the isomorphism φ between factor-set $G/(q^c, q)$ and the family $G/q = \{xq : x \in S\}$ defined by $\varphi(xq^c) = xq$. Besides, if we define a operation on G/q by

$$xq + yq = \varphi(xq^c) + \varphi(yq^c) = \varphi((x + y)q^c) = (x + y)q \ \ (x, y \in G),$$

then the family G/q is a groupoid too. Further on, there exist the canonical surjective homomorphism π from G onto G/q given by $\pi = \varphi \circ \mathfrak{i}_G$, where $\mathfrak{i}_G : G \longrightarrow G/(q^c, q)$ is the canonical epimorphism.

An ordered groupoid $(G, +, \nleq)$ under co-quasiorder $\nleq$ is said to be *stable* with respect to co-quasiorder $\nleq$ if for each $x, y, z \in G$, we have

$$y \nleq z \Longrightarrow (x + y \nleq x + z \ \wedge \ y + x \nleq z + x).$$

In the present paper we first introduce the concept of acting semigroup with apartness $(S, =, \neq, \cdot)$ on ordered groupoid with apartness $(G, =, \neq, +)$ under co-order $\nleq$. Using this concepts we show that if a commutative semigroup $(S, =, \neq, \cdot)$ acts on an ordered groupoid $(G, =, \neq, +)$ under co-quasirder $\nleq_G$, then there exists a commutative semigroup $\mathfrak{S}$, constructed by S, acting on the groupoid $(G \times S)/q$ ordered under $\nleq_q$, where $(x, a)q \nleq_q (y, b)q \Longleftrightarrow bx \nleq_G ay$ and $q = \rho \cup \rho^{-1}$, $((x, a), (y, b)) \in \rho \Longleftrightarrow bx \nleq_G ay$ for any $(x, a), (y, b) \in G \times S$.

3. The main results

Definition 3.1. Let $(G, =, \neq, +, \nleq_G)$ be an ordered groupoid under a co-order $\nleq_G$ and let $(S, =, \neq, \cdot)$ a commutative semigroup. Suppose there exists a strongly extensional mapping $*: S \times G \ni (a, x) \longmapsto ax \in G$ satisfying the properties:

(i) $(\forall a \in S)(\forall x, y \in G)(a(x + b) = ax + bx)$,

(ii) $(\forall a, b \in S)(\forall x \in G)((ab)x = a(bx))$,

(iii) $(\forall a \in S)(\forall x, y \in S)(ax \nleqslant_G ax \Longleftrightarrow x \nleqslant_G y)$.

Then we say that S *acts on* G (by $*$) and write $(G, S, *)$.

Clearly, if $(G, S, *)$, $a \in S$ and $x, y \in G$, then we have $ax \neq ay$ if and only if $x \neq y$. Indeed: $ax \neq ay$ is equivalent to $ax \nleqslant_G ay \vee ay \nleqslant_G ax$. Thus, by (iii), we have $x \nleqslant_G y \vee y \nleqslant_G x$. Finally, we have $x \neq y$. The opposite implication we give by analogy to previous.

Also, by (ii), for $a, b \in S$ and $x \in G$ we have $b(ax) = (ba)x = (ab)x = a(bx)$.

Theorem 3.2. *Let be $(S, G, *)$ an action of a semigroup S on a commutative groupoid G ordered under a co-iorder $\nleqslant_G$. Then the relation ρ on $G \times S$, defined by*

$$((x, a), (y, b)) \in \rho \Longleftrightarrow bx \nleqslant_G ay$$

is a co-qusiorder on $G{\times}S$.

Proof. (a). Let (x, a), (y, b) and (c, z) be arbitrary elements of $G \times S$ such that $(x, a)\rho(z, c)$. Thus $cx \nleqslant_G az$ and, by (iii), $bcx \nleqslant_G baz$. Hence, by co-transitivity of $\nleqslant_G$ we have

$$bcx \nleqslant_G cay \;\vee\; cay \nleqslant_G baz$$

and, by (iii)

$$bx \nleqslant_G ay \;\vee\; cy \nleqslant_G bz.$$

This means

$$(x, a)\rho(y, b) \;\vee\; (y, b)\rho(z, c).$$

So, the relation ρ is a co-transitive relation on $G \times S$.

(b). Let (x, a), (y, b) and (u, d) be arbitrary elements of $G \times S$ such that $(x, a)\rho(y, b)$. According to part (a) of this proof, we have

$$(x, a)\rho(u, d) \;\vee\; (u, d)\rho(y, b).$$

This means

$$dx \nleqslant_G au \;\vee\; bu \nleqslant_G du.$$

By co-transitivity of the co-order $\nleqslant_G$, we have

$$(dx \nleqslant_G du \;\vee\; du \nleqslant_G au) \;\vee\; (bu \nleqslant_G by \;\vee\; by \nleqslant_G dy).$$

From this, we conclude

$$(x \neq u \;\vee\; d \neq a) \;\vee\; (u \neq y\; b \neq d).$$

Therefore, we have $((x,a),(y,b)) \neq ((u,d),(u,d))$. Therefore, the relation ρ is a consistent relation on $G \times S$.

Since the relation ρ is consistent and co-transitive, then it is a co-quasiorder on $G \times S$. □

In the following we always denote by ρ the co-quasiorder on $G \times S$ defined in Theorem 3.2. Since ρ is a co-quasiorder on $G \times S$, the relation $q = \rho \cup \rho^{-1}$ is an co-equality relation on $G \times S$ and we can construct the factor-set $(G \times S)/(q^c, q)$ with

$$(x,a)q^c \cong (y,b)q^c \Longleftrightarrow ((x,a),(y,b)) \in q^c$$

$$(x,a)q^c \ncong (y,b)q^c \Longleftrightarrow ((x,a),(y,b)) \in q$$

and the family $(G \times S)/q$ with

$$(x,a)q \cong (y,b)q \Longleftrightarrow ((x,a),(y,b)) \in q^c$$

$$(x,a)q \ncong (y,b)q \Longleftrightarrow ((x,a),(y,b)) \in q.$$

Theorem 3.3. *Let $(G,S,*)$ be an action from a commutative semigroup S on a groupid G. Then the function*

$$\otimes : (G \times S)/q \times (G \times S)/q \longrightarrow (G \times S)/q$$

define by

$$(x,a)q \otimes (y,b)q \cong (bx + ay, ab)q$$

is the binary internal operation in $(G \times S)/q$.

Proof. (a). Let $(x,a)q, (x',a')q, (y,b)q, (y',b')q$ be arbitrary elements of $(G \times S)/q$ such that $(x,a)q \cong (x',a')q$ and $(y,b)q \cong (y',b')q$ and let $((u,c),(v,d)) \in q$. Thus

$$(u,c)q(bx + ay, ab) \vee (bx + ay, ab)q(b'x' + a'y', a'b') \vee (b'x' + a'y', a'b')q(v,d).$$

If we suppose that the second case is holds, we will have

$$(bx + ay, ab)\rho(b'x' + a'y', a'b') \ \vee (b'x' + a'y', a'b')\rho(bx + ay, ab)$$

and

$$a'b'(bx + ay) \nleq_G ab(b'x' + a'y') \ \vee \ ab(b'x' + a'y') \nleq_G a'b'(bx + ay).$$

From, for example, $a'b'(bx + ay) \nleq_G ab(b'x' + a'y')$ we conclude

$$a'b'bx + a'b'ay \nleq_G abb'x' + aba'y'.$$

Thus,

$$a'b'bx \nleq_G abb'x' \ \vee \ a'b'ay \nleq_G aba'y'.$$

and

$$a'x \not\leqslant_G ax' \ \vee \ b'y \not\leqslant_G by'.$$

This means

$$(x,a)\rho(x',a') \ \vee \ (y,b)\rho(y',b').$$

Finally, we have

$$(x,a)q(x',a') \ \vee \ (y,b)q(y',b')$$

which is in contradiction with $(x,a)q \cong (x',a')q$ and $(y,b)q \cong (y',b')q$. Therefore, have to be $(bx+ay,ab)q \ncong (u,c)q$ or $(b'x'+a'y',a'b') \ncong (v,d)q$. So,

$$(bx+ay,ab)q \cong (b'x'+a'y',a'b')q$$

which means that $\otimes$ is a function.

(b). Let $(x,a)q,(x',a')q,(y,b)q,(y',b')q$ be arbitrary elements of $(G \times S)/q$ such that

$$(x,a)q \otimes (y,b)q \ncong (x',a')q \otimes (y',b')q.$$

This means

$$(bx+ay,ab)q \ncong (b'x'+a'y',a'b')q,$$

$$((bx+ay,ab),(b'x'+a'y',a'b')) \in q$$

and

$$(bx+ay,ab)\rho(b'x'+a'y',a'b') \ \vee \ (b'x'+a'y',a'b')\rho(bx+ay,ab).$$

Thus

$$a'b'(bx+ay) \not\leqslant_G ab(b'x'+a'y') \ \vee \ ab(b'x'+a'y') \not\leqslant_G a'b'(bx+ay),$$

$$a'b'bx+a'b'ay \not\leqslant_G abb'x'+aba'y' \ \vee \ abb'x'+aba'y' \not\leqslant_G a'b'bx+a'b'ay.$$

Hence

$$a'b'bx \not\leqslant_G abb'x' \ \vee \ a'b'ay \not\leqslant_G aba'y' \ \vee \ abb'x' \not\leqslant_G a'b'bx \ \vee \ aba'y' \not\leqslant_G a'b'ay.$$

and

$$a'x \not\leqslant_G ax' \ \vee \ b'y \not\leqslant_G by' \ \vee \ ax' \not\leqslant_G a'x \ \vee \ by' \not\leqslant_G b'y.$$

Therefore

$$(x,a)\rho(x',a') \ \vee \ (y,b)\rho(y',b') \ \vee \ (x',a')\rho(x,a) \ \vee \ (y',b')\rho(y,b)$$

and

$$((x,a),(x',a')) \in q \ \vee \ ((y,b),(y',b')) \in q.$$

Finally, from

$$(a,x)q \ncong (x',a')q \ \vee \ (y,b)q \ncong (y',b')q$$

we conclude that the mapping $\otimes$ is a strongly extensional function.

So, the mapping $\otimes$ is an internal binary operation in $(G \times S)/q$ and

$$((G \times S)/q, \cong, \ncong, \otimes)$$

is a groupoid. □

Theorem 3.4. *Let* $(G, S, *)$ *be an action from a commutative semigroup* S *on a groupid* G*. Then the relation* $\nleqslant_q$ *on* $(G \times S)/q$ *given by*

$$(x, a)q \nleqslant_q (y, b)q \Longleftrightarrow bx \nleqslant_G ay$$

is a co-order on the gruoid $((G \times S)/q, \cong, \ncong, \otimes)$.

Proof. Since ρ is a co-quasiorder on $G \times S$, then the relation $\nleqslant_q$ is a co-order in the set $(G \times S)/q$. It remains to prove that the operation $\otimes$ is compatible with the ordering. Let $(x, a)q, (y, b)q, (z, c)q$ be arbitrary elements of $(G \times S)/q$ such that

$$(x, a)q \otimes (z, c)q \nleqslant_q (y, b)q \otimes (z, c)q.$$

Hence

$$(cx + az, ac)q \nleqslant_q (cy + bz, bc)q,$$
$$bc(cx + az) \nleqslant_G ac(cy + bz)$$

and

$$bccx + bcaz \nleqslant_G accy + acbz.$$

Thus

$$bccx \nleqslant_G accy$$

and

$$bx \nleqslant_G ay.$$

Therefore, we have

$$(x, a)q \nleqslant_q (y, b)q.$$

The proof for implication

$$(z, c)q \otimes (x, a) \nleqslant_q (z, c)q \otimes (y, b)q \Longrightarrow (x, a)q \nleqslant_q (y, b)q$$

we can give by analogy to previous proof. So, the relation $\nleqslant_q$ is compatible with the groupoid operation $\otimes$ and it is a co-order in the groupoid $((G \times S)/q, \cong, \ncong, \otimes)$. $\square$

Lemma 3.5. *Let* $(G, S, *)$ *be an action from a commutative semigroup* S *on a groupoid* G *and* $a \in S$*. Them the mapping*

$$F_a : (x, b)q \longmapsto (ax, b)q$$

is an isomorphism on $((G \times S)/q, \cong, \ncong, \otimes, \nleqslant_q)$.

Proof. It is obviously that F_a is surjective.

(i). Let $(x, b)q$, $(y, c)q$ be arbitrary elements of $(G \times S)/q$ such that $(x, b)q \cong (y, c)q$ and let $((u, d), (v, e))$ be an arbitrary element of q. We have $(x, b)q^c(y, c)q$. Secondly, from $((u, d), (v, e)) \in q$ follows

$$((u, d), (ax, b)) \in q \ \vee \ ((ax, b), (ay, c)) \in q \ \vee \ ((ay, c), (v, e)) \in q.$$

Suppose that $((ax,b),(ay,c)) \in q$ holds. Thus

$$(ax,b)\rho(ay,c) \ \vee \ (ay,c)\rho(ax,b).$$

This means $cax \nleq_G bay \ \vee \ bay \nleq_G cax$ and $cx \nleq_G by \ \vee \ by \nleq_G cx$. Hence

$$(x,b)\rho(y,c) \ \vee \ (y,c)\rho(x,b)$$

and $(x,b)q(y,c)$ which is in contradiction with $(x,b)q^c(y,c)q$. Finally have to be $(u,d)q \ncong (ax,b)q$ or $(ay,c)q \ncong (v,e)q$. This means $(ax,b)q^c(ay,c)$ and $(ax,b)q \cong (ay,c)q$. So, the mapping F_a is well defined.

(ii). Let $(x,b)q$ and $(y,c)q$ be arbitrary elements of $(G \times S)/q$ such that $(ax,b)q \ncong (ay,c)q$. Thus $(ax,b)q(ay,c)$ and

$$(ax,b)\rho(ay,c) \ \vee \ (ay,c)\rho(ax,b).$$

Hence

$$cax \nleq_G bay \ \vee \ bay \nleq_G cax,$$
$$cx \nleq_G by \ \vee \ by \nleq_G cx$$

and

$$(x,b)\rho(y,c) \ \vee \ (y,c)\rho(x,b).$$

This means $((x,b),(cy,c)) \in q$ and $(x,b)q \ncong (y,c)q$. Therefore, the mapping F_a $(a \in S)$ is a strongly extensional function.

(iii). Let $(x,b)q$ and $(y,c)q$ be arbitrary elements of $(G \times S)/q$ such that $(x,b)q \ncong (y,c)q$. This means $(x,b)\rho(y,c)$ $(y,c)\rho(x,b)$ and $cx \nleq_G by \ \vee \ by \nleq_G cx$. Thus

$$acx \nleq_G aby \ \vee \ aby \nleq_G acx.$$

Hence $(ax,b)\rho(ay,c)$ or $(ay,c)\rho(ax,b)$ and $(ax,b)q(ay,c)$. So, we have $F_a((x,b)q) \ncong F_a((y,c)q)$ which means that F_a is an embedding mapping.

(iv). Let $(x,b)q$ and $(y,c)q$ be arbitrary elements of $(G \times S)/q$ such that

$$F_a((x,b)q) \cong (ax,b)q \cong (ay,c)q \cong F_a((y,c)q)$$

and $((u,d),(v,e))$ be an arbitrary element of q. Thus

$$(u,d)q(x,b) \ \vee \ (x,b)q(y,c) \ \vee \ (y,c)q(v,e).$$

Since the mapping F_a is an embedding, from $(x,b)q(y,c)$ follows $F_a((x,b)q) \ncong F_a((y,c)q)$ which is in contradiction with $F_a((x,b)q) \cong F_a((y,c)q)$. So, have to be $(u,d)q(x,b) \ \vee \ (y,c)q(v,e)$ which means $(x,b)q^c(y,c)$, i.e., $(x,b)q \cong (y,c)q$. Therefore, F_a is an injective mapping.

(v). The mapping F_a is isotone and reverse isotone mapping. Indeed:

$$\begin{aligned} (x,b)q \nleq_q (y,c)q &\iff cx \nleq_G by \\ &\iff acx \nleq_G aby \\ &\iff (ax,b)q \nleq_q (ay,c)q \\ &\iff F_a((x,b)q) \nleq_q F_a((y,c)q). \end{aligned}$$

□

For ease of writing, we put $\mathfrak{S} = \{F_a : a \in S\}$ with

$$F_a = F_b \Longleftrightarrow (\forall (x,c)q \in (G\times S)/q)(F_a((x,c)q) \cong F_b((x,c)q))$$
$$F_a \neq F_b \Longleftrightarrow (\exists (x,c)q \in (G\times S)/q)(F_a((x,c)q) \ncong F_b((x,c)q)).$$

Without serious difficulties we can construct a proof for the following theorem.

Theorem 3.6. *Let $(G, S, *)$ be an action from a commutative semigroup S on a groupoid G. Then the family $\mathfrak{S}$ is commutative semigroup.*

Proof. Let $(x,c)q$ be an arbitrary element of $(G \times S)/q$ and $a, b \in S$. Then $(F_a \circ F_b)((x,c)q) \cong F_a(F_b((x,c)q)) \cong F_a((bx,c)q) \cong (a(bx),c)q \cong ((ab)x,c)q \cong F_{ab}((x,c)q)$. Since the composition of mappings is associative, $(\mathfrak{S}, \circ)$ is a semigroup. Since S is commutative, the semigroup $\mathfrak{S}$ is commutative also. □

At end of this research, we give the following theorem.

Theorem 3.7. *Let $(G, S, *)$ be an action from a commutative semigroup S on a groupoid G. If we consider the mapping $\circledast$ from $\mathfrak{S}\times(G\times S)/q$ to $(G\times S)/q$ defined by*

$$\circledast : \mathfrak{S} \times (G \times S)/q \ni (F_a, (x,b)q) \longmapsto F_a((x,b)q) \in (G \times S)/q,$$

then $(\mathfrak{S}, (G \times S)/q, \circledast)$ is an action from commutative semigroup $\mathfrak{S}$ on groupoid $(G \times S)/q$ ordered under co-order $\nleqslant_q$.

Proof. To prove this theorem, it is sufficient to check the axioms in Definition 3.1.

(i). Let $F_a \in \mathfrak{S}$ and $(x,b)q, (y,c)q \in (G \times S)/q$ be arbitrary elements. Then

$F_a \circledast ((x,b)q \otimes (y,c)q)) \cong F_a \circledast ((cx+by, bc)q) \cong (a(cx+by), bc)q \cong (a(cx) + a(by), bc)q \cong (c(ax)+b(ay), bc)q \cong (ax,b)q \otimes (ay,c)q \cong F_a((x,b)q) \otimes F_a((y,c)q) \cong F_a \circledast ((x,b)q) \otimes F_a \circledast ((y,c)q)$.

(ii). Let $F_a, F_b \in \mathfrak{S}$ and $(x,c)q \in (G \times S)/q$ be arbitrary elements. Then

$(F_a \circ F_b) \circledast ((x,c)q) \cong (F_a \circ F_b)((x,c)q) \cong F_a(F_b((x,c)q) \cong F_a \circledast (F \circledast ((x,c)q))$.

(iii). Let $F_a \in \mathfrak{S}$ and $(x,b)q, (y,c)q \in (G \times S)/q$ be arbitrary elements. Then

$$\begin{aligned}
F_a \circledast ((x,b)q) \nleqslant_q F_a((y,c)q) &\Longleftrightarrow F_a((x,b)q) \nleqslant_q F_a((y,c)q)\\
&\Longleftrightarrow (ax,b)q \nleqslant_q (ay,c)q\\
&\Longleftrightarrow c(ax) \nleqslant_G b(ay)\\
&\Longleftrightarrow a(cx) \nleqslant_G a(by)\\
&\Longleftrightarrow cx \nleqslant_G by\\
&\Longleftrightarrow (x,b)q \nleqslant_q (y,c)q.
\end{aligned}$$
□

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