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# Planar and outerplanar indices of zero divisor graphs of partially ordered sets

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**Abstract.** For poset $P$ with the least element 0, the zero divisor graph of $P$, denoted by $\Gamma(P)$, is an undirected graph with vertex set $Z^*(P)$ and, for two distinct vertices $x$ and $y$, $x$ is adjacent to $y$ in $\Gamma(P)$ if and only if $\{x,y\}^\ell = \{0\}$. In this paper, we study the planar and outerplanar indices of $\Gamma(P)$ and completely investigate these indices of $\Gamma(P)$ when Atom(P) is finite.

## 1. Introduction

In 1988, the concept of a zero divisor graph was introduced by Beck in [3]. For a commutative ring $R$ with identity, he defined $\Gamma(R)$ to be the graph whose vertices are elements of $R$ and in which two vertices $x$ and $y$ are adjacent if and only if $xy = 0$. In [2], Anderson and Livingston introduced and studied the zero divisor graph whose vertices are the non-zero zero divisors of $R$. Recently, there has been considerable researches done on associating graphs with algebraic structures (e.g. [6], [10] and [13]).

In [8], Halaš and Jukl introduced the zero-divisor graphs of posets. For all $x, y \in P$, let $\{x,y\}^\ell$ denote the set of lower bounds for the set $\{x,y\}$. They defined the *zero divisor graph* as a simple graph with vetex set consist of all the elements of $P$ and two distinct vertices $x$ and $y$ are adjacent if and only if $\{x,y\}^\ell = \{0\}$. Since the vertex 0 is adjacent to all other vertices, the authors in [1], omit 0 from the vertex set of this graph and denoted this graph by $G^*(P)$. They studied some properties of $G^*(P)$ and investigated when $G^*(P)$ is planar. Recently, a different method of associating a zero-divisor graph to a poset $P$ was proposed by Lu and Wu in [12]. The graph defined by them is slightly different from the one defined in [8] and [1]. The vertex set of the graph defined in [12] consists of all non-zero zero divisors of $P$.

In this paper, we deal with zero divisor graphs of posets based on the terminology of [12]. An element $x \in P$ is called a *zero divisor* of $P$ if there exists $y \in P^*$ such that $\{x,y\}^\ell = \{0\}$. We denote the set of zero divisors of $P$ by $Z(P)$ and we consider $Z^*(P) := Z(P) \setminus \{0\}$. The zero divisor graph of $P$, denoted by $\Gamma(P)$, is the graph obtained by setting all the elements of $Z^*(P)$ to be the vertices and defining distinct vertices $x$ and $y$ to be adjacent if and only if $\{x,y\}^\ell = \{0\}$. In

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[9], the authors studied the planarity of the line graph of the graph $\Gamma(P)$. In this paper, we continue their work and we study the planarity of iterated line graphs of the graph $\Gamma(P)$. In Section 2, we study the planar index of the iterated line graphs of the zero divisor graph of $P$. We give a full characterization of all zero divisor graphs with respect to planar index. Also, we study the outerplanarity of iterated line graphs of the graph $\Gamma(P)$. In Section 3, we show that when $\Gamma(P)$ is outerplanar and study the outerplanar index of its iterated line graphs.

We use the standard terminology of graphs [4], and of partially ordered sets [5]. In a partially ordered set $(P, \leqslant)$ (poset, briefly) with a least element 0, an element $a$ is called an *atom* if $a \neq 0$ and, for an element $x$ in $P$, the relation $0 \leqslant x \leqslant a$ implies either $x = 0$ or $x = a$. We use the notation $\mathrm{Atom}(P)$ for the set of atoms in $P$. Assume that $S$ is a subset of $P$. Then an element $x$ in $P$ is a *lower bound* of $S$ if $x \leqslant s$ for all $s \in S$. The set of all lower bounds of $S$ is denoted by $S^\ell$ and

$$S^\ell := \{x \in P \mid x \leqslant s, \text{ for all } s \in S\}.$$

## 2. Planar index of $\Gamma(P)$

From now on, $(P, \leq)$ is a partially ordered set with the least element 0 and with $\mathrm{Atom(P)} = \{a_1, a_2, \ldots, a_n\}$. The following notation was stated in [1].

**Notation 1.** *Let $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$. The notation $P_{i_1 i_2 \ldots i_k}$ stands for the following set:*

$$\{x \in P; \quad x \in \cap_{s=1}^{k} \{a_{i_s}\}^u \setminus \cup_{j \neq i_1, i_2, \ldots, i_k} \{a_j\}^u\}.$$

In [1], the authors showed that no two distinct elements in $P_{i_1 i_2 \ldots i_k}$ are adjacent in graph $G^*(P)$. Also, if the index sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_{k'}\}$ of $P_{i_1 i_2 \ldots i_k}$ and $P_{j_1 j_2 \ldots j_{k'}}$, respectively, are distinct, then $P_{i_1 i_2 \ldots i_k} \cap P_{j_1 j_2 \ldots j_{k'}} = \emptyset$. Moreover

$$P^* = \cup_{k=1, 1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n}^{n} P_{i_1 i_2 \ldots i_k}.$$

It is easy to see that $P_{12\ldots n}$ is the set of isolated points in $G^*(P)$ and

$$Z^*(P) = \cup_{k=1, 1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n}^{n} P_{i_1 i_2 \ldots i_k} \setminus P_{12\ldots n}.$$

In this section we want to study the planar index of the $\Gamma(P)$. The *planar index* of the graph $G$ was defined as the smallest $k$ such that $L^k(G)$ is non-planar. We denote the planar index of $G$ by $\xi(G)$. If $L^k(G)$ is planar for all $k \geqslant 0$, we define $\xi(G) = \infty$. In [7], the authors gave a full characterization of graphs with respect to their planar index.

**Theorem 2.1.** [Theorem 10, [7]] *Let $G$ be a connected graph. Then:*

(*i*) $\xi(G) = 0$ *if and only if $G$ is non-planar.*

(*ii*) $\xi(G) = \infty$ *if and only if* $G$ *is either a path, a cycle, or* $K_{1,3}$.

(*iii*) $\xi(G) = 1$ *if and only if* $G$ *is planar and either* $\Delta(G) \geqslant 5$ *or* $G$ *has a vertex of degree* 4 *which is not a cut-vertex.*

(*iv*) $\xi(G) = 2$ *if and only if* $L(G)$ *is planar and* $G$ *contains one of the graphs* $H_i$ *in Figure 1 as a subgraph.*

(*v*) $\xi(G) = 4$ *if and only if* $G$ *is one of the graphs* $X_k$ *or* $Y_k$ (*Figure 1*) *for some* $k \geqslant 2$.

(*vi*) $\xi(G) = 3$ *otherwise.*

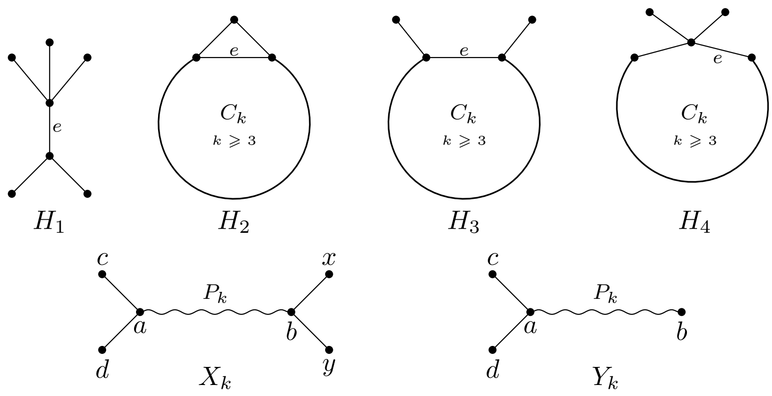


Figure 1

In Section 3 of [1], the planarity of the graph $G(P)$ was studied. In fact, they studied the planarity of $\Gamma(P)$. Since isolated points do not affect planarity, the authors ignored the set $P_{12\ldots n}$ from the vertex set of $G^*(P)$. By using [Section 3, [1]] and Theorem 2.1, we have the following theorem.

**Theorem 2.2.** *Let* $P$ *be a poset and* $Atom(P)$ *be a finite set with* $n$ *elements. Then:*

(*a*) $\xi(\Gamma(P)) = 0$ *if and only if* $\Gamma(P)$ *is non-planar.*

(*b*) $\xi(\Gamma(P)) = \infty$ *if and only if* $n = 1$ *or* $Z(P)$ *is one of the of Figure* 2.

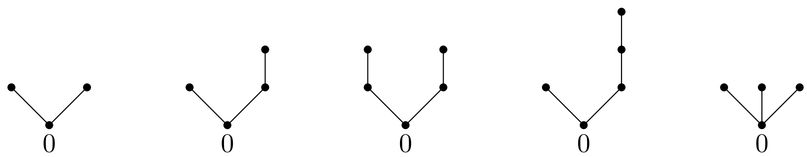


Figure 2

(*c*) $\xi(\Gamma(P)) = 1$ *if and only if*

(*c-1*) $n = 2$ *and one of the following holds:*

(*c-1-1*) $|P_1| = 1$ *and* $|P_2| \geqslant 5$.

(*c-1-2*) $|P_1| = 2$ *and* $|P_2| \geqslant 4$.

(*c-2*) $n = 3$ *and one of the following holds:*

(*c-2-1*) $|\cup_{i=1}^{3} P_i| = 3$ *and* $|P_{ij}| \geqslant 3$ *for some* $1 \leqslant i < j \leqslant 3$.

(*c-2-2*) $|\cup_{i=1}^{3} P_i| = 4$ *and* $|P_{ij}| \geqslant 2$ *for some* $1 \leqslant i < j \leqslant 3$.

(*c-2-3*) $|\cup_{i=1}^{3} P_i| = 5$, $|P_1| = 3$ *and* $P_{23} = \emptyset$.

(*c-2-4*) $|P_1| = |P_2| = 2$, $|P_3| = 1$ *and* $P_{13} = \emptyset$ *or* $P_{23} = \emptyset$.

(*c-2-5*) $|\cup_{i=1}^{3} P_i| = 6$, $|P_2| = |P_3| = 1$ *and* $P_{23} = \emptyset$.

(*c-2-6*) $|\cup_{i=1}^{3} P_i| = 6$, $|P_i| = 2$ *for all* $1 \leqslant i \leqslant 3$, *and* $P_{ij} = \emptyset$ *for all* $1 \leqslant i < j \leqslant 3$.

(*c-2-7*) $|\cup_{i=1}^{3} P_i| \geqslant 7$, $|P_2| = |P_3| = 1$ *and* $P_{23} = \emptyset$.

(*c-3*) $n = 4$ *and one of the following holds:*

(*c-3-1*) $|\cup_{i=1}^{4} P_i| = 4$ *and* $P_{i'j'} = \emptyset$ *whenever* $P_{ij} \neq \emptyset$ *for all* $1 \leqslant i < j \leqslant 4$ *where* $\{i', j'\} = \{1, 2, 3, 4\} \setminus \{i, j\}$ *and* $|P_{ij}| \geqslant 1$ *for some* $1 \leqslant i < j \leqslant 3$ *or* $|P_{ijk}| \geqslant 2$ *for some* $1 \leqslant i < j < k \leqslant 3$.

(*c-3-2*) $|\cup_{i=1}^{4} P_i| = 5$, $|P_1| = 2$, $P_{ij} = \emptyset$ *for all* $2 \leqslant i < j \leqslant 4$ *and* $P_{234} = \emptyset$.

(*d*) $\xi(\Gamma(P)) = 2$ *if and only if*

(*d-1*) $n = 3$ *and one of the following holds:*

(*d-1-1*) $|\cup_{i=1}^{3} P_i| = 3$, $|P_{ij}| \leqslant 2$ *for all* $1 \leqslant i < j \leqslant 3$ *and one of the sets* $P_{ij}$ *has two elements.*

(*d-1-2*) $|\cup_{i=1}^{3} P_i| = 3$, $|P_{ij}| \leqslant 1$ *for all* $1 \leqslant i < j \leqslant 3$ *and two of the sets* $P_{ij}$ *has one elements.*

(*d-1-3*) $|\cup_{i=1}^{3} P_i| = 4$, $|P_{ij}| \leqslant 1$ *for all* $1 \leqslant i < j \leqslant 3$.

(*d-2*) $n = 4$, $|\cup_{i=1}^{3} P_i| = 4$, $P_{ij} = \emptyset$ *for all* $1 \leqslant i < j \leqslant 3$ *and* $|P_{ijk}| \leqslant 1$ *for all* $1 \leqslant i < j < k \leqslant 3$.

(*e*) $\xi(\Gamma(P)) = 3$ *if and only if* $Z(P)$ *is one of the following of Figure* 3*:*

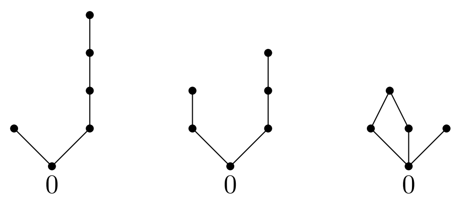


Figure 3

*Proof.* We know that if $\Gamma(P)$ is non-planar then $\xi(\Gamma(P)) = 0$. Thus we may assume that $\Gamma(P)$ is planar. By [1, Lemma 3.1], if $\Gamma(P)$ is planar, then $n \leqslant 4$. Now we have the following cases:

CASE 1. $n = 1$. Note that $\Gamma(P)$ has no edges at all if and only if $n = 1$. Therefore in this case $\xi(\Gamma(P)) = \infty$.

CASE 2. $n = 2$. By Corollarly 2.8 of [1], the graph $\Gamma(P)$ is isomorphic to a complete bipartite graph which its parts are $P_1$ and $P_2$. Therefore, by [1, Proposition 3.2], $\Gamma(P)$ is planar if and only if $|P_1| \leqslant 2$ or $|P_2| \leqslant 2$. Now, we have the following subcases:

(2-1) Suppose that $|P_1| = 1$. If $|P_2| \leqslant 3$, then $\xi(\Gamma(P)) = \infty$. If $|P_2| = 4$, then the line graph of the graph $\Gamma(P)$ is isomorphic to $K_4$ and so it is planar. Also, the line graph of the $\Gamma(P)$ has $H_2$ as a subgraph and $L^2(\Gamma(P))$ is planar. Thus $\xi(\Gamma(P)) = 3$. Otherwise $|P_2| \geqslant 5$. In this situation, $\Delta(\Gamma(P)) \geqslant 5$. Therefore $\xi(\Gamma(P)) = 1$.

(2-2) Suppose that $P_1$ has two elements, say $P_1 = \{a_1, a_1'\}$. If $|P_2| \leqslant 2$, then $\xi(\Gamma(P)) = \infty$. If $P_2$ has three elements, say $P_2 = \{a_2, a_2', a_2''\}$, then the graph $\Gamma(P)$ is isomorphic to $K_{2,3}$. Since $\Delta(\Gamma(P)) = 3$, the line graph of the graph $\Gamma(P)$ is planar. Also, by Figure 4, $L^2(\Gamma(P))$ is planar and $L(\Gamma(P))$ has $H_2$ as a subgraph. So $\xi(\Gamma(P)) = 3$. Otherwise $|P_2| \geqslant 4$. Then, by Theorem 2.1, $\xi(\Gamma(P)) = 1$.

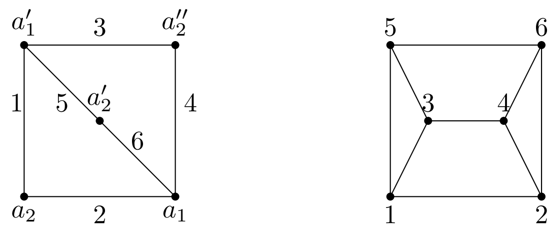


Figure 4: $\Gamma(P)$ and its line graph

CASE 3. $n = 3$. By [1, Theorem 3.3], we have the following subcases:

(3-1) $|\cup_{i=1}^{3} P_i| = 3$. The graph $\Gamma(P)$ is pictured in Figure 5. By Figure 5, if one of the sets $P_{12}$, $P_{13}$ or $P_{23}$ has at least 3 elements, then $\Delta(\Gamma(P)) \geqslant 5$. By Theorem 2.1, $\xi(\Gamma(P)) = 1$. Therefore $|P_{ij}| \leqslant 2$ for all $1 \leqslant i < j \leqslant 3$. We can conclude that $L(\Gamma(P))$ is planar. Now, if only one of the sets $P_{12}$, $P_{13}$ or $P_{23}$ has two elements, then $L(\Gamma(P))$ is planar and $\Gamma(P)$ has $H_4$ as a subgraph, we have that $\xi(\Gamma(P)) = 2$. If only two of the sets $P_{12}$, $P_{13}$ or $P_{23}$ has one element, then $L(\Gamma(P))$ is planar and $\Gamma(P)$ has $H_3$ as a subgraph, we have that $\xi(\Gamma(P)) = 2$. Also, if only one of the sets $P_{12}$, $P_{13}$ or $P_{23}$ has one element, then $L(L(\Gamma(P)))$ is planar and the line graph of the $\Gamma(P)$ has $H_2$ as a subgraph and so $\xi(\Gamma(P)) = 3$. If only two of the sets or all of the sets $P_{12}$, $P_{13}$ or $P_{23}$ has one element, exactly, then $L(\Gamma(P))$ is planar and $\Gamma(P)$

has $H_3$ as a subgraph. Hence $\xi(\Gamma(P)) = 2$. At last, if all the sets $P_{12}$, $P_{13}$ and $P_{23}$ are empty, then $\Gamma(P) \cong C_3$, which implies that $\xi(\Gamma(P)) = \infty$.

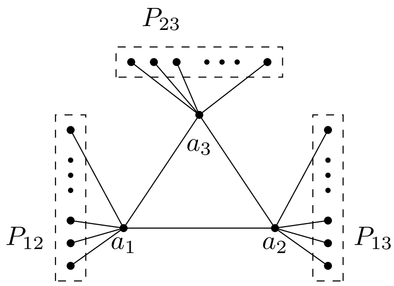


Figure 5

(3-2) $|\cup_{i=1}^{3} P_i| = 4$. Without loss the generality, we may assume $P_1 = \{a_1, a_1'\}$. With this assumption, $\Gamma(P)$ is the graph which was drawn in Figure 6. By Figure 6, if one the sets $P_{12}$ or $P_{13}$ has at least 2 elements, then $\Delta(\Gamma(P)) \geqslant 5$. Also, if $P_{23}$ has at least two elements, then $\Gamma(P)$ has a vertex of degree 4 which is not a cut vertex. So $\xi(\Gamma(P)) = 1$. Otherwise $|P_{ij}| \leqslant 1$ for all $1 \leqslant i < j \leqslant 3$. We can conclude that $L(\Gamma(P))$ is planar. Since $\Gamma(P)$ has $H_2$ as a subgraph, we have that $\xi(\Gamma(P)) = 2$.

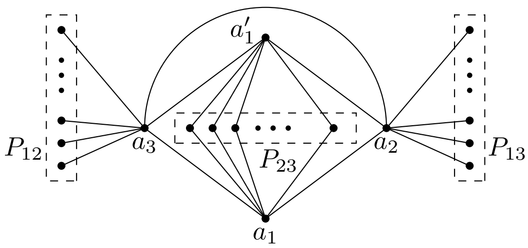


Figure 6

(3-3) $|\cup_{i=1}^{3} P_i| = 5$ and one of the following holds:

(3-3-1) One of the sets $P_i$, say $P_1$, has three elements and $P_{23} = \emptyset$. By Figure 7, if one of the sets $P_{13}$ or $P_{12}$ is non-empty, then $\Delta(\Gamma(P)) \geqslant 5$. So $\xi(\Gamma(P)) = 1$. Now, assume that $P_{12}$ and $P_{13}$ are empty. Since the $\Gamma(P)$ has a vertex of degree 4 which is not a cut vertex, we have that $\xi(\Gamma(P)) = 1$.

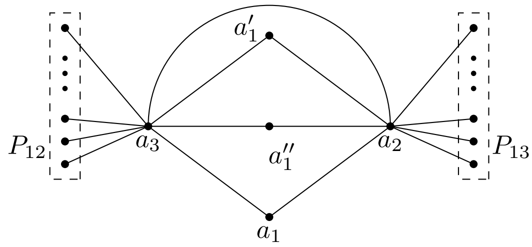


Figure 7

(3-3-2) $|P_i| \leq 2$, for all $1 \leq i \leq 3$, and $P_{13} = \emptyset$ or $P_{23} = \emptyset$. Without loss the generality, we may assume that $P_1 = \{a_1, a_1'\}$, $P_2 = \{a_2, a_2'\}$, $P_3 = \{a_3\}$ and $P_{13} = \emptyset$. Now, by Figure 8, if $P_{12} \neq \emptyset$ or $|P_{23}| \geqslant 2$, then $\Delta(\Gamma(P)) \geqslant 5$. So $\xi(\Gamma(P)) = 1$. Also, $|P_{23}| = 1$, the graph $\Gamma(P)$ has a vertex of degree 4 which is not a cut vetex. Hence $\xi(\Gamma(P)) = 1$. Otherwise, $P_{12} = \emptyset$ and $P_{23} = \emptyset$. In this case the vertex $a_3$ has degree 4 and it is not a cut vertex. Hence $\xi(\Gamma(P)) = 1$.

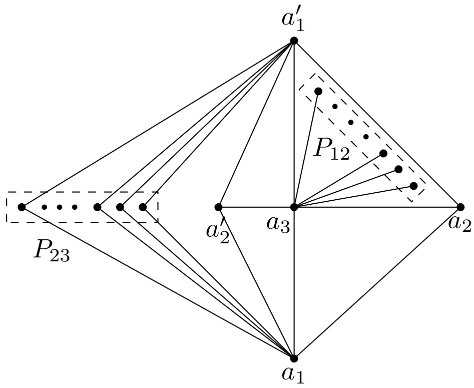


Figure 8

(3-4) Suppose that $| \cup_{i=1}^{3} P_i| = 6$. Now, by Theorem 3.3 of [1], we must consider one of the following cases:

(3-4-1) $|P_2| = |P_3| = 1$ and $P_{23} = \emptyset$. In this case we have that $\Delta(\Gamma(P)) \geqslant 5$ and so $L(\Gamma(P))$ is not planar which implies that $\xi(\Gamma(P)) = 1$.

(3-4-2) $|P_i| = 2$ for all $i = 1, 2, 3$ and $P_{ij} = \emptyset$ for all $1 \leq i < j \leq 3$. By Figure 9, we can see that $\Gamma(P)$ is a 4-regular graph but none of the vertices is a cut vertex. So $\xi(\Gamma(P)) = 1$.

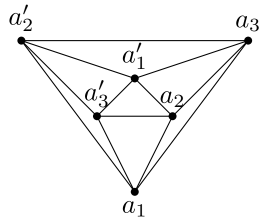


Figure 9

(3-5) At last suppose that $|\cup_{i=1}^{3} P_i| \geq 7$. By Theorem 3.3 of [1], since $\Gamma(P)$ is planar, we have that $|P_2| = |P_3| = 1$ and $P_{23} = \emptyset$. In this case $\Delta(\Gamma(P)) \geqslant 5$. So $\xi(\Gamma(P)) = 1$.

CASE 4. Suppose that $n = 4$. Now we have the following subcases:

(4-1) Suppose that $|\cup_{i=1}^{4} P_i| = 4$. Then, by [1, Theorem 3.5], $\Gamma(P)$ is planar if and only if $P_{i'j'} = \emptyset$ whenever $P_{ij} \neq \emptyset$, where $\{i', j'\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. If $|P_{ijk}| \geqslant 2$ for some $1 \leq i < j < k \leq 4$ or $|P_{ij}| \geqslant 2$ for some $1 \leq i < j \leq 4$, then $\Delta(\Gamma(P)) \geqslant 5$. So we can conclude that $\xi(\Gamma(P)) = 1$. Now suppose that $|P_{ij}| = 1$ for some $1 \leq i < j \leq 4$. Then $a_{i'}$ and $a_{j'}$, when $\{i', j'\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, are adjacent to the element of $P_{ij}$. So the graph $\Gamma(P)$ has a vertex of degree 4 which is not a cut vertex which implies that $\xi(\Gamma(P)) = 1$. Finally, if $P_{ij} = \emptyset$ for all $1 \leq i < j \leq 4$ and $|P_{ijk}| \leqslant 1$ for all $1 \leq i < j < k \leq 4$, then $L(\Gamma(P))$ is planar and $\Gamma(P)$ has $H_2$ as a subgraph. So $\xi(\Gamma(P)) = 2$.

(4-2) Let $|\cup_{i=1}^{4} P_i| = 5$ and $|P_1| = 2$. Then, by [1, Theorem 3.7], $\Gamma(P)$ is planar if and only if $P_{ij} = \emptyset$, for $2 \leq i < j \leq 4$, and $P_{234} = \emptyset$. If $P_{1j} \neq \emptyset$, for some $2 \leqslant j \leqslant 4$ or $P_{1jk} \neq \emptyset$ for some $2 \leqslant j < k \leqslant 4$, then $\Delta(\Gamma(P)) \geqslant 5$ and so $\xi(\Gamma(P)) = 1$. Otherwise, $P_{1j} = \emptyset$ for all $2 \leqslant j \leqslant 4$ and $P_{1jk} = \emptyset$ for all $2 \leqslant j < k \leqslant 4$. In this case, the degree of the vertices $a_2$, $a_3$ and $a_4$ are 4 and these vertices are not cut vertices. Thus the line graph $\Gamma(P)$ is not planar which implies that $\xi(\Gamma(P)) = 1$. $\square$

**Corollary 2.3.** *Let $P$ be a poset with least element 0 and $|Atom(P)| = n$. Then* $\xi(\Gamma(P)) \in \{0, 1, 2, 3, \infty\}$.

## 3. Outerplanar index of $\Gamma(P)$

In this section, we study the outerplanar index of the zero divisor graph of a poset. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of the complete graph $K_4$ or the complete bipartite graph $K_{2,3}$.

The outerplanar index of a graph $G$, which is denoted by $\zeta(G)$, is the smallest integer $k$ such that the $k$th iterated line graph of $G$ is non-outerplanar. If $L^k(G)$ is outerplanar for all $k \geqslant 0$, we define $\zeta(G) = \infty$. In [11], the authors gave a full characterization of all graphs with respect to their outerplanar index.

**Theorem 3.1.** *Let $G$ be a connected graph. Then:*

*(i)* $\zeta(G) = 0$ *if and only if $G$ is non-outerplanar.*

*(ii)* $\zeta(G) = \infty$ *if and only if $G$ is a path, a cycle, or $K_{1,3}$.*

*(iii)* $\zeta(G) = 1$ *if and only if $G$ is planar and $G$ has a subgraph homeomorphic to $K_{1,4}$ or $K_1 + P_3$ in Figure* 10.

*(iv)* $\zeta(G) = 2$ *if and only if $L(G)$ is planar and $G$ has a subgraph isomorphic to one of the graphs $G_2$ or $G_3$ in Figure* 10.

*(v)* $\zeta(G) = 3$ *if and only if $G \in I(d_1, d_2, \ldots, d_t)$ where $d_i \geqslant 2$ for $i = 2, \ldots, t-1$, and $d_1 \geqslant 1$ (Figure* 10*).*

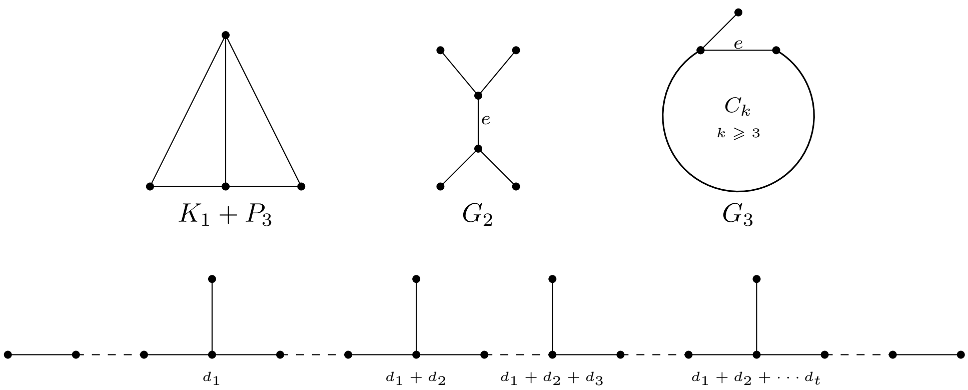


Figure 10

At first, we investigate the outerplanarity of $\Gamma(P)$. We know that the induced subgraph on $\mathrm{Atom}(P)$ is a complete graph. So, if $\Gamma(P)$ is outerplanar then $|\mathrm{Atom}(P)| \leqslant 3$. Thus we must study the cases that $|\mathrm{Atom}(P)|$ is equal to $1, 2$ and 3. When $|\mathrm{Atom}(P)| = 1$, the graph $\Gamma(P)$ has no edges at all which implies that $\Gamma(P)$ is an outerplanar graph. In the following proposition, we state the necessary and sufficient condition for outerplanarity of $\Gamma(P)$ when $|\mathrm{Atom}(P)| = 2$.

**Proposition 3.2.** *Suppose that $|Atom(P)| = 2$. Then $\Gamma(P)$ is outerplanar if and only if one of the following happens:*

(*a*) *one of the sets* $P_1$ *and* $P_2$ *has one element, exactly.*

(*b*) $|P_1| = 2$ *and* $|P_2| = 2$.

*Proof.* Since $\Gamma(P)$ is a complete bipartite graph, we are done. □

**Proposition 3.3.** *Suppose that* $|Atom(P)| = 3$. *Then* $\Gamma(P)$ *is outerplanar if and only if one of the following happens:*

(*a*) $|\cup_{i=1}^{3} P_i| = 3$.

(*b*) $|\cup_{i=1}^{3} P_i| = 4$, $|P_i| = 2$ *and* $P_{jk} = \emptyset$ *for* $j, k \in \{1, 2, 3\} \setminus \{i\}$.

*Proof.* Let $|\text{Atom}(P)| = 3$ and $\Gamma(P)$ is outerplanar. If $|\cup_{i=1}^{3} P_i| \geqslant 5$, then it is not hard to see that $\Gamma(P)$ has a $K_{2,3}$ as a subgraph. So we must investigate the following cases:

1. $|\cup_{i=1}^{3} P_i| = 3$. Now, by Figure 5, $\Gamma(P)$ is outerplanar.

2. $|\cup_{i=1}^{3} P_i| = 4$. In this case one of the sets $P_i$, say $P_1$ has two elements, exactly. Suppose that $P_1 = \{a_1, a_1'\}$ and $x \in P_{23}$. Then, by setting $V_1 := \{a_1, a_1'\}$ and $V_2 := \{a_2, a_3, x\}$, we can find a copy of $K_{2,3}$ in the contraction of $\Gamma(P)$. So $\Gamma(P)$ is not outerplanar. Now suppose that $P_{23}$ is empty. Then, by Figure 11, $\Gamma(P)$ is outerpalanr.

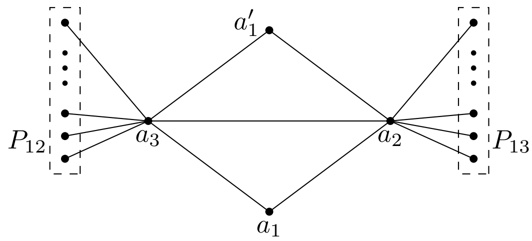


Figure 11

□

In next theorem, we investigate the outerplanar index of the zero divisor graph of a poset when $\text{Atom}(P)$ is a finite set.

**Theorem 3.4.** *Let* $P$ *be a poset and* $|Atom(P)| = n$. *Then:*

(*a*) $\zeta(\Gamma(P)) = 0$ *if and only if* $\Gamma(P)$ *is non-outerplanar.*

(*b*) $\zeta(\Gamma(P)) = \infty$ *if and only if* $n = 1$ *or* $Z(P)$ *is one of the Figure* 2.

(*c*) $\zeta(\Gamma(P)) = 1$ *if and only if*

(*c-1*) $n = 2$, $|P_1| = 1$ *and* $|P_2| \geqslant 4$.

(*c-2*) $n = 3$, $|\cup_{i=1}^{3} P_i| = 3$ *and* $|P_{ij}| \geqslant 2$ *for some* $1 \leqslant i < j \leqslant 3$.

(*c-3*) $n = 3$, $|\cup_{i=1}^{3} P_i| = 4$ *and if* $|P_i| = 2$, *then for* $j, k \in \{1, 2, 3\} \setminus \{i\}$, *the set* $P_{jk}$ *is empty.*

(*d*) $\zeta(\Gamma(P)) = 2$ *if and only if* $n = 3$, $|\cup_{i=1}^{3} P_i| = 3$, $|P_{ij}| = 1$ *for some* $1 \leqslant i < j \leqslant 3$.

*Proof.* We know $\zeta(\Gamma(P)) = 0$ if $\Gamma(P)$ is non-outerplanar. Thus we may assume that $\Gamma(P)$ is outerplanar. If $\Gamma(P)$ is outerplanar, then $n \leqslant 3$. Now we have the following cases:

CASE 1. $n = 1$. Since $\Gamma(P)$ has no edges at all, we have that $\zeta(\Gamma(P)) = \infty$.

CASE 2. $n = 2$. Note that the graph $\Gamma(P)$ is isomorphic to a complete bipartite graph which its parts are $P_1$ and $P_2$. We have the following subcases:

(2-1) Suppose that $|P_1| = 1$. If $|P_2| \leqslant 3$, then $\zeta(\Gamma(P)) = \infty$. If $|P_2| \geqslant 4$, then the line graph of the graph $\Gamma(P)$ has a copy of $K_4$ and so it is not outerplanar. Therefore $\zeta(\Gamma(P)) = 1$.

(2-2) $|P_1| = |P_2| = 2$. In this case $\Gamma(P) \cong C_4$, which implies that $\zeta(\Gamma(P)) = \infty$.

CASE 3. $n = 3$. By Proposition 3.3, we have the following subcases:

(3-1) $|\cup_{i=1}^{3} P_i| = 3$. The graph $\Gamma(P)$ is pictured in Figure 5. By Figure 5, if one the sets $P_{12}$, $P_{13}$ or $P_{23}$ has at least 2 elements, then $\Gamma(P)$ has $K_{1,4}$ as a subgraph. Now, by Theorem 3.1, we have that $\zeta(\Gamma(P)) = 1$. If one of the sets $P_{ij}$ has one elements for some $1 \leqslant i < j \leqslant 3$, then $\Gamma(P)$ has $G_3$ as a subgraph, we have that $\zeta(\Gamma(P)) = 2$. At last, if $P_{ij}$ is empty for all $1 \leqslant i < j \leqslant 3$, $\Gamma(P)$ is a cycle with 3 vertices. So $\zeta(\Gamma(P)) = \infty$.

(3-2) $|\cup_{i=1}^{3} P_i| = 4$ and if $|P_i| = 2$, then for $j, k \in \{1, 2, 3\} \setminus \{i\}$, the set $P_{jk}$ is empty. Without loss the generality, we may assume that $P_1 = \{a_1, a_1'\}$. With this assumption, $\Gamma(P)$ is the graph which was drawn in Figure 11. By Figure 11, $\Gamma(P)$ has a copy of $K_1 + P_3$ which implies that $\zeta(\Gamma(P)) = 1$. □

**Corollary 3.5.** *Let* $P$ *be a poset with least element* 0 *and* $|Atom(P)| = n$. *Then* $\zeta(\Gamma(P)) \in \{0, 1, 2, \infty\}$.

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