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# There exist semigroups which have bi-bases with different cardinalities

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**Abstract.** Kummoon and Changphas in Quasigroups and Related Systems 25(2017), 87 – 94 state the following question: *"Is it true that for any two bi-bases of a semigroup have the same cardinality?"*

In this paper, we provide a semigroup of order $n$ for every $n \geq 5$ which has two bi-bases with different cardinalities that is shown the answer of question is negative.

## 1. Introduction

Let $S$ be a semigroup, and $A, B$ non-empty subsets of $S$. The set product $AB$ of $A$ and $B$ is defined to be the set of all elements $ab$ with $a$ in $A$ and $b$ in $B$. That is

$$AB = \{ab \mid a \in A, b \in B\}.$$

Kummoon and Changphas in [1] introduced the concept which is called bi-base of semigroups and proved some properties.

**Definition.** Let $S$ be a semigroup. A subset $B$ of $S$ is called a *bi-base* of $S$ if it satisfies the following two conditions:

$(i)$ $S = B \cup BB \cup BSB$;

$(ii)$ if $A$ is a subset of $B$ such that $S = A \cup AA \cup ASA$, then $A = B$.

## 2. Main results

In [1] the authors asked the following question:

*Is it true that for any two bi-bases of a semigroup have the same cardinality?*

We would like to answer the question by providing a semigroup of order $n \geqslant 5$ which has two bi-bases with different cardinalities.

**Answer.** Let $S_n = \{1, 2, \dots, n\}$ for every $n \geqslant 5$ and consider the following binary operation on $S_n$:

$$x \cdot y = \begin{cases} 1, & \text{if } x \notin \{n-2, n\} \text{ and } y \notin \{n-1, n\}, \\ n-1, & \text{if } x \notin \{n-2, n\} \text{ and } y \in \{n-1, n\}, \\ n-2, & \text{if } x \in \{n-2, n\} \text{ and } y \notin \{n-1, n\}, \\ n, & \text{if } x \in \{n-2, n\} \text{ and } y \in \{n-1, n\}. \end{cases}$$

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To verify the associativity condition let $x, y, z \in S_n$. Then there are four cases:

*Case 1.* If $x \notin \{n-2, n\}$ and $z \notin \{n-1, n\}$ then $x \cdot (y \cdot z) = 1 = (x \cdot y) \cdot z$.

*Case 2.* If $x \notin \{n-2, n\}$ and $z \in \{n-1, n\}$ then $x \cdot (y \cdot z) = n-1 = (x \cdot y) \cdot z$.

*Case 3.* If $x \in \{n-2, n\}$ and $z \notin \{n-1, n\}$ then $x \cdot (y \cdot z) = n-2 = (x \cdot y) \cdot z$.

*Case 4.* If $x \in \{n-2, n\}$ and $z \in \{n-1, n\}$ then $x \cdot (y \cdot z) = n = (x \cdot y) \cdot z$.

In each case $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ so $(S_n, \cdot)$ is a semigroup.

Now, let $A \subseteq \{2, 3, \ldots, n-3, n-1\}$ then $AA = ASA = A$ so every bi-base of $(S_n, \cdot)$ contains $\{2, 3, \ldots, n-3, n-1\}$. Also, if $A = \{2, n\}$ or $A = \{n-2, n-1\}$ then $AA = ASA = \{1, n-2, n-1, n\}$. Therefore, the subsets $B = \{2, 3, \ldots, n-3, n\}$ and $B' = \{2, 3, \ldots, n-1\}$ are two bi-bases of $(S_n, \cdot)$ with cardinality $n-3$ and $n-2$, respectively.

**Example.** Consider $n = 5$. Then the Cayley table of $(S_5, \cdot)$ is as follows

| $\cdot$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 4 | 4 |
| 2 | 1 | 1 | 1 | 4 | 4 |
| 3 | 3 | 3 | 3 | 5 | 5 |
| 4 | 1 | 1 | 1 | 4 | 4 |
| 5 | 3 | 3 | 3 | 5 | 5 |

Also, the subsets $B = \{2, 5\}$ and $B' = \{2, 3, 4\}$ are two bi-bases of $(S_5, \cdot)$.

# References

[1] **P. Kummoon and T. Changphas**, *On bi-bases of a semigroup*, Quasigroups and Related Systems **25** (2017), 87 – 94.

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