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Action of the group $\langle x, y : x^2 = y^6 = 1 \rangle$ on imaginary quadratic fields

Abdul Razaq

Abstract. Let $H =< x, y : x^2 = y^6 = 1 >$ be acting on $\mathbb{Q}\left(\sqrt{-n}\right)$ and denote the subset $\left\{\frac{a+\sqrt{-n}}{3c} : a, \frac{a^2+n}{3c}, c \in \mathbb{Z}\setminus\{0\}\right\}$ of $\mathbb{Q}\left(\sqrt{-n}\right)$ by $\mathbb{Q}^*\left(\sqrt{-n}\right)$. Also $d(n)$ denotes the arithmetic function which is defined as the number of positive divisors of n which are multiple of 3. In this paper, we show that the total number of orbits of $\mathbb{Q}^*\left(\sqrt{-n}\right)$ under the action of H are

$$\begin{cases} 4 & \text{if } n = 3, \\ d(n) & \text{if } n \equiv 0 \pmod 3, \text{ but } n \neq 3, \\ 2d(n+1) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

1. Introduction

Let F be an extension field of degree two over the field $\mathbb{Q}$ of rational numbers. Then any element $x \in F\setminus\mathbb{Q}$ is of degree two over $\mathbb{Q}$ and is a primitive element of F (that is $F = \mathbb{Q}[x]$ and $\{1, x\}$ is a base of F over $\mathbb{Q}$). Let $p(x) = x^2 + bx + c$, where $b, c \in \mathbb{Q}$, be the minimal polynomial of such an element $x \in F$. Then $2x = -b \pm \sqrt{b^2 - 4c}$ and so, $F = \mathbb{Q}(\sqrt{b^2 - 4c})$. Since $b^2 - 4c$ is a rational number $\frac{u}{v} = \frac{uv}{v^2}$ with $u, v \in \mathbb{Z}$, we obtain $F = \mathbb{Q}(\sqrt{uv})$. In fact it is possible to write $F = \mathbb{Q}(\sqrt{n})$, where n is a square-free integer. If n is a negative square-free integer, then $\mathbb{Q}(\sqrt{n})$ is called an *imaginary quadratic field* and the elements of $\mathbb{Q}(\sqrt{n})$ are of the form $a + b\sqrt{n}$ with $a, b \in \mathbb{Q}$. The imaginary quadratic fields are usually denoted by $\mathbb{Q}(\sqrt{-n}) = \left\{a + b\sqrt{-n} : a,\ b \in \mathbb{Q}\right\}$, where n is a square-free positive integer. Imaginary quadratic fields are the only type (apart from $\mathbb{Q}$) with a finite unit group. This group has order 4 for $\mathbb{Q}(\sqrt{-1})$ (and generator $\sqrt{-1}$), order 6 for $\mathbb{Q}(\sqrt{-3})$ $\left(\text{and generator } \frac{1+\sqrt{-3}}{2}\right)$, and order 2 (and generator -1) for all other imaginary quadratic fields. We denote the subset $\left\{\frac{a+\sqrt{-n}}{3c} : a, \frac{a^2+n}{3c} \in \mathbb{Z} \text{ and } c \in \mathbb{Z}\setminus\{0\}\right\}$ of $\mathbb{Q}\left(\sqrt{-n}\right)$ by $\mathbb{Q}^*\left(\sqrt{-n}\right)$. Some fundamental properties of imaginary quadratic fields have been discussed in [2] and [3].

Let G be a group generated by the linear fractional transformations x and y satisfying the relations $x^2 = y^m = 1$. If $y : z \longrightarrow \frac{az+b}{cz+d}$ is to act on all imaginary

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quadratic fields, then a, b, c, d must be rational numbers and can taken to be integers, so that $\frac{(a+d)^2}{ad-bc}$ is rational. But if $y : z \longrightarrow \frac{az+b}{cz+d}$ is of order m, one must have $\frac{(a+d)^2}{ad-bc} = w + w^{-1} + 2$, where w is a primitive mth root of unity. Now $w + w^{-1}$ is rational, for a primitive mth root w, only if $m = 1, 2, 3, 4$ or 6. So these are the only possible orders of y. The group $< x, y >$ is cyclic of order 2, when $m = 1$. When $m = 2$, it is an infinite dihedral group and does not give inspiring information while studying its action on imaginary quadratic numbers. For $m = 3$, the group $< x, y >$ is the modular group $PSL(2, \mathbb{Z})$ and its action on real quadratic numbers has been discussed in detail in [4] and [5].

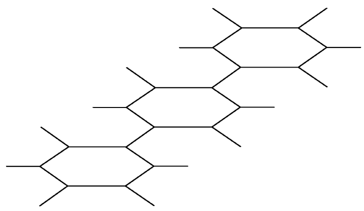
In this paper, we are interested in the action of the group $H =< x, y : x^2 = y^6 = 1 >$, where $(z)x = \frac{-1}{3z}$ and $(z)y = \frac{-1}{3(z+1)}$ are linear fractional transformations, on $\mathbb{Q}^*\left(\sqrt{-n}\right) = \left\{\frac{a+\sqrt{-n}}{3c} : a, \frac{a^2+n}{3c} \in \mathbb{Z} \text{ and } c \in \mathbb{Z}\setminus\{0\}\right\}$. Note that, $\mathbb{Q}^*\left(\sqrt{-n}\right)$ remains invariant under the action of H. We show that the total number of orbits of $\mathbb{Q}^*\left(\sqrt{-n}\right)$ under the action of H are

$$\begin{cases} 4 & \text{if } n = 3 \\ d(n) & \text{if } n \equiv 0 \pmod 3\text{, but } n \neq 3 \\ 2d(n+1) & \text{if } n \equiv 2 \pmod 3 \end{cases}.$$

2. Coset Diagrams

We use coset diagrams for the group H and study its action on the projective line over imaginary quadratic fields. The coset diagrams for the group H are defined as follows. The six cycles of transformation y are represented by six unbroken edges of a hexagon (may be irregular) permuted counter-clockwise by y. Any two vertices which are interchanged by involution x, is joined by an edge. The fixed points of x and y, if they exist, are denoted by heavy dots. This graph can be interpreted as a coset diagram,with the vertices identified with the cosets of $Stab_v(H)$, the stabilizer of some vertex v of the graph, or as 1-skeleton of the cover of the fundamental complex of the presentation which corresponds to the subgroup $Stab_v(H)$. For more details about coset diagrams, one can refer to [1],[6],[7] and [8].

A general fragment of the coset diagram of the action of H on $\mathbb{Q}^*\left(\sqrt{-n}\right)$ will look as follows.



Definition 2.1. If $\alpha = \frac{a+\sqrt{-n}}{3c} \in \mathbb{Q}^*\left(\sqrt{-n}\right)$ is such that $ac < 0$ then α is called a *totally negative imaginary quadratic number* and it is called a *totally positive imaginary quadratic number* if $ac > 0$.

As $d = \frac{a^2+n}{3c}$, so dc is always positive. Thus d and c will have the same sign. Hence an imaginary quadratic number $\alpha = \frac{a+\sqrt{-n}}{3c} \in \mathbb{Q}^*\left(\sqrt{-n}\right)$ is totally negative if either $a < 0$ and $d, c > 0$ or $a > 0$ and $d, c < 0$. Similarly $\alpha = \frac{a+\sqrt{-n}}{3c} \in \mathbb{Q}^*\left(\sqrt{-n}\right)$ is totally positive if either $a, d, c > 0$ or $a, d, c < 0$.

For $\alpha = \frac{a+\sqrt{-n}}{3c} \in \mathbb{Q}^*\left(\sqrt{-n}\right)$, norm of α is denoted by $\parallel \alpha \parallel$ and $\parallel \alpha \parallel = \mid a \mid$.

3. Main results

Theorem 3.1. *If* $\alpha = \frac{a+\sqrt{-n}}{3c} \in Q^*\left(\sqrt{-n}\right)$, *then* n *does not change its value in the orbit* αH.

Proof. Let $\alpha = \frac{a+\sqrt{-n}}{3c}$ and $d = \frac{a^2+n}{3c}$. Since $(\alpha)x = \frac{-1}{3\alpha} = \frac{-1}{3\left(\frac{a+\sqrt{-n}}{3c}\right)} = \frac{-c}{a+\sqrt{-n}} = = \frac{-c\left(a-\sqrt{-n}\right)}{a^2+n} = \frac{-a+\sqrt{-n}}{3d}$, therefore the new values of a and c for $(\alpha)x$ are $-a$ and d respectively. The new value of d for $(\alpha)x$ is $\frac{a^2+n}{3d} = \frac{a^2+n}{3\left(\frac{a^2+n}{3c}\right)} = c$. Since $(\alpha)y = \frac{-1}{3(\alpha+1)} = \frac{-1}{3\left(\frac{a+\sqrt{-n}}{3c}+1\right)} = \frac{-1}{3\left(\frac{a+\sqrt{-n}+3c}{3c}\right)} = \frac{-3c\left(a+3c-\sqrt{-n}\right)}{3[(a+3c)^2+n]} = \frac{-a-3c+\sqrt{-n}}{3(2a+d+3c)}$, therefore the new values of a and c for $(\alpha)y$ are $-a-3c$ and $(2a+d+3c)$ respectively. Moreover, the new value of d for $(\alpha)y$ is $\frac{(-a-3c)^2+n}{3(2a+d+3c)} = \frac{a^2+n+9c^2+6ac}{3(2a+d+3c)} = c$. Similarly we can calculate the new values of a, d and c for $(\alpha)\, y^j$, where $j = 2, 3, 4, 5$.

α	a	d	$3c$
$(\alpha)x$	$-a$	c	$3d$
$(\alpha)y$	$-a-3c$	c	$3\,(2a+d+3c)$
$(\alpha)y^2$	$-5a-3d-6c$	$2a+d+3c$	$3\,(4a+3d+4c)$
$(\alpha)y^3$	$-7a-6d-6c$	$4a+3d+4c$	$3\,(4a+4d+3c)$
$(\alpha)y^4$	$-5a-6d-3c$	$4a+4d+3c$	$3\,(2a+3d+c)$
$(\alpha)y^5$	$-a-3d$	$2a+3d+c$	$3d$
$(\alpha)yx$	$a+3c$	$(2a+d+3c)$	$3c$
$(\alpha)y^2x$	$5a+3d+6c$	$4a+3d+4c$	$3\,(2a+d+3c)$
$(\alpha)y^3x$	$7a+6d+6c$	$4a+4d+3c$	$3\,(4a+3d+4c)$
$(\alpha)y^4x$	$5a+6d+3c$	$2a+3d+c$	$3\,(4a+4d+3c)$
$(\alpha)y^5x$	$a+3d$	d	$3\,(2a+3d+c)$
$(\alpha)xy$	$a-3d$	d	$3\,(-2a+3d+c)$
$(\alpha)xy^2$	$5a-6d-3c$	$-2a+3d+c$	$3\,(-4a+4d+3c)$
$(\alpha)xy^3$	$7a-6d-6c$	$-4a+4d+3c$	$3\,(-4a+3d+4c)$
$(\alpha)xy^4$	$5a-3d-6c$	$-4a+3d+4c$	$3\,(-2a+d+3c)$
$(\alpha)xy^5$	$a-3c$	$-2a+d+3c$	$3c$

(Table 1)

From above information we see that all the elements in αH are of the form $\frac{a+\sqrt{-n}}{3c}$. Hence non square positive integer n does not change its value in αH. □

Theorem 3.2. *The fixed points under the action of H on $Q^*\left(\sqrt{-n}\right)$ exist only if $n = 3$.*

Proof. Let g be a linear fractional transformation in H. Therefore $(z)g$ can be taken as $\frac{az+b}{cz+d}$, where $ad-bc = 1$ or 3. Let $\frac{az+b}{cz+d} = z$ which yields quadratic equation $cz^2 + (d-a)z - b = 0$. It has imaginary roots only if $(a+d)^2 - 4(ad-bc) < 0$. If $ad - bc = 1$, then $(a+d)^2 < 4$ implies $a + d = 0, \pm 1$, and if $ad - bc = 3$, then $(a+d)^2 < 12$ implies $a + d = 0, \pm 1, \pm 2, \pm 3$. Hence we have the following cases.

(i) If $a + d =$trace$(g) = 0$, then g is involution and hence it is conjugate to the linear fractional transformation x or y^3.

(ii) If trace$(g) = \pm 1$ and $det\,(g) = 1$, then $(\text{trace}\,(g))^2 = \det(g)$ implying that order of g is 3 and hence g is conjugate to y^2 or y^4.

(iii) If trace$(g) = \pm 3$ and $det\,(g) = 3$, then $(\text{trace}\,(g))^2 = 3\det(g)$ implying that order of g will be six and hence it is conjugate to the linear fractional transformation y or y^5.

(iv) If trace$(g) = \pm 1$, $det\,(g) \neq 1$ or trace$(g) = \pm 3$, $det\,(g) \neq 3$ or trace$(g) = \pm 2$, then the order of g is infinite and it is conjugate to the linear fractional transformation $(xy)^n$.

Hence fixed points of g are imaginary if it is conjugate to the linear fractional transformation x, y, y^2, y^3, y^4 or y^5. Since fixed points of x and y are $\pm\frac{\sqrt{-3}}{3}$ and $\frac{-3\pm\sqrt{-3}}{6}$ respectively, and the conjugates of x and y having the same discriminant. Hence fixed points exist only if $n = 3$. □

Example 3.3. Let $g = xyx \in H$. Then $(z)g = z$ yields the quadratic equation $3z^2 - 3z + 1 = 0$, which has roots $\frac{3\pm\sqrt{-3}}{6}$ which are fixed points of $g = xyx$.

Example 3.4. Let $g = yxy^{-1} \in H$. Then $(z)g = z$ yields the quadratic equation $3z^2 + 6z + 4 = 0$. This equation has roots $\frac{-3\pm\sqrt{-3}}{3}$, which are fixed points of $g = yxy^{-1}$.

Theorem 3.5.

(i) *x maps a totally negative imaginary quadratic number onto a totally positive imaginary quadratic number and vice versa.*

(ii) *If $\alpha = \frac{a+\sqrt{-n}}{3c} \in Q^*\left(\sqrt{-n}\right)$ is totally positive imaginary quadratic number, then $(\alpha)y^j$ is totally negative imaginary quadratic number for $j = 1, 2, 3, 4, 5$.*

Proof. (i) Let α be a totally negative imaginary quadratic number, then $ac < 0$ implies that either $a > 0$ and $c, d < 0$ or $a < 0$ and $c, d > 0$. Now we have the following table.

α	a	d	$3c$
$(\alpha)x$	$-a$	c	$3d$

If $a < 0$ and $c, d > 0$, then from above information we can see that new values of a, d, c for $(\alpha)x$ are all positive. This implies that $(\alpha)x$ is totally positive imaginary quadratic number.

On the other hand, if $a > 0$ and $c, d < 0$ then new values of a, d, c are all negative. So $(\alpha)x$ is a totally positive imaginary quadratic number.

Similarly x maps a totally positive imaginary quadratic number to a totally negative imaginary quadratic number.

(ii) Following table gives the new values of a, d, c for $(\alpha)y^j$, where $j = 1, 2, 3, 4, 5$.

α	a	d	$3c$
$(\alpha)y$	$-a-3c$	c	$3\,(2a+d+3c)$
$(\alpha)y^2$	$-5a-3d-6c$	$2a+d+3c$	$3\,(4a+3d+4c)$
$(\alpha)y^3$	$-7a-6d-6c$	$4a+3d+4c$	$3\,(4a+4d+3c)$
$(\alpha)y^4$	$-5a-6d-3c$	$4a+4d+3c$	$3\,(2a+3d+c)$
$(\alpha)y^5$	$-a-3d$	$2a+3d+c$	$3d$

Since α is a totally positive, so either $a, d, c > 0$ or $a, d, c < 0$. If $a, d, c > 0$, then $(\alpha)y^j$ are all totally negative imaginary quadratic numbers. Now if $a, d, c < 0$, then again from above table, we can see $(\alpha)y^j$ are all totally negative imaginary quadratic numbers. Thus $(\alpha)y^j$ are all totally negative imaginary quadratic numbers. □

Theorem 3.6.

(i) *If* $\alpha = \frac{a+\sqrt{-n}}{3c} \in Q^*\left(\sqrt{-n}\right)$, *then* $\|\alpha\| = \|(\alpha)x\|$.

(ii) *If* $\alpha = \frac{a+\sqrt{-n}}{3c} \in Q^*\left(\sqrt{-n}\right)$ *is totally positive imaginary quadratic number, then* $\|\alpha\| < \|(\alpha)y^j\|$ *for* $j = 1, 2, 3, 4, 5$.

Proof. (i) Consider the following table.

α	a	d	$3c$
$(\alpha)x$	$-a$	c	$3d$

which implies $\|\alpha\| = \mid a \mid = \|(\alpha)x\|$.

(ii) The values of $(\alpha)y^j$ for $j = 1, 2, 3, 4, 5$ are given in the following table.

α	a	d	$3c$
$(\alpha)y$	$-a-3c$	c	$3\,(2a+d+3c)$
$(\alpha)y^2$	$-5a-3d-6c$	$2a+d+3c$	$3\,(4a+3d+4c)$
$(\alpha)y^3$	$-7a-6d-6c$	$4a+3d+4c$	$3\,(4a+4d+3c)$
$(\alpha)y^4$	$-5a-6d-3c$	$4a+4d+3c$	$3\,(2a+3d+c)$
$(\alpha)y^5$	$-a-3d$	$2a+3d+c$	$3d$

Since α is a totally positive imaginary quadratic number, so $ac > 0$. Therefore either $a, d, c > 0$ or $a, d, c < 0$. This implies $\|(\alpha)y\| = \mid a+c \mid > \mid a \mid$. Also, $\|(\alpha)y^2\| = \mid 5a+3d+2c \mid > \mid a \mid$, $\|(\alpha)y^3\| = \mid 7a+6d+2c \mid > \mid a \mid$, $\|(\alpha)y^4\| = \mid 5a+6d+c \mid > \mid a \mid$, $\|(\alpha)y^5\| = \mid a+3d \mid > \mid a \mid$.

Thus $\|\alpha\| < \|(\alpha)y^j\|$ for $j = 1, 2, 3, 4, 5$. □

Theorem 3.7. *If* $\alpha = \frac{a+\sqrt{-n}}{3c} \in Q^*\left(\sqrt{-n}\right)$, *then denominator of every element in* αH *has the same sign.*

Proof. Consider the following table.

α	a	d	$3c$
$(\alpha)x$	$-a$	c	$3d$
$(\alpha)y$	$-a-3c$	c	$3(2a+d+3c)$
$(\alpha)y^2$	$-5a-3d-6c$	$2a+d+3c$	$3(4a+3d+4c)$
$(\alpha)y^3$	$-7a-6d-6c$	$4a+3d+4c$	$3(4a+4d+3c)$
$(\alpha)y^4$	$-5a-6d-3c$	$4a+4d+3c$	$3(2a+3d+c)$
$(\alpha)y^5$	$-a-3d$	$2a+3d+c$	$3d$

If $\alpha = \frac{a+\sqrt{-n}}{3c} \in Q^*\left(\sqrt{-n}\right)$ with $c > 0$, then d is also positive. So it can be easily observed from the above information that every element in αH has positive denominator. If $\alpha = \frac{a+\sqrt{-n}}{3c} \in Q^*\left(\sqrt{-n}\right)$ with $c < 0$, then d is also negative. So it can be easily observed from the above information that every element in αH has negative denominator. □

Theorem 3.8. *If* $\alpha = \frac{a+\sqrt{-n}}{3c} \in Q^*\left(\sqrt{-n}\right)$, *then there exists a sequence of positive integers* $\|\alpha_0\|, \|\alpha_1\|, \|\alpha_2\|, \ldots, \|\alpha_m\|$ *such that* $\|\alpha_0\| > \|\alpha_1\| > \|\alpha_2\| > \ldots > \|\alpha_m\|$, *where* $\|\alpha_m\| = \begin{cases} 0,3 & \text{if } n = 3 \\ 0 & \text{if } n \equiv 0 \,(\mathrm{mod}\, 3), \text{ but } n \neq 3 \\ 1 & \text{if } n \equiv 2 \,(\mathrm{mod}\, 3) \end{cases}$.

Proof. Let $\alpha = \alpha_1$ be a totally positive imaginary quadratic number so $(\alpha_1)x$ is a totally negative imaginary quadratic number and $\| (\alpha_1)x \| = \| \alpha_1 \|$. Since $(\alpha_1)x$ is a totally negative imaginary quadratic number, then by Theorem 3.5 (ii), one of $(\alpha_1)xy^j$ for $j = 1, 2, 3, 4, 5$ is a totally positive imaginary quadratic number. If $(\alpha)xy^j = \alpha_2$ is a totally positive imaginary quadratic number, then by Theorem 3.6 $\| \alpha_2 \| < \| (\alpha_1)x \| = \| \alpha_1 \|$. Similarly, we obtain another totally positive imaginary quadratic number α_3 in the adjacent hexagon to that containing α_2 such that $\|\alpha_0\| > \|\alpha_1\| > \|\alpha_2\|$. Ultimately, we get a decreasing sequence of positive integers $\|\alpha_0\|, \|\alpha_1\|, \|\alpha_2\|, \ldots, \|\alpha_m\|$ such that $\|\alpha\| = \|\alpha_0\| > \|\alpha_1\| > \|\alpha_2\| \ldots > \|\alpha_m\|$. After a finite number of steps it must terminate.

(i) If $n = 3$, then after a finite number of steps we reach to α_m such that $\|\alpha_m\| = 0$ or 3. If $\alpha_m = \frac{-3\pm\sqrt{-3}}{6}$, then because $\frac{-3\pm\sqrt{-3}}{6}$ are fixed points of y, therefore, we can not reach at an imaginary quadratic number whose norm is equal to zero. Otherwise we reach at $\alpha_m = \frac{\sqrt{-3}}{\pm 3}$.

(ii) If $n \equiv 0 \,(\mathrm{mod}\, 3)$, but $n \neq 3$, then we reach at an imaginary quadratic number α_m such that $\|\alpha_m\| = 0$.

(iii) If $n \equiv 2 \;(\mathrm{mod}\, 3)$, then we reach at an imaginary quadratic number α_m such that $\|\alpha_m\| = 1$. □

Example 3.9. Let $\alpha_1 = \frac{7+\sqrt{-2}}{3}$, which is totally positive imaginary quadratic number. Then $(\alpha_1)x = \frac{-7+\sqrt{-2}}{51}$, which is totally negative imaginary quadratic number. Also in the hexagon containing $(\alpha_1)\,x$, $(\alpha_1)\,xy^5 = \frac{4+\sqrt{-2}}{3}$ is totally positive imaginary quadratic number. Take $\alpha_1 = \frac{4+\sqrt{-2}}{3}$, so $\|\alpha_1\| > \|\alpha_2\|$. Now $(\alpha_2)x = \frac{-4+\sqrt{-2}}{18}$ is totally negative imaginary quadratic number, then in the hexagon containing $(\alpha_2)x$, $(\alpha_2)xy^5 = \frac{1+\sqrt{-2}}{3}$ is totally positive imaginary quadratic number. Take $\alpha_3 = \frac{1+\sqrt{-2}}{3}$, implying that $\|\alpha_0\| > \|\alpha_1\| > \|\alpha_3\|$.

Theorem 3.10. *There are exactly four orbits of $Q^*\left(\sqrt{-3}\right)$ under the action of H.*

Proof. Since we know that there exists a sequence of positive integers $\|\alpha_0\|$, $\|\alpha_1\|$, $\|\alpha_2\|, \ldots, \|\alpha_m\|$ such that $\|\alpha_0\| > \|\alpha_1\| > \|\alpha_2\| > \|\alpha_3\| > \|\alpha_4\| > \ldots > \|\alpha_m\|$, where $\|\alpha_m\| = 0$ or 3. If $\alpha_m = \pm\frac{\sqrt{-3}}{3}$ or $\frac{-3\pm\sqrt{-3}}{6}$, then $\pm\frac{\sqrt{-3}}{3}$ and $\frac{-3\pm\sqrt{-3}}{6}$ are fixed points of x and y respectively. Therefore in this case there are four orbits of $Q^*\left(\sqrt{-3}\right)$. That is, $\frac{\sqrt{-3}}{3}H$, $\frac{-\sqrt{-3}}{3}H$, $\frac{-3+\sqrt{-3}}{6}H$ and $\frac{3+\sqrt{-3}}{-6}H$. Hence there are exactly four orbits of $Q^*\left(\sqrt{-3}\right)$ under the action of H. □

Theorem 3.11. *Let $\alpha \in Q^*\left(\sqrt{-n}\right)$, where $n \neq 3$.*

(i) If $\alpha = \frac{\sqrt{-n}}{3}$, where $n \equiv 0 \pmod 3$ then $\frac{\sqrt{-n}}{3}$ and $\frac{\sqrt{-n}}{n}$ lie in αH.

(ii) If $\alpha = \frac{1+\sqrt{-n}}{3}$, where $n \equiv 2 \pmod 3$ then $\frac{1+\sqrt{-n}}{3}$ and $\frac{-1+\sqrt{-n}}{n+1}$ lie in αH.

(iii) If $\alpha = \frac{-1+\sqrt{-n}}{3}$, where $n \equiv 2 \pmod 3$ then $\frac{-1+\sqrt{-n}}{3}$ and $\frac{1+\sqrt{-n}}{n+1}$ lie in αH.

(iv) If $\alpha = \frac{\sqrt{-n}}{3c}$, where $n \equiv 0 \pmod 3$ and $c \neq \pm 1, \pm\frac{n}{3}$, $n = 3cc_1$ then $\frac{\sqrt{-n}}{3c}$ and $\frac{\sqrt{-n}}{3c_1}$ lie in αH.

(v) If $\alpha = \frac{1+\sqrt{-n}}{3c}$ where $n \equiv 2 \pmod 3$ and $n+1 = 3cc_1$, then $\frac{1+\sqrt{-n}}{3c}$ and $\frac{-1+\sqrt{-n}}{3c_1}$ lie in αH.

(vi) If $\alpha = \frac{-1+\sqrt{-n}}{3c}$ where $n \equiv 2 \pmod 3$ and $n+1 = 3cc_1$, then $\frac{-1+\sqrt{-n}}{3c}$ and $\frac{1+\sqrt{-n}}{3c_1}$ lie in αH.

Proof. (i) If $\alpha = \frac{\sqrt{-n}}{3}$, then we have the following information.

α	0	$\frac{n}{3}$	3
$(\alpha)x$	0	1	n
$(\alpha)y$	-3	1	$n+9$
$(\alpha)y^2$	$-6-n$	$\frac{n+9}{3}$	$3\,(4+n)$
$(\alpha)y^3$	$-2n-6$	$4+n$	$9+4n$
$(\alpha)y^4$	$-2n-3$	$\frac{9+4n}{3}$	$3\,(n+1)$
$(\alpha)y^5$	$-n$	$n+1$	n

Hence from the above table, we see that $\frac{\sqrt{-n}}{3}$ and $\frac{\sqrt{-n}}{n}$ lie in the same orbit.

(ii) If $\alpha = \frac{1+\sqrt{-n}}{3}$, then we have the following information.

α	1	$\frac{n+1}{3}$	3
$(\alpha)x$	-1	1	$1+n$
$(\alpha)y$	-4	1	$16+n$
$(\alpha)y^2$	$-12-n$	$\frac{16+n}{3}$	$27+3n$
$(\alpha)y^3$	$-15-2n$	$9+n$	$25+4n$
$(\alpha)y^4$	$-10-2n$	$\frac{25+4n}{3}$	$12+3n$
$(\alpha)y^5$	$-2-n$	$4+n$	$1+n$

Hence from the above table, we see that $\frac{1+\sqrt{-n}}{3}$ and $\frac{-1+\sqrt{-n}}{n+1}$ lie in αH.

(iii) If $\alpha = \frac{-1+\sqrt{-n}}{3}$, then we have the following information.

α	-1	$\frac{n+1}{3}$	3
$(\alpha)x$	1	1	$1+n$
$(\alpha)y$	-2	1	$4+n$
$(\alpha)y^2$	$-2-n$	$\frac{4+n}{3}$	$3(1+n)$
$(\alpha)y^3$	$-1-2n$	$1+n$	$1+4n$
$(\alpha)y^4$	$-2n$	$\frac{1+4n}{3}$	$3n$
$(\alpha)y^5$	$-n$	n	$n+1$

Hence from the above table, we see that $\frac{-1+\sqrt{-n}}{3}$ and $\frac{1+\sqrt{-n}}{n+1}$ lie in the same orbit.

(iv) If $\alpha = \frac{\sqrt{-n}}{3c}$, then we have the following information.

α	0	c_1	$3c$
$(\alpha)x$	0	c	$3\ c_1$
$(\alpha)y$	$-3c$	c	$3(3c+c_1)$
$(\alpha)y^2$	$-6c-3c_1$	$3c+c_1$	$3(4c+3c_1)$
$(\alpha)y^3$	$-6c-6c_1$	$4c+3c_1$	$3(3c+4c_1)$
$(\alpha)y^4$	$-3c-6c_1$	$3c+4c_1$	$3(c+3c_1)$
$(\alpha)y^5$	$-3c_1$	$c+3c_1$	$3c_1$

Hence from the above table, we see that $\frac{\sqrt{-n}}{3c}$ and $\frac{\sqrt{-n}}{3c_1}$ lie in the same orbit.

(v) If $\alpha = \frac{1+\sqrt{-n}}{3c}$, then we have the following information.

α	1	c_1	$3c$
$(\alpha)x$	-1	c	$3\ c_1$
$(\alpha)y$	$-1-3c$	c	$3(2+3c+c_1)$
$(\alpha)y^2$	$-5-6c-3c_1$	$2+3c+c_1$	$3(4+4c+3c_1)$
$(\alpha)y^3$	$-7-6c-6c_1$	$4+4c+3c_1$	$3(4+3c+4c_1)$
$(\alpha)y^4$	$-5-3c-6c_1$	$4+3c+4c_1$	$3(2+c+3c_1)$
$(\alpha)y^5$	$-1-3c_1$	$2+c+3c_1$	$3c_1$

Hence from the above table, we see that $\frac{1+\sqrt{-n}}{3c}$ and $\frac{-1+\sqrt{-n}}{3c_1}$ lie in αH.

(vi) If $\alpha = \frac{-1+\sqrt{-n}}{3c}$, then we have the following information.

α	-1	c_1	$3c$
$(\alpha)x$	1	c	$3\,c_1$
$(\alpha)y$	$1-3c$	c	$3\,(2+3c+c_1\,)$
$(\alpha)y^2$	$5-6c-3c_1$	$2+3c+c_1$	$3\,(4+4c+3c_1\,)$
$(\alpha)y^3$	$7-6c-6c_1$	$4+4c+3c_1$	$3\,(4+3c+4c_1\,)$
$(\alpha)y^4$	$5-3c-6c_1$	$4+3c+4c_1$	$3\,(2+c+3c_1\,)$
$(\alpha)y^5$	$1-3c_1$	$2+c+3c_1$	$3c_1$

Hence from the above table, we see that $\frac{-1+\sqrt{-n}}{3c}$ and $\frac{1+\sqrt{-n}}{3c_1}$ lie in αH. □

Example 3.12. By using Theorem 9, the orbits of $Q^*\left(\sqrt{-30}\right)$ are

(i) $\frac{\sqrt{-30}}{3}$ and $\frac{\sqrt{-30}}{30}$ lie in $\frac{\sqrt{-30}}{3}H$. (ii) $\frac{\sqrt{-30}}{-3}$ and $\frac{\sqrt{-30}}{-30}$ lie in $\frac{\sqrt{-30}}{-3}H$.

(iii) $\frac{\sqrt{-30}}{6}$ and $\frac{\sqrt{-30}}{15}$ lie in $\frac{\sqrt{-30}}{6}H$. (iv) $\frac{\sqrt{-30}}{-6}$ and $\frac{\sqrt{-30}}{-15}$ lie in $\frac{\sqrt{-30}}{-6}H$.

So, there are four orbits of $Q^*\left(\sqrt{-30}\right)$.

Example 3.13. By using Theorem 9, the orbits of $Q^*\left(\sqrt{-11}\right)$ are

(i) $\frac{1+\sqrt{-11}}{3}$ and $\frac{-1+\sqrt{-11}}{12}$ lie in $\frac{1+\sqrt{-11}}{3}H$.

(ii) $\frac{1+\sqrt{-11}}{-3}$ and $\frac{-1+\sqrt{-11}}{-12}$ lie in $\frac{1+\sqrt{-11}}{-3}H$.

(iii) $\frac{-1+\sqrt{-11}}{3}$ and $\frac{1+\sqrt{-11}}{12}$ lie in $\frac{-1+\sqrt{-11}}{3}H$.

(iv) $\frac{-1+\sqrt{-11}}{-3}$ and $\frac{1+\sqrt{-11}}{-12}$ lie in $\frac{-1+\sqrt{-11}}{-3}H$.

(v) $\frac{1+\sqrt{-11}}{6}$ and $\frac{-1+\sqrt{-11}}{6}$ lie in $\frac{1+\sqrt{-11}}{6}H$.

(vi) $\frac{1+\sqrt{-11}}{-6}$ and $\frac{-1+\sqrt{-11}}{-6}$ lie in $\frac{1+\sqrt{-11}}{-3}H$.

So, there are six orbits of $Q^*\left(\sqrt{-11}\right)$.

Definition 3.14. If n is a positive integer, then $d\,(n)$ denotes the arithmetic function defined by the number of positive divisors of n which are multiple of 3.

Theorem 3.15. *If $n \neq 3$ then the total number of orbits of $Q^*\left(\sqrt{-n}\right)$ under the action of H are*

$$\begin{cases} d\,(n) & \text{if } n \equiv 0 \,(\text{mod}\,3), \text{ but } n \neq 3. \\ 2d\,(n+1) & \text{if } n \equiv 2 \,(\text{mod}\,3). \end{cases}$$

Proof. If $n \equiv 0\,(\text{mod}\,3)$, then the divisors of n which are multiples of 3 are $\pm 3, \pm m_1, \pm m_2, \pm m_3, \ldots, \pm n$. Then by Theorem 3.11 (i) there exist two orbits of $Q^*\left(\sqrt{-n}\right)$ corresponding to the divisors $\pm 3, \pm n$ of n. We therefore left with

$2d(n)-4$ divisors of n. Then by Theorem 3.11 (iv), there exist $\frac{2d(n)-4}{2}$ orbits corresponding to the remaining $2d(n)-4$ divisors of n. Hence there are $2+\frac{2d(n)-4}{2}=d(n)$ orbits of $Q^*\left(\sqrt{-n}\right)$.

If $n \equiv 2 \pmod 3$, then the divisors of $n+1$ which are multiples of 3 are $\pm 3, \pm n_1$, $\pm n_2, \pm n_3, \ldots, \pm(n+1)$. By Theorem 3.11 (ii) and (iii), there exist four orbits corresponding to the divisors $\pm 3, \pm(n+1)$ of $n+1$. Thus we are left with $2d(n+1)-4$ divisors of $n+1$. By Theorem 3.11 (v) and (vi) corresponding to the remaining $2d(n+1)-4$ divisors of $n+1$, there exist $2d(n+1)-4$ orbits. Hence there are $4+2d(n+1)-4=2d(n+1)$ orbits of $Q^*\left(\sqrt{-n}\right)$. □

Example 3.16. Consider $Q^*\left(\sqrt{-30}\right)$. Then the positive divisors of 30 which are multiple of 3 are $3, 6, 15, 30$. Therefore $d(30)=4$, which implies that the total number of orbits are four.

Example 3.17. In $Q^*\left(\sqrt{-11}\right)$. The number of positive divisors of 12 which are multiple of three are $3, 6, 12$. Therefore $d(12)=3$. Hence the total number of orbits are $2d(12)=2\times 3=6$.

Corollary 3.18. *The action of H on $Q^*\left(\sqrt{-n}\right)$ is intransitive.*

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Department of Mathematics, University of Education, Lahore, Jauharabad, Pakistan
E-mail: abdul.razaq@ue.edu.pk