Planar and outerplanar indices of zero divisor graphs of partially ordered sets

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Abstract. For poset $P$ with the least element $0$, the zero divisor graph of $P$, denoted by $\Gamma(P)$, is an undirected graph with vertex set $Z^*(P)$ and, for two distinct vertices $x$ and $y$, $x$ is adjacent to $y$ in $\Gamma(P)$ if and only if $\{x, y\}^\ell = \{0\}$. In this paper, we study the planar and outerplanar indices of $\Gamma(P)$ and completely investigate these indices of $\Gamma(P)$ when $\text{Atoms}(P)$ is finite.

1. Introduction

In 1988, the concept of a zero divisor graph was introduced by Beck in [3]. For a commutative ring $R$ with identity, he defined $\Gamma(R)$ to be the graph whose vertices are elements of $R$ and in which two vertices $x$ and $y$ are adjacent if and only if $xy = 0$. In [2], Anderson and Livingston introduced and studied the zero divisor graph whose vertices are the non-zero zero divisors of $R$. Recently, there has been considerable research done on associating graphs with algebraic structures (e.g. [6], [10] and [13]).

In [8], Halaš and Jukl introduced the zero-divisor graphs of posets. For all $x, y \in P$, let $\{x, y\}^\ell$ denote the set of lower bounds for the set $\{x, y\}$. They defined the zero divisor graph as a simple graph with vertex set consist of all the elements of $P$ and two distinct vertices $x$ and $y$ are adjacent if and only if $\{x, y\}^\ell = \{0\}$. Since the vertex $0$ is adjacent to all other vertices, the authors in [1], omit $0$ from the vertex set of this graph and denoted this graph by $G^*(P)$. They studied some properties of $G^*(P)$ and investigated when $G^*(P)$ is planar. Recently, a different method of associating a zero-divisor graph to a poset $P$ was proposed by Lu and Wu in [12]. The graph defined by them is slightly different from the one defined in [8] and [1]. The vertex set of the graph defined in [12] consists of all non-zero zero divisors of $P$.

In this paper, we deal with zero divisor graphs of posets based on the terminology of [12]. An element $x \in P$ is called a zero divisor of $P$ if there exists $y \in P^*$ such that $\{x, y\}^\ell = \{0\}$. We denote the set of zero divisors of $P$ by $Z(P)$ and we consider $Z^*(P) := Z(P) \setminus \{0\}$. The zero divisor graph of $P$, denoted by $\Gamma(P)$, is the graph obtained by setting all the elements of $Z^*(P)$ to be the vertices and defining distinct vertices $x$ and $y$ to be adjacent if and only if $\{x, y\}^\ell = \{0\}$. In

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[9], the authors studied the planarity of the line graph of the graph \( \Gamma(P) \). In this paper, we continue their work and we study the planarity of iterated line graphs of the graph \( \Gamma(P) \). In Section 2, we study the planar index of the iterated line graphs of the zero divisor graph of \( P \). We give a full characterization of all zero divisor graphs with respect to planar index. Also, we study the outerplanarity of iterated line graphs of the graph \( \Gamma(P) \). In Section 3, we show that when \( \Gamma(P) \) is outerplanar and study the outerplanar index of its iterated line graphs.

We use the standard terminology of graphs [4], and of partially ordered sets [5]. In a partially ordered set \((P, \leq)\) (poset, briefly) with a least element 0, an element \( a \) is called an atom if \( a \neq 0 \) and, for an element \( x \) in \( P \), the relation \( 0 \leq x \leq a \) implies either \( x = 0 \) or \( x = a \). We use the notation \( \text{Atom}(P) \) for the set of atoms in \( P \). Assume that \( S \) is a subset of \( P \). Then an element \( x \) in \( P \) is a lower bound of \( S \) if \( x \leq s \) for all \( s \in S \). The set of all lower bounds of \( S \) is denoted by \( S^\downarrow \) and

\[
S^\downarrow := \{ x \in P \mid x \leq s \, \text{for all } s \in S \}. \]

2. Planar index of \( \Gamma(P) \)

From now on, \((P, \leq)\) is a partially ordered set with the least element 0 and with \( \text{Atom}(P) = \{ a_1, a_2, \ldots, a_n \} \). The following notation was stated in [1].

**Notation 1.** Let \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \). The notation \( P_{i_1i_2\ldots i_k} \) stands for the following set:

\[
\{ x \in P; \ x \in \bigcap_{s=1}^{k} \{ a_{i_s} \}^u \cup \bigcup_{j \neq i_1, i_2, \ldots, i_k} \{ a_j \}^u \}. \]

In [1], the authors showed that no two distinct elements in \( P_{i_1i_2\ldots i_k} \) are adjacent in graph \( G^*(P) \). Also, if the index sets \( \{ i_1, i_2, \ldots, i_k \} \) and \( \{ j_1, j_2, \ldots, j_k' \} \) of \( P_{i_1i_2\ldots i_k} \) and \( P_{j_1j_2\ldots j_k'} \), respectively, are distinct, then \( P_{i_1i_2\ldots i_k} \cap P_{j_1j_2\ldots j_k'} = \emptyset \). Moreover

\[
P^* = \bigcup_{k=1}^{n} \bigcup_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} P_{i_1i_2\ldots i_k}. \]

It is easy to see that \( P_{12\ldots n} \) is the set of isolated points in \( G^*(P) \) and

\[
Z^*(P) = \bigcup_{k=1}^{n} \bigcup_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} P_{i_1i_2\ldots i_k} \setminus P_{12\ldots n}. \]

In this section we want to study the planar index of the \( \Gamma(P) \). The planar index of the graph \( G \) was defined as the smallest \( k \) such that \( L^k(G) \) is non-planar. We denote the planar index of \( G \) by \( \xi(G) \). If \( L^k(G) \) is planar for all \( k \geq 0 \), we define \( \xi(G) = \infty \). In [7], the authors gave a full characterization of graphs with respect to their planar index.

**Theorem 2.1.** [Theorem 10, [7]] Let \( G \) be a connected graph. Then:

(i) \( \xi(G) = 0 \) if and only if \( G \) is non-planar.
(ii) \( \xi(G) = \infty \) if and only if \( G \) is either a path, a cycle, or \( K_{1,3} \).

(iii) \( \xi(G) = 1 \) if and only if \( G \) is planar and either \( \Delta(G) \geq 5 \) or \( G \) has a vertex of degree 4 which is not a cut-vertex.

(iv) \( \xi(G) = 2 \) if and only if \( L(G) \) is planar and \( G \) contains one of the graphs \( H_i \) in Figure 1 as a subgraph.

(v) \( \xi(G) = 4 \) if and only if \( G \) is one of the graphs \( X_k \) or \( Y_k \) (Figure 1) for some \( k \geq 2 \).

(vi) \( \xi(G) = 3 \) otherwise.

\[\begin{align*}
H_1 & \quad C_k \quad k \geq 3 \\
H_2 & \quad C_k \quad k \geq 3 \\
H_3 & \quad C_k \quad k \geq 3 \\
H_4 & \\
\end{align*}\]

Figure 1

In Section 3 of [1], the planarity of the graph \( G(P) \) was studied. In fact, they studied the planarity of \( \Gamma(P) \). Since isolated points do not affect planarity, the authors ignored the set \( P_{12\ldots n} \) from the vertex set of \( G^*(P) \). By using [Section 3, [1]] and Theorem 2.1, we have the following theorem.

**Theorem 2.2.** Let \( P \) be a poset and \( \text{Atom}(P) \) be a finite set with \( n \) elements. Then:

(a) \( \xi(\Gamma(P)) = 0 \) if and only if \( \Gamma(P) \) is non-planar.

(b) \( \xi(\Gamma(P)) = \infty \) if and only if \( n = 1 \) or \( Z(P) \) is one of the of Figure 2.

\[\begin{align*}
0 & \quad 0 & \quad 0 & \quad 0 & \quad 0 \\
\end{align*}\]

Figure 2
(c) \( \zeta(\Gamma(P)) = 1 \) if and only if

\[ (c-1) \quad n = 2 \] and one of the following holds:

- \((c-1-1) \quad |P_1| = 1 \) and \(|P_2| \geq 5 \).
- \((c-1-2) \quad |P_1| = 2 \) and \(|P_2| \geq 4 \).

\[ (c-2) \quad n = 3 \] and one of the following holds:

- \((c-2-1) \quad \bigcup_{i=1}^{j} P_i = 3 \) and \(|P_{ij}| \geq 3 \) for some \( 1 \leq i < j \leq 3 \).
- \((c-2-2) \quad \bigcup_{i=1}^{j} P_i = 4 \) and \(|P_{ij}| \geq 2 \) for some \( 1 \leq i < j \leq 3 \).
- \((c-2-3) \quad \bigcup_{i=1}^{j} P_i = 5 \), \(|P_1| = 3\) and \(P_{23} = \emptyset\).
- \((c-2-4) \quad |P_1| = |P_2| = 2\), \(|P_3| = 1\) and \(P_{13} = \emptyset\) or \(P_{23} = \emptyset\).
- \((c-2-5) \quad \bigcup_{i=1}^{j} P_i = 6\), \(|P_2| = |P_3| = 1\) and \(P_{23} = \emptyset\).
- \((c-2-6) \quad \bigcup_{i=1}^{j} P_i = 6\), \(|P_1| = 2\) for all \( 1 \leq i \leq 3 \), and \(P_{ij} = \emptyset\) for all \( 1 \leq i < j \leq 3 \).

\[ (c-2-7) \quad \bigcup_{i=1}^{j} P_i \nsubseteq 7\), \(|P_2| = |P_3| = 1\) and \(P_{23} = \emptyset\).

\[ (c-3) \quad n = 4 \] and one of the following holds:

- \((c-3-1) \quad \bigcup_{i=1}^{j} P_i = 4\) and \(P_{ij} = \emptyset\) whenever \(P_{ij} \neq \emptyset\) for all \( 1 \leq i < j \leq 4 \) whenever \( P_{ij} \neq \emptyset \) for all \( 1 \leq i < j \leq 4 \) and \( P_{ij} \neq \emptyset \) for some \( 1 \leq i < j \leq 4 \) whenever \( P_{ij} \neq \emptyset \) for some \( 1 \leq i < j \leq 4 \).
- \((c-3-2) \quad \bigcup_{i=1}^{j} P_i = 5\), \(|P_1| = 2\), \(P_{ij} = \emptyset\) for all \( 2 \leq i < j \leq 4 \) and \(P_{234} = \emptyset\).

\[ (d) \quad \zeta(\Gamma(P)) = 2 \] if and only if

\[ (d-1) \quad n = 3 \] and one of the following holds:

- \((d-1-1) \quad \bigcup_{i=1}^{j} P_i = 3\), \(|P_{ij}| \leq 2\) for all \( 1 \leq i < j \leq 3 \) and one of the sets \(P_{ij}\) has two elements.
- \((d-1-2) \quad \bigcup_{i=1}^{j} P_i = 3\), \(|P_{ij}| \leq 1\) for all \( 1 \leq i < j \leq 3 \) and two of the sets \(P_{ij}\) have one element.
- \((d-1-3) \quad \bigcup_{i=1}^{j} P_i = 4\), \(|P_{ij}| \leq 1\) for all \( 1 \leq i < j \leq 3 \).

\[ (d-2) \quad n = 4 \), \(|P_i| = 4\), \(P_{ij} = \emptyset\) for all \( 1 \leq i < j \leq 3 \) and \(|P_{ijk}| \leq 1\) for all \( 1 \leq i < j < k \leq 3 \).

\[ (e) \quad \zeta(\Gamma(P)) = 3 \] if and only if \( Z(P) \) is one of the following of Figure 3:

![Figure 3](image-url)
Proof. We know that if \( \Gamma(P) \) is non-planar then \( \xi(\Gamma(P)) = 0 \). Thus we may assume that \( \Gamma(P) \) is planar. By [1, Lemma 3.1], if \( \Gamma(P) \) is planar, then \( n \leq 4 \).

Now we have the following cases:

Case 1. \( n = 1 \). Note that \( \Gamma(P) \) has no edges at all if and only if \( n = 1 \). Therefore in this case \( \xi(\Gamma(P)) = \infty \).

Case 2. \( n = 2 \). By Corollary 2.8 of [1], the graph \( \Gamma(P) \) is isomorphic to a complete bipartite graph which its parts are \( P_1 \) and \( P_2 \). Therefore, by [1, Proposition 3.2], \( \Gamma(P) \) is planar if and only if \( |P_1| \leq 2 \) or \( |P_2| \leq 2 \). Now, we have the following subcases:

(2-1) Suppose that \( |P_1| = 1 \). If \( |P_2| \leq 2 \), then \( \xi(\Gamma(P)) = \infty \). If \( |P_2| = 4 \), then the line graph of the graph \( \Gamma(P) \) is isomorphic to \( K_4 \) and so it is planar. Also, the line graph of the \( \Gamma(P) \) has \( H_2 \) as a subgraph and \( L^2(\Gamma(P)) \) is planar. Thus \( \xi(\Gamma(P)) = 3 \). Otherwise \( |P_2| \geq 5 \). In this situation, \( \Delta(\Gamma(P)) \geq 5 \). Therefore \( \xi(\Gamma(P)) = 1 \).

(2-2) Suppose that \( P_1 \) has two elements, say \( P_1 = \{a_1, a'_1\} \). If \( |P_2| \leq 2 \), then \( \xi(\Gamma(P)) = \infty \). If \( P_2 \) has three elements, say \( P_2 = \{a_2, a'_2, a''_2\} \), then the graph \( \Gamma(P) \) is isomorphic to \( K_{2,3} \). Since \( \Delta(\Gamma(P)) = 3 \), the line graph of the graph \( \Gamma(P) \) is planar. Also, by Figure 4, \( L^2(\Gamma(P)) \) is planar and \( L(\Gamma(P)) \) has \( H_2 \) as a subgraph. So \( \xi(\Gamma(P)) = 3 \). Otherwise \( |P_2| \geq 4 \). Then, by Theorem 2.1, \( \xi(\Gamma(P)) = 1 \).

Case 3. \( n = 3 \). By [1, Theorem 3.3], we have the following subcases:

(3-1) \( |\bigcup_{i=1}^n P_i| = 3 \). The graph \( \Gamma(P) \) is pictured in Figure 5. By Figure 5, if one of the sets \( P_{12}, P_{13} \) or \( P_{23} \) has at least 3 elements, then \( \Delta(\Gamma(P)) \geq 5 \). By Theorem 2.1, \( \xi(\Gamma(P)) = 1 \). Therefore \( |P_{ij}| \leq 2 \) for all \( 1 \leq i < j \leq 3 \). We can conclude that \( L(\Gamma(P)) \) is planar. Now, if only one of the sets \( P_{12}, P_{13} \) or \( P_{23} \) has two elements, then \( L(\Gamma(P)) \) is planar and \( \Gamma(P) \) has \( H_4 \) as a subgraph, we have that \( \xi(\Gamma(P)) = 2 \). If only two of the sets \( P_{12}, P_{13} \) or \( P_{23} \) has one element, then \( L(\Gamma(P)) \) is planar and \( \Gamma(P) \) has \( H_2 \) as a subgraph, we have that \( \xi(\Gamma(P)) = 2 \). Also, if only one of the sets \( P_{12}, P_{13} \) or \( P_{23} \) has one element, then \( L(L(\Gamma(P))) \) is planar and the line graph of the \( \Gamma(P) \) has \( H_2 \) as a subgraph and so \( \xi(\Gamma(P)) = 3 \). If only two of the sets or all of the sets \( P_{12}, P_{13} \) or \( P_{23} \) has one element, exactly, then \( L(L(\Gamma(P))) \) is planar and \( \Gamma(P) \)
has $H_3$ as a subgraph. Hence $\xi(\Gamma(P)) = 2$. At last, if all the sets $P_{12}$, $P_{13}$ and $P_{23}$ are empty, then $\Gamma(P) \cong C_3$, which implies that $\xi(\Gamma(P)) = \infty$.

(3-2) $|\bigcup_{i=1}^{3} P_i| = 4$. Without loss the generality, we may assume $P_1 = \{a_1, a'_1\}$. With this assumption, $\Gamma(P)$ is the graph which was drawn in Figure 6. By Figure 6, if one the sets $P_{12}$ or $P_{13}$ has at least 2 elements, then $\Delta(\Gamma(P)) \geq 5$. Also, if $P_{23}$ has at least two elements, then $\Gamma(P)$ has a vertex of degree 4 which is not a cut vertex. So $\xi(\Gamma(P)) = 1$. Otherwise $|P_{ij}| \leq 1$ for all $1 \leq i < j \leq 3$. We can conclude that $L(\Gamma(P))$ is planar. Since $\Gamma(P)$ has $H_2$ as a subgraph, we have that $\xi(\Gamma(P)) = 2$.

(3-3) $|\bigcup_{i=1}^{3} P_i| = 5$ and one of the following holds:

(3-3-1) One of the sets $P_i$, say $P_1$, has three elements and $P_{23} = \emptyset$. By Figure 7, if one of the sets $P_{13}$ or $P_{12}$ is non-empty, then $\Delta(\Gamma(P)) \geq 5$. So $\xi(\Gamma(P)) = 1$. Now, assume that $P_{12}$ and $P_{13}$ are empty. Since the $\Gamma(P)$ has a vertex of degree 4 which is not a cut vertex, we have that $\xi(\Gamma(P)) = 1$. 

Figure 5

(3-3) $|\bigcup_{i=1}^{3} P_i| = 5$ and one of the following holds:

(3-3-1) One of the sets $P_i$, say $P_1$, has three elements and $P_{23} = \emptyset$. By Figure 7, if one of the sets $P_{13}$ or $P_{12}$ is non-empty, then $\Delta(\Gamma(P)) \geq 5$. So $\xi(\Gamma(P)) = 1$. Now, assume that $P_{12}$ and $P_{13}$ are empty. Since the $\Gamma(P)$ has a vertex of degree 4 which is not a cut vertex, we have that $\xi(\Gamma(P)) = 1$. 

Figure 6
Zero divisor graphs of posets

179

Figure 7

(3-3-2) \(|P_i| \leq 2\), for all \(1 \leq i \leq 3\), and \(P_{13} = \emptyset\) or \(P_{23} = \emptyset\). Without loss the generality, we may assume that \(P_1 = \{a_1, a'_1\}\), \(P_2 = \{a_2, a'_2\}\), \(P_3 = \{a_3\}\) and \(P_{13} = \emptyset\). Now, by Figure 8, if \(P_{12} \neq \emptyset\) or \(|P_{23}| \geq 2\), then \(\Delta(\Gamma(P)) \geq 5\). So \(\xi(\Gamma(P)) = 1\). Also, \(|P_{23}| = 1\), the graph \(\Gamma(P)\) has a vertex of degree 4 which is not a cut vertex. Hence \(\xi(\Gamma(P)) = 1\).

Otherwise, \(P_{12} = \emptyset\) and \(P_{23} = \emptyset\). In this case the vertex \(a_3\) has degree 4 and it is not a cut vertex. Hence \(\xi(\Gamma(P)) = 1\).

Figure 8

(3-4) Suppose that \(|\bigcup_{i=1}^3 P_i| = 6\). Now, by Theorem 3.3 of [1], we must consider one of the following cases:

(3-4-1) \(|P_2| = |P_3| = 1\) and \(P_{23} = \emptyset\). In this case we have that \(\Delta(\Gamma(P)) \geq 5\) and so \(L(\Gamma(P))\) is not planar which implies that \(\xi(\Gamma(P)) = 1\).

(3-4-2) \(|P_i| = 2\) for all \(i = 1, 2, 3\) and \(P_{ij} = \emptyset\) for all \(1 \leq i < j \leq 3\). By Figure 9, we can see that \(\Gamma(P)\) is a 4-regular graph but none of the vertices is a cut vertex. So \(\xi(\Gamma(P)) = 1\).
(3-5) At last suppose that $|\cup_{i=1}^{3} P_i| \geq 7$. By Theorem 3.3 of [1], since $\Gamma(P)$ is planar, we have that $|P_2| = |P_3| = 1$ and $P_{23} = \emptyset$. In this case $\Delta(\Gamma(P)) \geq 5$.

So $\xi(\Gamma(P)) = 1$.

Case 4. Suppose that $n = 4$. Now we have the following subcases:

(4-1) Suppose that $|\cup_{i=1}^{4} P_i| = 4$. Then, by [1, Theorem 3.5], $\Gamma(P)$ is planar if and only if $P_{ij} = \emptyset$ whenever $P_{ij} \neq \emptyset$, where $\{i', j'\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. If $|P_{ijk}| \geq 2$ for some $1 \leq i < j < k \leq 4$ or $|P_{ij}| \geq 2$ for some $1 \leq i < j \leq 4$, then $\Delta(\Gamma(P)) \geq 5$. So we can conclude that $\xi(\Gamma(P)) = 1$. Now suppose that $|P_{ij}| = 1$ for some $1 \leq i < j \leq 4$. Then $a_{ij}$ and $a_{ij}'$, when $\{i', j'\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, are adjacent to the element of $P_{ij}$. So the graph $\Gamma(P)$ has a vertex of degree 4 which is not a cut vertex which implies that $\xi(\Gamma(P)) = 1$. Finally, if $P_{ij} = \emptyset$ for all $1 \leq i < j \leq 4$ and $|P_{ijk}| \leq 1$ for all $1 \leq i < j < k \leq 4$, then $L(\Gamma(P))$ is planar and $\Gamma(P)$ has $H_2$ as a subgraph. So $\xi(\Gamma(P)) = 2$.

(4-2) Let $|\cup_{i=1}^{4} P_i| = 5$ and $|P_4| = 2$. Then, by [1, Theorem 3.7], $\Gamma(P)$ is planar if and only if $P_{ij} = \emptyset$, for $2 \leq i < j \leq 4$, and $P_{234} = \emptyset$. If $P_{ij} \neq \emptyset$, for some $2 \leq i < j \leq 4$ or $P_{ijk} \neq \emptyset$ for some $2 \leq j < k \leq 4$, then $\Delta(\Gamma(P)) \geq 5$ and so $\xi(\Gamma(P)) = 1$. Otherwise, $P_{ij} = \emptyset$ for all $2 \leq i < j \leq 4$ and $P_{ijk} = \emptyset$ for all $2 \leq j < k \leq 4$. In this case, the degree of the vertices $a_2$, $a_3$, and $a_4$ are 4 and these vertices are not cut vertices. Thus the line graph $\Gamma(P)$ is not planar which implies that $\xi(\Gamma(P)) = 1$.

Corollary 2.3. Let $P$ be a poset with least element $0$ and $|\text{Atom}(P)| = n$. Then $\xi(\Gamma(P)) \in \{0, 1, 2, 3, \infty\}$.

3. Outerplanar index of $\Gamma(P)$

In this section, we study the outerplanar index of the zero divisor graph of a poset. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is
outerplanar if and only if it does not contain a subdivision of the complete graph $K_4$ or the complete bipartite graph $K_{2,3}$.

The outerplanar index of a graph $G$, which is denoted by $\zeta(G)$, is the smallest integer $k$ such that the $k$th iterated line graph of $G$ is non-outerplanar. If $L^k(G)$ is outerplanar for all $k \geq 0$, we define $\zeta(G) = \infty$. In [11], the authors gave a full characterization of all graphs with respect to their outerplanar index.

**Theorem 3.1.** Let $G$ be a connected graph. Then:

(i) $\zeta(G) = 0$ if and only if $G$ is non-outerplanar.

(ii) $\zeta(G) = \infty$ if and only if $G$ is a path, a cycle, or $K_{1,3}$.

(iii) $\zeta(G) = 1$ if and only if $G$ is planar and $G$ has a subgraph homeomorphic to $K_{1,4}$ or $K_1 + P_3$ in Figure 10.

(iv) $\zeta(G) = 2$ if and only if $L(G)$ is planar and $G$ has a subgraph isomorphic to one of the graphs $G_2$ or $G_3$ in Figure 10.

(v) $\zeta(G) = 3$ if and only if $G \in \mathcal{I}(d_1, d_2, \ldots, d_t)$ where $d_i \geq 2$ for $i = 2, \ldots, t-1$, and $d_1 \geq 1$ (Figure 10).

![Figure 10](image)

At first, we investigate the outerplanarity of $\Gamma(P)$. We know that the induced subgraph on $\text{Atom}(P)$ is a complete graph. So, if $\Gamma(P)$ is outerplanar then $|\text{Atom}(P)| \leq 3$. Thus we must study the cases that $|\text{Atom}(P)|$ is equal to 1, 2 and 3. When $|\text{Atom}(P)| = 1$, the graph $\Gamma(P)$ has no edges at all which implies that $\Gamma(P)$ is an outerplanar graph. In the following proposition, we state the necessary and sufficient condition for outerplanarity of $\Gamma(P)$ when $|\text{Atom}(P)| = 2$.

**Proposition 3.2.** Suppose that $|\text{Atom}(P)| = 2$. Then $\Gamma(P)$ is outerplanar if and only if one of the following happens:
(a) one of the sets $P_1$ and $P_2$ has one element, exactly.

(b) $|P_1| = 2$ and $|P_2| = 2$.

**Proof.** Since $\Gamma(P)$ is a complete bipartite graph, we are done.

**Proposition 3.3.** Suppose that $|\text{Atom}(P)| = 3$. Then $\Gamma(P)$ is outerplanar if and only if one of the following happens:

(a) $|\bigcup_{i=1}^3 P_i| = 3$.

(b) $|\bigcup_{i=1}^3 P_i| = 4$, $|P_1| = 2$ and $P_{jk} = \emptyset$ for $j, k \in \{1, 2, 3\} \setminus \{i\}$.

**Proof.** Let $|\text{Atom}(P)| = 3$ and $\Gamma(P)$ is outerplanar. If $|\bigcup_{i=1}^3 P_i| \geq 5$, then it is not hard to see that $\Gamma(P)$ has a $K_{2,3}$ as a subgraph. So we must investigate the following cases:

1. $|\bigcup_{i=1}^3 P_i| = 3$. Now, by Figure 5, $\Gamma(P)$ is outerplanar.

2. $|\bigcup_{i=1}^3 P_i| = 4$. In this case one of the sets $P_i$, say $P_1$ has two elements, exactly. Suppose that $P_1 = \{a_1, a_1'\}$ and $x \in P_{23}$. Then, by setting $V_1 := \{a_1, a_1'\}$ and $V_2 := \{a_2, a_3, x\}$, we can find a copy of $K_{2,3}$ in the contraction of $\Gamma(P)$. So $\Gamma(P)$ is not outerplanar. Now suppose that $P_{23}$ is empty. Then, by Figure 11, $\Gamma(P)$ is outerplanar.

![Figure 11](image)

In next theorem, we investigate the outerplanar index of the zero divisor graph of a poset when $\text{Atom}(P)$ is a finite set.

**Theorem 3.4.** Let $P$ be a poset and $|\text{Atom}(P)| = n$. Then:

(a) $\zeta(\Gamma(P)) = 0$ if and only if $\Gamma(P)$ is non-outerplanar.

(b) $\zeta(\Gamma(P)) = \infty$ if and only if $n = 1$ or $Z(P)$ is one of the Figure 2.

(c) $\zeta(\Gamma(P)) = 1$ if and only if

\[ (c-1) \text{ } n = 2, \quad |P_1| = 1 \text{ and } |P_2| \geq 4. \]
(c-2) $n = 3$, $|\bigcup_{i=1}^{3} P_i| = 3$ and $|P_{ij}| \geq 2$ for some $1 \leq i < j \leq 3$.

(c-3) $n = 3$, $|\bigcup_{i=1}^{3} P_i| = 4$ and if $|P_i| = 2$, then for $j, k \in \{1, 2, 3\} \setminus \{i\}$, the set $P_{jk}$ is empty.

(d) $\zeta(\Gamma(P)) = 2$ if and only if $n = 3$, $|\bigcup_{i=1}^{3} P_i| = 3$, $|P_{ij}| = 1$ for some $1 \leq i < j \leq 3$.

Proof. We know $\zeta(\Gamma(P)) = 0$ if $\Gamma(P)$ is non-outerplanar. Thus we may assume that $\Gamma(P)$ is outerplanar. If $\Gamma(P)$ is outerplanar, then $n \leq 3$. Now we have the following cases:

Case 1. $n = 1$. Since $\Gamma(P)$ has no edges at all, we have that $\zeta(\Gamma(P)) = \infty$.

Case 2. $n = 2$. Note that the graph $\Gamma(P)$ is isomorphic to a complete bipartite graph which its parts are $P_1$ and $P_2$. We have the following subcases:

(2-1) Suppose that $|P_1| = 1$. If $|P_2| \leq 3$, then $\zeta(\Gamma(P)) = \infty$. If $|P_2| \geq 4$, then the line graph of the graph $\Gamma(P)$ has a copy of $K_4$ and so it is not outerplanar. Therefore $\zeta(\Gamma(P)) = 1$.

(2-2) $|P_1| = |P_2| = 2$. In this case $\Gamma(P) \cong C_4$, which implies that $\zeta(\Gamma(P)) = \infty$.

Case 3. $n = 3$. By Proposition 3.3, we have the following subcases:

(3-1) $|\bigcup_{i=1}^{3} P_i| = 3$. The graph $\Gamma(P)$ is pictured in Figure 5. By Figure 5, if one of the sets $P_{12}$, $P_{13}$ or $P_{23}$ has at least 2 elements, then $\Gamma(P)$ has $K_{1,4}$ as a subgraph. Now, by Theorem 3.1, we have that $\zeta(\Gamma(P)) = 1$. If one of the sets $P_{ij}$ has one elements for some $1 \leq i < j \leq 3$, then $\Gamma(P)$ has $G_3$ as a subgraph, we have that $\zeta(\Gamma(P)) = 2$. At last, if $P_{ij}$ is empty for all $1 \leq i < j \leq 3$, $\Gamma(P)$ is a cycle with 3 vertices. So $\zeta(\Gamma(P)) = \infty$.

(3-2) $|\bigcup_{i=1}^{3} P_i| = 4$ and if $|P_i| = 2$, then for $j, k \in \{1, 2, 3\} \setminus \{i\}$, the set $P_{jk}$ is empty. Without loss the generality, we may assume that $P_i = \{a_1, a_1\}$. With this assumption, $\Gamma(P)$ is the graph which was drawn in Figure 11. By Figure 11, $\Gamma(P)$ has a copy of $K_1 + P_3$ which implies that $\zeta(\Gamma(P)) = 1$.

Corollary 3.5. Let $P$ be a poset with least element 0 and $|\text{Atom}(P)| = n$. Then $\zeta(\Gamma(P)) \in \{0, 1, 2, \infty\}$.

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References


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