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Recognition by order and set of orders of vanishing elements of $C_n(2^\alpha)$, for some n and α

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Abstract. We say that an element g in a finite group G is a vanishing element of G if there exists an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$. The set of orders of vanishing elements of G is denoted by $\mathrm{Vo}(G)$. In [5], the authors put forward the following conjecture: If G is a finite group and M is a finite nonabelian simple group such that $\mathrm{Vo}(G) = \mathrm{Vo}(M)$ and $|G| = |M|$, then $G \cong M$. In this paper we answer positive to this conjecture for a family of classical simple groups $C_n(q)$, where $n = 2^m \geqslant 2$, $q = 2^\alpha$ and $q^n + 1$ is a prime.

1. Introduction

Let n be a positive integer. By $\pi(n)$ we mean the set of prime divisors of n. Let G be a finite group and $\pi(G)$ be the set of prime divisors of $|G|$. Denote by $\omega(G)$, the set of element orders of G. For a finite set of positive integers X, the prime graph $\Pi(X)$ is a graph whose vertices are the prime divisors of elements of X, and two distinct vertices p and q are adjacent if X has an element divisible by pq. We denote the graph $\Pi(\omega(G))$ by $GK(G)$ and we call it the prime graph or the Gruenberg-Kegel graph of G. The number of connected components of $GK(G)$ is denoted by $t(G)$, and the connected components of $GK(G)$ is denoted by $\pi_1(G), \ldots, \pi_{t(G)}(G)$. If there is no ambiguity, we use the notation π_i instead of $\pi_i(G)$. If $2 \in \pi(G)$, we assume that $2 \in \pi_1(G)$. It is easy to see that $|G|$ can be written as the product of coprime positive integers m_i such that $\pi(m_i) = \pi_i(G)$, for $i = 1, \ldots, t(G)$. These integers are called the order components of G.

We denote by $\mathrm{Irr}(G)$ the set of complex irreducible characters of G. We call an element $g \in G$, a *vanishing element*, if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$. Put $\mathrm{Vo}(G)$, the set of orders of all vanishing elements of G. The prime graph $\Pi(\mathrm{Vo}(G))$ is denoted by $\Gamma(G)$ and is called the *vanishing prime graph* of G. Note that for every finite group G, $\Gamma(G)$ is a subgraph of $GK(G)$. There is a strong relation between the structure of a group G and the set $\mathrm{Vo}(G)$. For example, if a finite group G does not have any vanishing element whose order is divisible by p, where $p \in \pi(G)$, then G has a normal Sylow p-subgroup (cf. [2]). In [7], it is proved that if x is a non-vanishing element of a solvable group G, then for some n, x^{2^n} is an element of the Fitting subgroup $F(G)$ and conjectured that $x \in F(G)$. In [13], this

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conjecture has been proved in a special case that if G is solvable and no Mersenne prime divides $|G|$, then every non-vanishing element of G is an element of $F(G)$. In [14], it is proved that the finite simple group A_5 is recognizable by its set of orders of vanishing elements. But not all finite simple groups are characterizable by their set of orders of vanishing elements. For example $\mathrm{Vo}(L_3(5)) = \mathrm{Vo}(\mathrm{Aut}(L_3(5)))$, but $L_3(5) \ncong \mathrm{Aut}(L_3(5))$. In [5], M. Foroudi Ghasemabadi et al. proposed the following conjecture that finite nonabelian simple groups are recognizable by their order and their set of orders of vanishing elements:

Conjecture. *Let G be a finite group and M be a finite nonabelian simple group such that* $\mathrm{Vo}(G) = \mathrm{Vo}(M)$ *and* $|G| = |M|$. *Then* $G \cong M$.

They proved this conjecture for $M = A_1(q)$, where $q \in \{5, 7, 8, 9, 17\}$, $A_4(4)$, A_7, $Sz(8)$ and $Sz(32)$. Also in [4], the conjecture has been proved where M is a sporadic simple group, an alternating group, $A_1(p)$, for an odd prime p, and finite simple K_3-groups and K_4-groups. In this paper, we show that this conjecture is true for classical simple groups $C_n(q)$, where $n = 2^m \geqslant 2$, $q = 2^\alpha$ and $q^n + 1$ is a prime. In fact, we prove the following theorem:

Main Theorem. *Let G be a group and $M = C_n(q)$, where $n = 2^m \geqslant 2$, $q = 2^\alpha$ and q^n+1 is a prime. Then $G \cong M$ if and only if* $\mathrm{Vo}(G) = \mathrm{Vo}(M)$ *and* $|G| = |M|$.

Let k and n be coprime integers. If there is an integer x such that $x^2 \equiv k \pmod n$, then k is called a quadratic residue mode n, otherwise k is called a quadratic nonresidue mode n. For a prime p, the symbol (a/p) is defined as follows: $(a/p) = 1$ if a is a quadratic residue mode p, $(a/p) = -1$ if a is a quadratic nonresidue mode p, and $(a/p) = 0$ if $p \mid a$. It is a well known result that $(-1/p) = (-1)^{(p-1)/2}$.

Let n and m be positive integers and p be a prime. We write $p^m \| n$, if $p^m \mid n$ but $p^{m+1} \nmid n$. We write $n_p = p^m$, if $p^m \| n$. All further notation can be found in [1], for instance.

2. Preliminary results

Definition 1. A finite group G is called a 2-*Frobenius group* if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

The following lemma (see [9]) summarizes the basic structural properties of a Frobenius group and a 2-Frobenius group:

Lemma 1.

(*a*) *Let G be a Frobenius group and let H, K be the Frobenius complement and the Frobenius kernel of G, respectively. Then $t(G) = 2$ and the prime graph components of G are $\pi(H)$ and $\pi(K)$. Moreover, K is nilpotent and hence $GK(K)$ is a complete graph.*

(*b*) *If* G *is a* 2*-Frobenius group then* $t(G) = 2$*. With the notations of Definition* 1*, we also have* $\pi_1 = \pi(G/K) \cup \pi(H)$ *and* $\pi_2 = \pi(K/H)$*.*

The next lemma is a consequence of Gruenberg–Kegel Theorem (see [12]):

Lemma 2. *If* G *is a finite group with disconnected prime graph* $GK(G)$*, then one of the following holds:*

(1) *the finite group* G *is a Frobenius group and* $t(G) = 2$*;*

(2) *the finite group* G *is a* 2*-Frobenius group and* $t(G) = 2$*;*

(3) *the finite group* G *has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$*, such that* H *and* G/K *are* π_1*-groups and* K/H *is a nonabelian simple group, where* H *is a nilpotent group and* $|G/K| \mid |\mathrm{Out}(K/H)|$*.*

Lemma 3. [2, 3] *If* G *is a finite nonabelian simple group except* A_7*, then* $GK(G) = \Gamma(G)$*.*

As a consequense of [8, Corollary 22.26], we get the following lemma:

Lemma 4. *If* $\chi \in Irr(G)$ *vanishes on a* p*-element for some prime* p*, then* $p|\chi(1)$*.*

Let p be a prime number. A character $\chi \in Irr(G)$ is said to be of p-defect zero if p is not a divisor of $|G|/\chi(1)$. It is a well-known result that if χ is of p-defect zero, then for every element $g \in G$ which order is divisible by p, we have $\chi(g) = 0$ (see for example [6, Theorem 8.17]).

Lemma 5. [9, Lemma 2.5] *Let* G *be a finite group with* $t(G) \geqslant 2$*, and let* N *be a normal subgroup of* G*. If* N *is a* π_i*-group for some prime graph component of* G *and* $m_1, m_2, \ldots, m_r$ *are some order components of* G *but not* π_i*-numbers, then* $m_1 m_2 \cdots m_r$ *is a divisor of* $|N| - 1$*.*

Lemma 6. [10, Lemma 8] *Assume* $q > 1$ *is a natural number,* $s = \prod_{i=1}^{n}(q^i - 1)$*,* p *is a prime,* $p \mid s$*. We denote the power of* p *in the standard factorization of* s *by* s_p*,* $e = \min\{d : p \mid q^d - 1\}$*,* $q^e = 1 + p^r k$*,* $p \nmid k$*. If* $p > 2$ *or* $r > 2$*, then* $s_p < q^{np/(p-1)}$*.*

Lemma 7. (Zsigmondy Theorem) [15] *Let* p *be a prime and let* n *be a positive integer. Then one of the following holds:*

(*i*) *there is a primitive prime* p' *for* $p^n - 1$*, that is ,* $p' \mid (p^n - 1)$ *but* $p' \nmid (p^m - 1)$*, for every* $1 \leqslant m < n$*,*

(*ii*) $p = 2$*,* $n = 1$ *or 6,*

(*iii*) p *is a Mersenne prime and* $n = 2$*.*

Lemma 8. [9, Lemma 2.9] *The equation* $p^m - q^n = 1$*, where* p *and* q *are primes and* $m, n > 1$ *has only one solution, namely* $3^2 - 2^3 = 1$*.*

3. Main results

Theorem 1. *Let G be a group and $M = C_n(q)$, where $n = 2^m \geqslant 2$, $q = 2^\alpha$ and $q^n + 1$ is odd prime. Then $G \cong M$ if and only if $Vo(G) = Vo(M)$ and $|G| = |M|$.*

Proof. If $G \cong M$, the result follows obviously. Let $Vo(G) = Vo(M)$ and $|G| = |M| = q^{n^2}(q^n - 1)\prod_{i=1}^{n-1}(q^{2i} - 1)(q^n + 1)$. Let $l = q^n + 1$. Since l is a prime number there exists a natural number s such that $n\alpha = 2^s$. We continue proof in the following steps:

• Step 1. The prime graph of G is disconnected.

According to Lemma 3, we have $\Gamma(G) = \Gamma(M) = GK(M)$. Hence, $\Gamma(G)$ has 2 connected components [11] and l is an isolated vertex in $\Gamma(G)$. So G has an l-element g such that $\chi(g) = 0$ for some irreducible complex character χ of G. Now, Lemma 4, implies that l divides $\chi(1)$. Since $l \| |M|$ and $|G| = |M|$, χ is an irreducible character of l-defect zero of G. So for every element $h \in G$ such that $l \mid o(h)$, we conclude that $\chi(h) = 0$. So, by the fact that l is an isolated vertex in $\Gamma(G)$, we conclude that l is an isolated vertex in $GK(G)$. Hence $t(G) \geq 2$.

• Step 2. G is not a Frobenius group.

Let G be a Frobenius group with complement H and kernel K. Consequently, $GK(G)$ has two connected components, namely $\pi(H)$ and $\pi(K)$. Since l is an isolated vertex in $GK(G)$, then l is a connected component. Since $|H| \mid (|K| - 1)$, we conclude that $|H| = l$. There exists a primitive prime divisor x of $q^n - 1$. Set $S \in \mathrm{Syl}_x(K)$, so $S \trianglelefteq G$ and $|S| \mid (q^n - 1)$. On the other hand, H acts fixed point freely on S, and consequently $|S| \equiv 1 \pmod{l}$, which is a contradiction.

• Step 3. G is not a 2-Frobenius group.

Let G be a 2-Frobenius group, so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi_2(G) = \pi(K/H)$ and $|G/K| \mid (|K/H| - 1)$. Therefore $|K/H| = l$ and $|G/K| \mid q^n$. Then $(q^n - 1) \mid |H|$. Let x be a primitive prime of $q^n - 1$ and $S \in \mathrm{Syl}_x(H)$. So similarly to Step 2, we get a contradiction.

• Step 4. G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups, K/H is a nonabelian simple group with disconnected prime graph, H is a nilpotent group and $|G/K| \mid |\mathrm{Out}(K/H)|$.

It follows from Lemma 2 and Steps 1, 2 and 3.

In the following, let K/H be the same as in Step 4.

• Step 5. K/H is not an sporadic group.

Let $K/H \cong M_{22}$. It is clear that l is not equal to 7 or 11. So $l = 2^{n\alpha} + 1 = 5$, hence $n\alpha = 2$. So $\alpha = 1$, in this case $|K/H| \nmid |G|$, which is a contradiction.
Similarly, K/H cannot be isomorphic to other sporadic groups.

• Step 6. K/H is not an alternating group.

Let $K/H \cong A_{p'}$, where $p' - 2$ is not odd prime.
Therefore $p' = l$, and so $|K/H| = (l)!/2 \mid |G|$, which is impossible.
Similarly, K/H can not be isomorphic to A_m, where $m \in \{p' + 1, p' + 2\}$ and m or $m - 2$ is not odd prime and K/H can not be isomorphic to $A_{p'}$, where p' and $p' - 2$ are prime numbers.

Therefore, K/H is a Lie type group, by classification of simple group.

• Step 7. K/H is isomorphic to $C_n(q)$, where $n = 2^m \geqslant 2$, $q = 2^\alpha$ and $q^n + 1$ is odd prime.

We get that K/H is a Lie type group. Now by [11, Tables 1a-1c], we consider each possibility for K/H, separately:

• Case 1. Let $K/H \cong A_{p'-1}(q')$, where $q' = r^f$ and p' is an odd prime. Therefore, $(q'^{p'} - 1)/((q' - 1)(p', q' - 1)) = l = 2^{n\alpha} + 1$. We know that

$$q'^{p'} - 1 \geqslant \frac{q'^{p'} - 1}{(q' - 1)(p', q' - 1)} = 2^{n\alpha} + 1 \quad \Rightarrow \quad q'^{p'} \geqslant 2^{n\alpha}.$$

On the other hand, let $S \in \mathrm{Syl}_r(G)$, so $q'^{p'(p'-1)/2} \mid |S|$. Assume that $r \neq 2$, so by Lemma 6, $|S| < 2^{2n\alpha r/(r-1)} \leqslant q'^{3p'}$. Consequently, $p'(p' - 1)/2 < 3p'$, which implies that $p' = 3$ or $p' = 5$.

Let $p' = 3$.

Consider $(p', q' - 1) = 1$. Then $(q'^3 - 1)/(q' - 1) = q'^2 + q' + 1 = 2^{n\alpha} + 1$. It follows that $q'(q' + 1) = 2^{n\alpha}$, which is a contradiction.

Let $(p', q' - 1) = 3$, then $(q'^2 + q' + 1)/3 = (r^{3f} - 1)/3(r^f - 1) = l$. We claim that $\pi(f) = \{3\}$. Let $f = 3^i t$, for some non-negative integers i and t also $3 \nmid t$. So $(r^{3^{i+1}t} - 1)/3(r^{3^i t} - 1) = (r^{3^{i+1}} - 1)(r^{3^{i+1}(t-1)} + \cdots + 1)/3(r^{3^i t} - 1) = l$. Therefore, $(r^{3^{i+1}} - 1) \mid (r^{3^i t} - 1)$, so by Lemma 7, we get that $3^{i+1} \mid 3^i t$, a contradiction. So $\pi(f) = \{3\}$, and consequently $\pi(G/K) \subseteq \{2, 3\}$, since $|G/K| \mid |\mathrm{Out}(K/H)|$. Let x be a primitive prime of $q^n - 1$ and $x \notin \{3, 5\}$. Therefore, x is a divisor of $(q'^2 + q' + 1)/3 - 2 = (q'^2 + q' - 5)/3$. It is easy to get that $x \nmid |K/H| = q'^3(q' - 1)(q'^2 - 1)(q'^2 + q' + 1)/3$. So $x \in \pi(H)$. Let $T \in \mathrm{Syl}_x(H)$. So $T \trianglelefteq G$ and $|T| \mid (q^n - 1)$. Now by Lemma 5 we have $|T| \equiv 1 \pmod{l}$, a contradiction. If $x = 3$, then $\alpha = 1$ and $n = 2$. In this case, $|K/H| \nmid |G|$, which is a contradiction. If $x = 5$, then $n\alpha = 4$. So $l = 17$, hence $(q'^2 + q' + 1)/3 = 17$, which is a contradiction.

Let $p' = 5$. Completely similar to the above we get a contradiction.

Therefore $r = 2$. If $(p', q' - 1) = 1$, then $2^{2f} + 2^f + 1 = 2^{n\alpha} + 1$, which is a contradiction. Otherwise, $(p', q' - 1) = 3$ and so $(2^{2f} + 2^f + 1)/3 = 2^{n\alpha} + 1$. It follows that $n = 1$, which is a contradiction.

Similarly, $K/H \ncong {}^2A_{p'-1}(q')$, where $q' = r^f$ and p' is an odd prime.

• Case 2. Let $K/H \cong A_{p'}(q')$, where $q' = r^f$ and $(q' - 1) \mid (p' + 1)$. So we have $(q'^{p'} - 1)/(q' - 1) = l = 2^{n\alpha} + 1$. Consequently, $q'^{p'-1} + q'^{p'-2} + \ldots + 1 = 2^{n\alpha} + 1$, which is a contradiction.

Similarly, $K/H \ncong {}^2A_{p'}(q')$, where $q' = r^f$ and $(q' + 1) \mid (p' + 1)$.

• Case 3. Let $K/H \cong B_{p'}(3)$.

Therefore, $(3^{p'} - 1)/2 = 2^{n\alpha} + 1$, hence $3^{p'} - 3 = 2^{n\alpha+1}$, which is a contradiction. Similarly, K/H cannot be isomorphic to $C_{p'}(q')$, where $q' = 2$ or 3, $D_{p'}(q')$, where $p' \geqslant 5$ $q' = 2$, 3 or 5, $D_{p'+1}(q')$, where $q' = 2$ or 3 and ${}^2D_p(3)$, where $5 \leqslant p \neq 2^{m'} + 1$.

• CASE 4. Let $K/H \cong B_{n'}(q')$, where $n' = 2^{m'} \geqslant 4$ and $q' = r^f$ is odd. We have

$$(q'^{n'} + 1)/2 = 2^{n\alpha} + 1 \quad \Rightarrow \quad q'^{n'} - 2^{n\alpha+1} = 1,$$

which is a contradiction, by Lemma 8.
Similarly, K/H cannot be isomorphic to ${}^2D_{n'}(3)$, where $n' = 2^{m'} + 1$ is not a prime number and $m' \geq 2$.

• CASE 5. Let $K/H \cong {}^2D_{n'}(q')$, where $n' = 2^{m'} \geqslant 4$.
Hence $(q'^{n'} + 1)/(2, q' + 1) = 2^{n\alpha} + 1$. If $(2, q' + 1) = 2$, then $q'^{n'} - 2^{n\alpha+1} = 1$, which is a contradiction, by Lemma 8. Therefore, $(2, q'+1) = 1$ and so $q'^{n'} = 2^{n\alpha}$. Hence $r = 2$ and $n'f = n\alpha$. Since $|K/H| \mid |G|$, so $n' - 1 \leqslant n$. We know that $n\alpha = 2^s$, therefore $n' \mid n$ and $\alpha \mid f$, so $n \geq 2n'$ and $\alpha \leqslant f/2$. Let x be a primitive prime divisor of $q^{n-1} - 1$. We have three following cases:

1. Let $x \in \pi(H)$ and $S \in Syl_x(H)$. So $S \trianglelefteq G$ and $|S| \mid (q^{n-1} - 1)$. On the other hand, we have $|S| \equiv 1 \pmod{l}$, which is a contradiction.

2. Let $x \in \pi(G/K)$. We have $\pi(G/K) \subseteq \pi(\mathrm{Out}(K/H)) = \pi(f) \cup \{2\}$, hence $x \mid f$. We know that $n\alpha = 2^s$, hence $n'f = 2^s$. It follows that $x = 2$, which is a contradiction.

3. Let $x \in (K/H)$, then there exists natural number t such that x is a primitive prime divisor of $2^{ft} - 1$. Consequently, $ft = (n-1)\alpha$. Hence $f(n' - t) = \alpha \leqslant f/2$, which is a contradiction.

• CASE 6. Let $K/H \cong {}^2D_{n'}(2)$, where $n' = 2^{m'} + 1 \geqslant 5$.
We have

$$2^{n'-1} + 1 = l \quad \Rightarrow \quad n' = n\alpha + 1.$$

Therefore, we get that $2^{n\alpha+1} + 1 \mid |K/H|$ and $2^{n\alpha+1} + 1 \nmid |G|$, which is impossible.

• CASE 7. Let $K/H \cong {}^2D_p(3)$, where $p = 2^{n'} + 1$ and $n' \geqslant 2$.
If $(3^p + 1)/4 = 2^{n\alpha} + 1$, then $3^p - 3 = 2^{n\alpha+2}$, which is a contradiction. Otherwise, $(3^{p-1} + 1)/2 = 2^{n\alpha} + 1$ so $3^{p-1} - 2^{n\alpha+1} = 1$, which is a contradiction, by Lemma 8.

• CASE 8. Let $K/H \cong G_2(q')$, where $q' \equiv \varepsilon \pmod{3}$, $\varepsilon = \pm 1$ and $q' = r^f > 2$. We have

$$q'^2 - \varepsilon q' + 1 = l \quad \Rightarrow \quad q'(q' - \varepsilon) = 2^{n\alpha},$$

which is impossible.
Similarly, K/H cannot be isomorphic to ${}^3D_4(q')$, $F_4(q')$, where q' is odd.

• CASE 9. Let $K/H \cong E_6(q')$, where $q' = r^f$.
Consequently, $(q'^6 + q'^3 + 1)/(3, q' - 1) = l$. If $(3, q' - 1) = 1$, then $q'^3(q'^3 + 1) = 2^{n\alpha}$, which is a contradiction. Otherwise, $(3, q' - 1) = 3$ and so $q'^3(q'^3 + 1) = 2(3.2^{n\alpha-1} + 1)$. Therefore $r \neq 2$ and we have

$$q'^9 > q'^9 - 1 > (q'^6 + q'^3 + 1)/3 = 2^{n\alpha} + 1 > 2^{n\alpha}.$$

Let $S \in Syl_r(G)$, hence $q'^{36} \mid |S|$. By Lemma 6, we have

$$|S| < q^{2nr/(r-1)} \leqslant 2^{3n\alpha} \leqslant q'^{27}.$$

Which implies that $36 \leqslant 27$, and it is a contradiction.
Similarly, $K/H \ncong {}^2E_6(q')$.

• Case 10. Let $K/H \cong A_1(q')$, where $q' = r^f$.
We consider three following cases:
1. Let $4 \mid (q'+1)$.

If $q' = l$, then $r^f - 2^{n\alpha} = 1$, so $f = 1$, by Lemma 8. Hence $r = l$ and so $(l+1)/2 = (2^{n\alpha-1}+1) \mid |G|$, which is a contradiction. Otherwise, $(q'-1)/2 = l$, hence l is a primitive prime divisor of $r^f - 1$, which implies that $f \mid (l-1)$. Therefore, $f \mid 2^{n\alpha}$ and hence $\pi(f) = \{2\}$. Consequently, $\pi(G/K) = \{2\}$, since $|G/K| \mid |\mathrm{Out}(K/H)|$. Moreover, $|K/H| = (2^{n\alpha+1}+3)(2^{n\alpha}+1)(2^{n\alpha+1}+4)$. Let x be a primitive prime divisor of $(2^{n\alpha}-1)$ and $x \notin \{3,5\}$, so $x \mid |H|$. Assume that $S \in Syl_x(H)$, so $S \trianglelefteq G$ and $|S| \mid (2^{n\alpha}-1)$. On the other hand, by Lemma 5, we have $|S| \equiv 1 \pmod{l}$, which is a contradiction. If $x = 3$, then $n\alpha = 2$, hence $n = 2$ and $\alpha = 1$. It follows that $|K/H| \nmid |G|$, which is a contradiction. So $x = 5$, then $l = 17$ which implies that $q' = 35$, which is a contradiction.

2. Let $4 \mid (q'-1)$.

If $q' = l$, then $r^f - 2^{n\alpha} = 1$, so $f = 1$, by Lemma 8. So similar to the above we get a contradiction. Otherwise, $(q'+1)/2 = l$ so $q' - 2^{n\alpha+1} = 1$, Therefore, $f = 1$, by Lemma 8 and we get a contradiction similar to the above.

3. Let $q' = 2^f$.

Assume that $q'+1 = 2^{n\alpha}+1$, so $q' = q^n$, hence $f = n\alpha = 2^s$, where s is a natural number. Let x be a primitive prime divisor of $q^{n-1}-1$. It is clear that $x \notin \pi(K/H)$, hence $x \in \pi(G/K) \cup \pi(H)$. If $x \in \pi(G/K)$, then $x \in \mathrm{Out}(K/H) = 2f$, which is a contradiction. Otherwise, $x \in \pi(H)$. Let $S \in Syl_x(H)$, so $S \trianglelefteq G$ and $|S| \mid (2^{(n-1)\alpha}-1)$. Also by Lemma 6, we know that $l \mid |S|-1$, which is impossible. Therefore, $q'-1 = l = 2^{n\alpha}+1$, which is a contradiction.

• Case 11. Let $K/H \cong {}^2B_2(q')$, where $q' = 2^{2n'+1} > 2$.
It is clear that $q'-1 \neq l$. Hence $q' \pm \sqrt{2q'} + 1 = l$ and so $q' \pm \sqrt{2q'} = 2^{n\alpha}$, which is impossible.
Similarly, K/H cannot be isomorphic to ${}^2F_4(q')$, where $q' = 2^{2n'+1} > 2$, $G_2(q')$, where $q' = 3^f$, ${}^2G_2(q')$, where $q' = 3^{2n'+1}$ and $E_8(q')$.

• Case 12. Let $K/H \cong F_4(q')$, where $q' = 2^{n'} > 2$.
If $q'^4+1 = l$, then $n' = n\alpha/4$. We know that $(q'^4-1)^2 \mid |K/H|$, hence $(2^{n\alpha}-1)^2 \mid |G|$, which is a contradiction. Consequently, $q'^4 - q'^2 + 1 = l$ hence $q'^2(q'^2-1) = 2^{n\alpha}$, that is impossible.

• Case 13. Let $K/H \cong C_{n'}(q')$, where $n' = 2^{m'} \geqslant 2$ and $q' = r^f$.
Therefore, $(q'^{n'}+1)/(2, q'-1) = l$. If $(2, q'-1) = 2$, then $q'^{n'} - 2^{n\alpha+1} = 1$. By Lemma 8, we get that $n = 2$, $\alpha = 1$, $q' = 3$ and $n' = 2$. Hence, $K/H \cong C_2(3)$ and $G \cong C_2(2)$. In this case, $|K/H| \nmid |G|$, which is a contradiction. Otherwise, $(2, q'-1) = 1$ and so $r = 2$ and $n'f = n\alpha$. Since $|K/H| \mid |G|$, so $n' \leqslant n$. Let $n' < n$. We know that $n\alpha = 2^s$, so $n' \mid n$. Consequently, $n \geqslant 2n'$ and so $\alpha \leqslant f/2$.

We know that $2^{(n-1)\alpha} - 1$ has a primitive prime divisor, we say it x. We claim that $x \in \pi(K/H)$. If $x \notin \pi(K/H)$, then $x \in \pi(G/K) \cup \pi(H)$. Assume that $x \in \pi(G/K)$, so $x \in \mathrm{Out}(K/H) = f$, which implies that $x = 2$ that is a contradiction.

Therefore, $x \in \pi(H)$. Let $S \in Syl_x(H)$, so $S \trianglelefteq G$, hence $|S| \mid (q^{n-1}-1)$. Moreover, we know that $l \mid (|S|-1)$, by Lemma 5, which is a contradiction. Therefore, $x \in \pi(K/H)$. Consequently, there exists a natural number t such that x is primitive prime divisor of $2^{tf}-1$. It follows that $(n-1)\alpha = tf$. So $(n'-t)f = \alpha \leqslant f/2$, which is a contradiction. Therefore, $n = n'$ and so $\alpha = f$. It follows that $K/H \cong C_n(q)$. So, $H = 1$ and $G = K$. Consequently, $G \cong C_n(q)$, as required. □

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