Quasigroups and Related Systems **26** (2018), 229 − 238

Basarab loop and its variance with inverse properties

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Abstract. A loop $(Q, \cdot)$ is called a Basarab loop if the identities: $(x \cdot yx^\rho)(xz) = x \cdot yz$ and $(yx) \cdot (x^\lambda z \cdot x) = yz \cdot x$ hold. It is a special type of a G-loop. It was shown that a Basarab loop $(Q, \cdot)$ has the cross inverse property if and only if $(Q, \cdot)$ is an abelian group or all left (right) translations of $(Q, \cdot)$ are right (left) regular. In a Basarab loop, the following properties are equivalent: flexibility property, right inverse property, left inverse property, inverse property, right alternative property, left alternative property and alternative property. The following were proved: a Basarab loop is a weak inverse property loop if it is flexible such that the middle inner mapping is contained in a permutation group; a Basarab loop is an automorphic inverse property loop if a semi-commutative law is obeyed such that the middle inner mapping is contained in a permutation group; a Basarab loop is an anti-automorphic inverse property loop if every element has a two-sided inverse such that the middle inner mapping is contained in a permutation group; a Basarab loop is a semi-automorphic inverse property loop if the Basarab loop is flexible, the middle inner mapping is contained in a permutation group such that a semi-cross inverse property holds; a Basarab loop with the m-inverse property such that a permutation condition is true is a cross inverse property loop if it is flexible. Necessary and sufficient conditions for a Basarab loop to be of exponent 2 or a centrum square were established.

1. Introduction

Let G be a non-empty set. Define a binary operation $(\cdot)$ on G. If $x \cdot y \in G$ for all $x, y \in G$, then the pair $(G, \cdot)$ is called a *groupoid* or *Magma*.

If each of the equations $a \cdot x = b$ and $y \cdot a = b$ has unique solutions in G for x and y respectively, then $(G, \cdot)$ is called a *quasigroup*.

If there exists a unique element $e \in G$ called the *identity element* such that for all $x \in G$, $x \cdot e = e \cdot x = x$, $(G, \cdot)$ is called a *loop*. We write xy instead of $x \cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x \cdot yz$ stands for $x(yz)$.

In a loop $(G, \cdot)$ with identity element e, the *left inverse element* of $x \in G$ is the element $xJ_\lambda = x^\lambda \in G$ such that $x^\lambda \cdot x = e$ while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that $x \cdot x^\rho = e$. If $x^\lambda = x^\rho$ for any $x \in G$, then we simply write $x^\lambda = x^\rho = x^{-1}$ or $J_\lambda = J_\rho = J$. Let x be a fixed

2010 Mathematics Subject Classification: 20N02, 20N05
Keywords: K-loops, Basarab loops, inverse property loops

element in a groupoid $(G,\cdot)$. The left and right translation maps of G, L_x and R_x respectively can be defined by $yL_x = x\cdot y$ and $yR_x = y\cdot x$.

Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist. Let $x\backslash y = yL_x^{-1} = xM_y$ and $x/y = xR_y^{-1} = yM_x^{-1}$ and note that $x\backslash y = z \Leftrightarrow x\cdot z = y$ and $x/y = z \Leftrightarrow z\cdot y = x$.

The group of all permutations on G is called the permutation group of G and denoted by $SYM(G)$. For an overview of the theory of loops, readers may check [9, 10, 12, 13, 14, 16, 30, 31].

The triple (A,B,C) of bijections of a loop $(G,\cdot)$ is called an *autotopism* if $xA\cdot yB = (x\cdot y)C$ for all $x,y\in G$. Such triples form a group $AUT(G,\cdot)$ called the *autotopism group* of $(G,\cdot)$. Furthermore, if $A=B=C$, then A is called an automorphism of $(G,\cdot)$. Such bijections form a group $AUM(G,\cdot)$ called the automorphism group of $(G,\cdot)$.

Definition 1.1. Let $(G,\cdot)$ be a quasigroup. Then

1. $U\in SYM(G)$ is called *λ-regular* if there exists $(U,I,U)\in AUT(G,\cdot)$; the set of all such mappings forms a group $\Lambda(G,\cdot)$,
2. a bijection U is called *ρ-regular* if there exists $(I,U,U)\in AUT(G,\cdot)$; the set of all such mappings forms a group $\mathcal{P}(G,\cdot)$.

Definition 1.2. A loop $(G,\cdot)$ is said to

– be a *left alternative property loop* (LAPL) if for all $x,y\in G$, $x\cdot xy = xx\cdot y$,
– be a *right alternative property loop* (RAPL) if for all $x,y\in G$, $yx\cdot x = y\cdot xx$,
– be an *alternative loop* if it is both left and right alternative,
– be *flexible* or *elastic* if $xy\cdot x = x\cdot yx$ holds for all $x,y\in G$,
– have the *left inverse property* (LIP) if for all $x,y\in G$, $x^\lambda\cdot xy = y$,
– have the *right inverse property* (RIP) if for all $x,y\in G$, $yx\cdot x^\rho = y$,
– have the *inverse property* if it has both left and right inverse properties.

There are some classes of loops which do not have the inverse property but have properties which can be considered as variations of the inverse property.

Definition 1.3. A loop $(G,\cdot)$ is called

– a *weak inverse property loop* (WIPL) if and only if it obeys the identity $x(yx)^\rho = y^\rho$ or $(xy)^\lambda x = y^\lambda$ for all $x,y\in G$,
– a *cross inverse property loop* (CIPL) if it obeys the identity $xy\cdot x^\rho = y$ or $x\cdot yx^\rho = y$ or $x^\lambda\cdot(yx) = y$ or $x^\lambda y\cdot x = y$ for all $x,y,\in G$,
– an *automorphic inverse property loop* (AIPL or D-*loop* [15]) if it obeys the identity $(xy)^\rho = x^\rho y^\rho$ or $(xy)^\lambda = x^\lambda y^\lambda$ for all $x,y,\in G$,
– an *anti-automorphic inverse property loop* (AAIPL) if it obeys the identity $(xy)^\rho = y^\rho x^\rho$ or $(xy)^\lambda = y^\lambda x^\lambda$ for all $x,y\in G$,
– a *semi-automorphic inverse property loop* (SAIPL) if it obeys the identity $(xy\cdot x)^\rho = x^\rho y^\rho\cdot x^\rho$ or $(xy\cdot x)^\lambda = x^\lambda y^\lambda\cdot x^\lambda$ for all $x,y,\in G$,
– a *m-inverse property loop* where $m\in\mathbb{Z}$ if it obeys any of the equivalent identities $(xy)J_\rho^m\cdot xJ_\rho^{m+1} = yJ_\rho^m$ and $xJ_\lambda^{m+1}\cdot(yx)J_\lambda^m = yJ_\lambda^m$.

As observed by Osborn [29], a loop is a WIPL and an AIPL if and only if it is a CIPL. The past efforts of Artzy [1, 2, 3, 4], Belousov and Curkan [11], Keedwell [24], Keedwell and Shcherbacov [25, 26, 27] are of great significance in the study of WIPLs, AIPLs, CIPQs and CIPLs, their generalizations (i.e., m-inverse loops and quasigroups, (r, s, t)-inverse quasigroups) and applications to cryptography. These were further generalized to (α, β, γ)-inverse quasigroups by Keedwell and Shcherbacov [28]. Some other contributions to the study of these class of loops and quasigroups can be found in Jaiyeola [17, 18, 19, 21, 22] and Jaiyeola and Adeniran [20].

Definition 1.4. A loop $(Q, \cdot)$ is called a *Basarab loop* (or *K-loop*), if the identities:

$$\underbrace{(x \cdot yx^{\rho}) \cdot xz = x \cdot yz}_{\text{BK1}}, \quad \underbrace{yx \cdot (x^{\lambda}z \cdot x) = yz \cdot x}_{\text{BK2}} \tag{1}$$

hold for all $x, y, z \in Q$. A loop $(Q, \cdot)$ is called an *automorphic inverse property Basarab loop* (or *IK-loop*) means $(Q, \cdot)$ is a Basarab loop and the mapping J of $(Q, \cdot)$ is an automorphism of the loop $(Q, \cdot)$.

The first publications introducing the class of loop called Basarab loop are the two prominent papers of Basarab [5, 6] in 1992. Later, IK-loops were studied in [7] and [8]. Just of recent, Jaiyéọlá and Effiong [23] investigated a Basarab loop and the generators of its total inner mapping group.

Theorem 1.5. (Jaiyéọlá and Effiong [23])
Let $(Q, \cdot)$ be a Basarab loop and let $U_x = M_x R_x^{-1}, V_x = M_x^{-1} L_x^{-1}, W_x = R_x M_x$ for any arbitrarily fixed $x \in Q$. The following are true for any $n \in \mathbb{N}$:

1. $J_{\rho}^n = T_x^{-1} U_x^n T_x$,
2. $J_{\lambda}^n = T_x V_x^{n-1} W_x^{-1}$.

In this present paper, some properties were investigated in a Basarab loop. The properties in Definition 1.2 were shown to be equivalent to each other. All the properties in Definition 1.3, except one (CIP) were shown to be true provided that some conditions were satisfied; among which is middle inner mapping is contained in a permutation group.

2. Main Results

Lemma 2.1. *A loop $(Q, \cdot)$ is a Basarab loop if and only if $(R_{x^{\rho}} L_x, L_x, L_x)$ and $(R_x, L_{x^{\lambda}} R_x, R_x)$ are in $AUT(Q, \cdot)$. Hence, $(T_x^{-1}, L_x, L_x), (R_x, T_x, R_x) \in AUT(Q, \cdot)$.*

Proof. Simply put BK1 and BK2 of (1) in autotopic forms. The last part follows from the facts that $T_x^{-1} = R_{x^{\rho}} L_x$ and $T_x = L_{x^{\lambda}} R_x$. □

Theorem 2.2. *Let $(Q,\cdot)$ be a Basarab loop. The following are equivalent:*

1. $(Q,\cdot)$ *is a cross inverse property loop,*
2. $(Q,\cdot)$ *is commutative,*
3. $(Q,\cdot)$ *is an abelian group,*
4. $L_x \in \mathcal{P}(Q,\cdot)$ *for all* $x \in Q$,
5. $R_x \in \Lambda(G,\cdot)$ *for all* $x \in Q$.

Proof. By BK1 of (1), $x \cdot y = (x \cdot yx^\rho) \cdot x$. If $(Q,\cdot)$ has the CIP, then $x \cdot y = (x \cdot yx^\rho) \cdot x \Rightarrow x \cdot y = y \cdot x$ which implies commutativity. The converse is also true.

By BK1 of (1), $(x \cdot yx^\rho)(xz) = x \cdot yz$. $(Q,\cdot)$ has CIP if and only if $y \cdot xz = x \cdot yz \Leftrightarrow (Q,\cdot)$ is an abelian group.

By Lemma 2.1, $(T_x^{-1}, L_x, L_x), (R_x, T_x, R_x) \in AUT(Q,\cdot)$.

$(T_x^{-1}, L_x, L_x) \in AUT(Q,\cdot) \Rightarrow (L_x R_x^{-1}, L_x, L_x) \in AUT(Q,\cdot)$. So, $(Q,\cdot)$ is commutative if and only if $L_x \in \mathcal{P}(Q,\cdot)$ for all $x \in Q$.

$(R_x, T_x, R_x) \in AUT(Q,\cdot) \Rightarrow (R_x, R_x L_x^{-1}, R_x) \in AUT(Q,\cdot)$. So, $(Q,\cdot)$ is commutative if and only if $R_x \in \Lambda(G,\cdot)$ for all $x \in Q$. □

Lemma 2.3. *Let $(Q,\cdot)$ be a Basarab loop and let $U_x = M_x R_x^{-1}$ and $V_x = M_x^{-1} L_x^{-1}$ be a mapping defined on $(Q,\cdot)$. Then the following are true for all $x \in Q$:*

1. $T_x^{-1} = J_\rho L_x M_x^{-1}$,
2. $T_x = J_\lambda R_x M_x$ *and* $L_{xJ_\lambda} = J_\lambda R_x U_x$.

Proof. From BK1 of (1), $(x \cdot yx^\rho) \cdot xy^\rho = x$, we get $x \cdot yx^\rho = x/(xy^\rho) = (xy^\rho)M_x^{-1} \Rightarrow yR_{x^\rho}L_x = yJ_\rho L_x M_x^{-1} \Rightarrow T_x^{-1} = J_\rho L_x M_x^{-1}$.

From BK2 of (1), we get $(z^\lambda x)(x^\lambda z \cdot x) = x$. So, $x^\lambda z \cdot x = z^\lambda x \backslash x \Rightarrow zL_{x^\lambda} \cdot x = (z^\lambda x)M_x \Rightarrow zL_{x^\lambda} R_x = zJ_\lambda R_x M_x \Rightarrow L_{x^\lambda} R_x = J_\lambda R_x M_x \Rightarrow L_{xJ_\lambda} R_x = J_\lambda R_x M_x \Rightarrow T_x = J_\lambda R_x M_x$.

$T_x = J_\lambda R_x M_x \Rightarrow L_{xJ_\lambda} R_x = J_\lambda R_x M_x \Rightarrow L_{xJ_\lambda} R_x R_x^{-1} = J_\lambda R_x M_x R_x^{-1} \Rightarrow L_{xJ_\lambda} = J_\lambda R_x M_x R_x^{-1} \Rightarrow L_{xJ_\lambda} = J_\lambda R_x U_x$. □

Theorem 2.4. *Let $(Q,\cdot)$ be a Basarab loop and let $U_x = M_x R_x^{-1}, V_x = M_x^{-1} L_x^{-1}$ for any arbitrarily fixed $x \in Q$. The following are equivalent.*

1. *Flexibility.*
2. *Right inverse property.*
3. *Left inverse property.*
4. *Inverse property.*
5. $L_{y^\lambda \cdot y^2} = L_{y^\lambda}^{-1}\ \forall\ y \in Q$.
6. $R_{y^2 \cdot y^\rho} = R_{y^\rho}^{-1}\ \forall\ y \in Q$.
7. $L_{y^\lambda} L_{y^\lambda \cdot y^2} = I = R_{y^\rho} R_{(y^2 \cdot y^\rho)}\ \forall\ y \in Q$.
8. *Right alternative property.*
9. *Left alternative property.*

10. *Alternative property.*

11. $V_x \ : \ y \mapsto (x\backslash y)^\lambda / x$.

12. $U_x \ : \ y \mapsto x\backslash (y/x)^\rho$.

13. $V_x \ : \ y \mapsto (x\backslash y)^\lambda / x$ *and* $U_x \ : \ y \mapsto x\backslash (y/x)^\rho$.

14. $x \cdot yx^\rho = xy \cdot x^\lambda$.

15. $x^\rho \cdot zx = x^\lambda z \cdot x$.

16. $x \cdot yx^\rho = xy \cdot x^\lambda$ *and* $x^\rho \cdot zx = x^\lambda z \cdot x$.

Proof. 1 $\Leftrightarrow$ 2. By Lemma 2.1, $R_{x^\rho} L_x R_x = L_x \Leftrightarrow R_{x^\rho} = L_x R_x^{-1} L_x^{-1}$. If $(Q,\cdot)$ is flexible, then $R_{x^\rho} = L_x R_x^{-1} L_x^{-1} = L_x (L_x R_x)^{-1} = L_x (R_x L_x)^{-1} = L_x L_x^{-1} R_x^{-1} = R_x^{-1} \Rightarrow R_{x^\rho} = R_x^{-1} \Rightarrow yx \cdot x^\rho = y \Rightarrow$ RIP.

Conversely, if $(Q,\cdot)$ has the RIP, then $R_{x^\rho} = R_x^{-1} \Rightarrow R_x^{-1} = L_x R_x^{-1} L_x^{-1} \Rightarrow L_x R_x = R_x L_x \Rightarrow$ flexibility.

1 $\Leftrightarrow$ 3. By Lemma 2.1, $L_{x^\lambda} R_x L_x = R_x \Leftrightarrow L_{xJ_\lambda} = R_x L_x^{-1} R_x^{-1}$. If $(Q,\cdot)$, then $L_{xJ_\lambda} = R_x L_x^{-1} R_x^{-1} = R_x (R_x L_x)^{-1} = R_x (L_x R_x)^{-1} = R_x R_x^{-1} L_x^{-1} = L_x^{-1} \Rightarrow L_{x^\lambda} = L_x^{-1} \Rightarrow x^\lambda \cdot xy = y \Rightarrow$ LIP.

Conversely, if $(Q,\cdot)$ has the LIP, then, $L_x^{-1} = R_x L_x^{-1} R_x^{-1} \Leftrightarrow L_x^{-1} R_x = R_x L_x^{-1} \Leftrightarrow R_x L_x = L_x R_x \Rightarrow$ flexibility.

1 $\Leftrightarrow$ 4. This follows from 1 $\Leftrightarrow$ 2 and 1 $\Leftrightarrow$ 3.

3 $\Leftrightarrow$ 5. Put $x = y^\lambda$ into BK2 of (1) to get $(y^\lambda \cdot yy)(y^\lambda z) = y^\lambda \cdot yz$. Thus, $(Q,\cdot)$ has the LIP $\Leftrightarrow (y^\lambda \cdot yy)(y^\lambda z) = z \Leftrightarrow L_{y^\lambda . y^2} = L_{y^\lambda}^{-1}$.

2 $\Leftrightarrow$ 6. Put $x = z^\rho$ in BK1 of (1) to get $(yz^\rho)(zz \cdot z^\rho) = yz \cdot z^\rho$. Thus, $(Q,\cdot)$ has the RIP $\Leftrightarrow (yz^\rho)(zz \cdot z^\rho) = y \Leftrightarrow R_{z^2 . z^\rho} = R_{z^\rho}^{-1}$.

4 $\Leftrightarrow$ 7. This follows from 3 $\Leftrightarrow$ 5 and 2 $\Leftrightarrow$ 6.

1 $\Leftrightarrow$ 8. Put $z = x$ in BK1 of (1) to get $(x \cdot yx^\rho)(xx) = x \cdot yx$. Then, $(Q,\cdot)$ is flexible $\Leftrightarrow (x \cdot yx^\rho)x^2 = xy \cdot x \Leftrightarrow yR_{x^\rho} L_x R_{x^2} = yL_x R_x \Leftrightarrow L_x R_x^{-1} R_{x^2} = L_x R_x \Leftrightarrow R_{x^2} = R_x R_x \Leftrightarrow yx^2 = yx \cdot x \Leftrightarrow$ $(Q,\cdot)$ has RAP.

1 $\Leftrightarrow$ 9. Put $y = x$ in BK2 of (1) to get $(xx)(x^\lambda z \cdot x) = xz \cdot x$. Then, $(Q,\cdot)$ is flexible $\Leftrightarrow (xx)(x^\lambda z \cdot x) = x \cdot zx \Leftrightarrow zL_{x^\lambda} R_x L_{x^2} = zR_x L_x \Leftrightarrow L_{x^\lambda} R_x L_{x^2} = R_x L_x \Leftrightarrow R_x L_x^{-1} L_{x^2} = R_x L_x \Leftrightarrow L_{x^2} = L_x^2 \Leftrightarrow x^2 y = x \cdot xy \Leftrightarrow$ $(Q,\cdot)$ has LAP.

1 $\Leftrightarrow$ 10. This follows from 1 $\Leftrightarrow$ 8 and 1 $\Leftrightarrow$ 9.

2 $\Leftrightarrow$ 11. By 1 of Lemma 2.3, $T_x^{-1} = R_{x^\rho} L_x = J_\rho L_x M_x^{-1} \Rightarrow R_{x^\rho} = J_\rho L_x M_x^{-1} L_x^{-1} \Rightarrow R_{x^\rho} = J_\rho L_x V_x$. Thus, $(Q,\cdot)$ has the RIP if and only if $R_x^{-1} = R_{x^\rho} \Leftrightarrow R_x^{-1} = J_\rho L_x V_x \Leftrightarrow I = R_x J_\rho L_x V_x \Leftrightarrow y = \big[x \cdot (yx)^\rho\big] V_x \Leftrightarrow V_x \ : \ y \mapsto (x\backslash y)^\lambda / x$.

3 $\Leftrightarrow$ 12. By 2 of Lemma 2.3, $L_{xJ_\lambda} = J_\lambda R_x U_x$. Thus, $(Q,\cdot)$ has the LIP if and only if $J_\lambda = L_x^{-1} \Leftrightarrow L_x^{-1} = J_\lambda R_x U_x \Leftrightarrow I = L_x J_\lambda R_x U_x \Leftrightarrow y = \big[(xy)^\lambda \cdot x\big] U_x \Leftrightarrow U_x \ : \ y \mapsto x\backslash (y/x)^\rho$.

4 $\Leftrightarrow$ 13. This follows from 2 $\Leftrightarrow$ 11 and 3 $\Leftrightarrow$ 12.

2 $\Leftrightarrow$ 14. From BK1 of (1), $(x \cdot yx^\rho)x = xy$. So, $x \cdot yx^\rho = xy \cdot x^\lambda \Leftrightarrow (xy \cdot x^\lambda)x = xy \Leftrightarrow$ RIP hold.

3 $\Leftrightarrow$ 15. From BK2 of (1), $x(x^\lambda z \cdot x) = zx$. So, $x^\rho \cdot zx = x^\lambda z \cdot x \Leftrightarrow x(x^\rho \cdot zx) = zx \Leftrightarrow$ LIP hold.

4 $\Leftrightarrow$ 16. This follows from 2 $\Leftrightarrow$ 14 and 3 $\Leftrightarrow$ 15. $\square$

Theorem 2.5. *Let* $Q_\rho^\lambda = \{A \in SYM(Q) : J_\lambda A J_\rho = A\}$, *where* $(Q,\cdot)$ *is a Basarab loop.*

1. *Then, any two of the following implies the third:*
   (*a*) $(Q,\cdot)$ *is a WIPL,*
   (*b*) $T_x \in Q_\rho^\lambda \leqslant SYM(Q)$,
   (*c*) $(Q,\cdot)$ *is flexible.*
2. *Then, any two of the following implies the third:*
   (*a*) $(Q,\cdot)$ *is an AIPL,*
   (*b*) $T_x \in Q_\rho^\lambda \leqslant SYM(Q)$,
   (*c*) $yx^\rho = x^\lambda y$ *for all* $x, y \in Q$.
3. *Then, any two of the following implies the third:*
   (*a*) $(Q,\cdot)$ *is an AAIPL,*
   (*b*) $T_x \in Q_\rho^\lambda \leqslant SYM(Q)$,
   (*c*) $x^\lambda = x^\rho$ *for all* $x \in Q$.
4. *Then, any three of the following implies the fourth:*
   (*a*) $(Q,\cdot)$ *is an SAIPL,*
   (*b*) $T_x^{-1}L_x \in Q_\rho^\lambda \leqslant SYM(Q)$,
   (*c*) $(Q,\cdot)$ *is flexible,*
   (*d*) $x \cdot yx = y$ *for all* $x, y \in Q$.
5. *Suppose that* $(Q,\cdot)$ *is an m-IPL such that* $U_x^m T_x = I$ $(W_x = V_x^{m-1})$ *for all* $x \in Q$. *Then,* $(Q,\cdot)$ *is flexible if and only if* $(Q,\cdot)$ *obeys* $x \cdot yx^\lambda = y$ $(x^\rho y \cdot x = y)$. *Hence,* $(Q,\cdot)$ *is a CIPL.*

Proof. From Lemma 2.3, we have the mappings $M_x = T_x J_\rho L_x$ and $M_x^{-1} = T_x^{-1} J_\lambda R_x$. So, $M_x^{-1} M_x = T_x^{-1} J_\lambda R_x T_x J_\rho L_x = I \Rightarrow J_\lambda R_x T_x J_\rho L_x = T_x \Rightarrow$

$$J_\lambda R_x T_x J_\rho L_x = T_x \tag{2}$$

1. $(Q,\cdot)$ is a WIPL $\Leftrightarrow$ $(xy)^\lambda x = y^\lambda \Leftrightarrow L_x J_\lambda R_x = J_\lambda$. From (2), we have $L_x J_\lambda R_x T_x J_\rho L_x = L_x T_x$. If $(Q,\cdot)$ is a WIPL, then $J_\lambda T_x J_\rho L_x = L_x T_x$. With flexibility, $J_\lambda T_x J_\rho = T_x \Rightarrow T_x \in Q_\rho^\lambda$. So, (a) and (c) implies (b). The other two implications are similarly deduced.

2. $(Q,\cdot)$ is a AIPL $\Leftrightarrow$ $(xy)^\rho = x^\rho \cdot y^\rho \Leftrightarrow R_{y^\rho} = J_\lambda R_y J_\rho$. From (2), we have $J_\lambda R_x J_\rho J_\lambda T_x J_\rho L_x = T_x$. If $(Q,\cdot)$ is a AIPL, then, $R_{x^\rho} J_\lambda T_x J_\rho L_x = T_x$. Furthermore, if $T_x \in Q_\rho^\lambda$, then $R_{x^\rho} T_x L_x = T_x \Rightarrow R_{x^\rho} R_x = L_{x^\lambda} R_x \Rightarrow yx^\rho = x^\lambda y$. So, (a) and (b) implies (c). The other two implications are similarly deduced.

3. $(Q,\cdot)$ is a AAIPL $\Leftrightarrow x^\rho\cdot y^\rho=(yx)^\rho\Leftrightarrow L_{x^\rho}=J_\lambda R_xJ_\rho\Leftrightarrow$. From (2), we have $J_\lambda R_xJ_\rho J_\lambda T_xJ_\rho L_x=T_x$. If $(Q,\cdot)$ is a AIPL and $T_x\in Q_\rho^\lambda$, then, $L_{x^\rho}T_xL_x=T_x\Rightarrow L_{x^\rho}R_x=T_x\Rightarrow L_{x^\rho}R_x=L_{x^\lambda}R_x\Rightarrow x^\rho=x^\lambda$. So, (a) and (b) implies (c). The other two implications are similarly deduced.

4. $(Q,\cdot)$ is a SAIPL $\Leftrightarrow x^\rho\cdot y^\rho x^\rho=(x\cdot yx)^\rho\Leftrightarrow R_{x^\rho}L_{x^\rho}=J_\lambda R_xL_xJ_\rho$. From (2), we have $J_\lambda R_x=T_xL_x^{-1}J_\lambda T_x^{-1}\Rightarrow J_\lambda R_xL_xJ_\rho=T_xL_x^{-1}J_\lambda T_x^{-1}L_xJ_\rho$. If $T_x^{-1}L_x\in Q_\rho^\lambda$, then $J_\lambda R_xL_xJ_\rho=T_xL_x^{-1}T_x^{-1}L_x=T_xR_x^{-1}L_x$. Furthermore, if $(Q,\cdot)$ is a SAIPL and flexible, then $R_{x^\rho}L_{x^\rho}=T_xR_x^{-1}L_x=R_x(R_xL_x)^{-1}L_x=R_x(L_xR_x)^{-1}L_x=I\Rightarrow R_{x^\rho}L_{x^\rho}=I\Rightarrow x\cdot yx=y$. So, (a), (b) and (c) implies (d). The other three implications are similarly deduced.

5. From Theorem 1.5, the following are

$$J_\rho^n=T_x^{-1}U_x^nT_x \text{ and } J_\lambda^n=T_xV_x^{n-1}W_x^{-1}$$

true for any $n\in\mathbb{N}$ and for any arbitrarily fixed $x\in Q$.

If $(Q,\cdot)$ a m-inverse property loop where $m\in\mathbb{Z}$, then $(xy)J_\rho^m\cdot xJ_\rho^{m+1}=yJ_\rho^m\Leftrightarrow J_\rho^m=L_xJ_\rho^mR_{xJ_\rho^{m+1}}$. So,

$$\begin{aligned}
T_x^{-1}U_x^mT_x&=L_xT_x^{-1}U_x^mT_xR_{xT_x^{-1}U_x^{m+1}T_x}\Rightarrow\\
R_x^{-1}U_x^mT_x&=L_xR_x^{-1}U_x^mT_xR_{xT_x^{-1}U_x^{m+1}T_x}\Rightarrow\\
(R_xL_x)^{-1}U_x^mT_x&=R_x^{-1}U_x^mT_xR_{xT_x^{-1}U_x^{m+1}T_x}\overset{\text{flexibility}}{\Longrightarrow}\\
(L_xR_x)^{-1}U_x^mT_x&=R_x^{-1}U_x^mT_xR_{xT_x^{-1}U_x^{m+1}T_x}\Rightarrow\\
L_x^{-1}U_x^mT_x&=U_x^mT_xR_{xT_x^{-1}U_x^{m+1}T_x}
\end{aligned}$$

Hence, with $U_x^mT_x=I$, $R_{xT_x^{-1}U_x}L_x=I\Rightarrow x\big(y\cdot xT_x^{-1}M_xR_x^{-1}\big)=y\Rightarrow x\cdot yx^\lambda=y$. The converse follows by doing a reverse. Thus, $(Q,\cdot)$ is flexible if and only if $(Q,\cdot)$ obeys $x\cdot yx^\lambda=y$. The proof for the second case is similar by using $J_\lambda^m=T_xV_x^{m-1}W_x^{-1}$ in $J_\lambda^m=R_xJ_\lambda^mL_{xJ_\lambda^{m+1}}$. □

Theorem 2.6. *Let $(Q,\cdot)$ be a Basarab loop.*

1. *If $x^\lambda=x^\rho$, $|T_x|=|M_x|=2$, then $(Q,\cdot)$ is commutative.*
2. *Any two of the following implies the third for all $x\in Q$:*
   - (a) *$(Q,\cdot)$ is commutative,*
   - (b) *$x^\lambda=x^\rho$,*
   - (c) *$T_x^2=M_x^2$.*
3. *If $(Q,\cdot)$ is commutative, then, any two of the following implies the third for all $x\in Q$:*
   - (a) *$x^\lambda=x^\rho$,*
   - (b) *$|T_x|=2$,*
   - (c) *$|M_x|=2$.*

4. $(Q,\cdot)$ *is centrum square if and only if* $(x\cdot yx^\rho)(xy)=(yx)(x^\lambda y\cdot x)$.

5. *The following are equivalent:*

   (*a*) $(Q,\cdot)$ *is of exponent* 2,

   (*b*) $M_x=R_x$,

   (*c*) $M_x=L_x^{-1}$.

6. *If* $(Q,\cdot)$ *has the LSIP (RSIP), then*

   (*a*) $x^\lambda=x^\rho$,

   (*b*) $(Q,\cdot)$ *has the LIP (RIP) if and only if* $x\cdot x^\lambda z=z$ $(zx^\rho\cdot x=z)$.

Proof. From Lemma 2.3, we have $L_x=J_\lambda T_x^{-1}M_x$ and $R_x=J_\rho T_x M_x^{-1}$.

1. If $x^\lambda=x^\rho$ and $|T_x|=|M_x|=2$, then $L_x=J_\lambda T_x M_x=J_\rho T_x M_x^{-1}=R_x$. So, $(Q,\cdot)$ is commutative.

2. If $(Q,\cdot)$ is commutative, then $L_x=R_x\Rightarrow J_\lambda T_x^{-1}M_x=R_x=J_\rho T_x M_x^{-1}$. Furthermore, $J_\lambda=J_\rho\Rightarrow T_x^2=M_x^2$. So, (a) and (b) implies (c). The other two implications are similarly deduced.

3. This is similar to 2.

4. Substitute $z=y$ into BK1 and BK2 of (1) to get

$$\underbrace{(x\cdot yx^\rho)\cdot xy=x\cdot yy}_{\text{a}},\quad \underbrace{yx\cdot(x^\lambda y\cdot x)=yy\cdot x}_{\text{b}} \tag{3}$$

respectively. Hence, the claim.

5. Going by (3)(a), $(Q,\cdot)$ is of exponent 2 if and only if $x\cdot yx^\rho=x/xy=xR_{xy}^{-1}=(xy)M_x^{-1}\Leftrightarrow R_{x^\rho}L_x=L_xM_x^{-1}\Leftrightarrow T_x^{-1}=L_xM_x^{-1}\Leftrightarrow L_xR_x^{-1}=L_xM_x^{-1}\Leftrightarrow M_x=R_x$.

Going by (3)(b), $(Q,\cdot)$ is of exponent 2 if and only if $x^\lambda z\cdot x=yx\backslash x=xL_{yx}^{-1}=(yx)M_x=yR_xM_x\Leftrightarrow L_{x^\lambda}R_x=R_xM_x\Leftrightarrow T_x=R_xM_x\Leftrightarrow R_xL_x^{-1}=R_xM_x\Leftrightarrow M_x=L_x^{-1}$.

6. $(Q,\cdot)$ has the LSIP if $x^\lambda\cdot xx=x$ and RSIP if $xx\cdot x^\rho=x$. Substitute $x=y^\lambda$ and $x=z^\rho$ into BK1 and BK2 of (1) respectively to get

$$\underbrace{(y^\lambda\cdot yy)\cdot y^\lambda z=y^\lambda\cdot yz}_{\text{a}},\quad \underbrace{yz^\rho\cdot(zz\cdot z^\rho)=yz\cdot z^\rho}_{\text{b}} \tag{4}$$

(*i*) If $(Q,\cdot)$ has the LSIP, then by (4)(a), $y\cdot y^\lambda z=y^\lambda\cdot yz$. So, substituting $z=e$, $y^\lambda=y^\rho$. A similar proof goes for when $(Q,\cdot)$ is a RSIP by using (4)(b).

(*ii*) If $(Q,\cdot)$ has the LSIP (RSIP), then by (4)(a)((b)), $(Q,\cdot)$ has the LIP (RIP) if and only if $x\cdot x^\lambda z=z$ $(zx^\rho\cdot x=z)$. □

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Received January 26, 2018

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