Quasigroups and Related Systems **26** (2018), 67 − 76

The Cayley graph of commutative ring on triangular subsets

Kazem Hamidizadeh and Gholamreza Aghababaei

Abstract. Let R be a commutative ring with nonzero identity, and T be a triangular subset of R^n. We investigate the structure of the Cayley graph $TCay(R^n, T^*)$, where $T^* = T \setminus \{0\}$ is the triangular subset of R^n.

1. Introduction

The investigation of algebraic structures of graphs is a very large and growing area of research. In particular, Cayley graphs and their generalizations have been a main topic in algebraic graph theory (see [1], [2], [3], [4]). Several other classes of graphs associated with algebraic structures, such as power graph, total graph and zero divisor graph, have been investigated in [5] and [6].

Let R be a commutative ring with nonzero identity, $L_n(R)$ be the set of all lower triangular $n \times n$ matrices, and U be a subset of R^n, where n is a positive integer. We say that U is a *triangular subset* of R^n if the following condition holds:

$$\text{for all } (u_1, \ldots, u_n) \in U, \;\; A \in L_n(R) \text{ and } (w_1, \ldots, w_n) \in R^n,$$

$$\text{if } A[(u_1, \ldots, u_n)]^T = [w_1, \ldots, w_n]^T, \text{ then } (w_1, \ldots, w_n) \in U.$$

If T be a triangular subset of R^n, then for every $(x_1, \ldots, x_n) \in T$, we have $Rx_1 \times \ldots \times Rx_n \subseteq T$. Hence $T = \bigcup_{i\in\Omega} \sqcap_{j=1}^{n} I_{ij}$, where $I_{i1} \subseteq \ldots \subseteq I_{in}$, for every $i \in \Omega$.

Let R be an arbitary commutative ring and T be a triangular subset of R^n. In this paper, we study the Cayley graph $TCay(R^n, T^*)$, which is an undirected graph with vertex set R^n, and two distinct vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if and only if $(x_1 - y_1, \ldots, x_n - y_n) \in T^*$. For simplicity our notations, we denote the graph $TCay(R^n, T^*)$ by $TCay(R^n)$. We study the structure of $TCay(R^n)$, in the cases that T is closed under addition and T is not closed under addition. In sections 2 and 3, we investigate the diameter and the girth of the $TCay(R^n)$, where the proofs of the results in these two sections are similar to that in [7]. In section 4, we investigate the planarity of graph $TCay(R^n)$.

Now, we recall some definitions and notations on graphs. We use the standard terminology of graphs in [9]. Let G be a simple graph. We say that G is *connected* if

2010 Mathematics Subject Classification: 05C10, 13A15.
Keywords: Cayley graph, commutative ring, triangular subset.

there is a path between any two distinct vertices of G, otherwise G is *disconnected*. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. For vertices x and y of G, we use the notation $x \sim y$ to denote that x and y are adjacent. Also, the length of a shortest path from x to y is denoted by $d(x, y)$ if a path from x to y exists. Also we define $d(x, y) = 0$, and $d(x, y) = \infty$ if there is no path between x and y. The *diameter* of G is $diam(G) = sup\{d(x, y) : x, y \in V(G)\}$. The *girth* of G, denoted by $gr(G)$, is length of a smallest cycle in G (if G contains no cycles, then $gr(G) = \infty$). A graph G is said to be *complete bipartite* if the vertices of G can be partitioned into two disjoint sets V_1, V_2 such that no two vertices in any V_1 or V_2 are adjacent, but for every $u \in V_1, v \in V_2$, the vertices u and v are adjacent. Then we use the symbol $K_{m,n}$ for the complete bipartite graph where the cardinal numbers of V_1 and V_2 are m, n, respectively. A graph with n vertices in which each pair of distinct vertices is joined by an edge is called a *complete graph*, and it is denoted by K_n. A graph G is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths.

We investigate this graph in case that $n \geqslant 2$. First, assume that T is closed under addition.

2. The case that T is closed under addition

The proofs of the following theorems are similar to that in [7], and hence we omit the proofs.

Theorem 2.1. *Let R be a commutative ring and T be a triangular subst of R^n. Then $TCay(T)$ is disjoint from $TCay(R^n \setminus T)$.*

Proof. This is clear according to the definitions. □

Theorem 2.2. *Let R be a commutative ring, T be a triangular subset of R^n, which is closed under addition, $|T| = \alpha$ and $|R^n/T| = \beta$. Then $TCay(T)$ is a complete graph K_α and $TCay(R^n \setminus T)$ is the union of $\beta - 1$ disjoint K_α.*

Theorem 2.3. *Let R be a commutative ring, T be a triangular subset of R^n that closed under addition, then the following statements hold.*

(1) *$TCay(R^n \setminus T)$ is complete if and only if $R^n/T \cong \mathbb{Z}_2$.*

(2) *$TCay(R^n \setminus T)$ is connected if and only if $R_n/T \cong \mathbb{Z}_2$.*

The following corollary follows from Theorems 2.1 and 2.2.

Corollary 2.4. *Let R be a commutative ring, T be a triangular subset of R^n that closed under addition, then the following statements hold.*

(1) *$diam(TCay(R^n \setminus T)) = 1$ if and only if $R^n/T \cong \mathbb{Z}_2$ and $| T | \geqslant 2$. Otherwise $diam(TCay(R^n \setminus T)) = \infty$.*

(2) *$gr(TCay(R^n \setminus T)) = 3$ if and only if $|T| \geqslant 3$. Otherwise $gr(TCay(R^n \setminus T)) = \infty$.*

(3) *$gr(TCay(T)) = 3$ if and only if $|T| \geqslant 3$. Otherwise $gr(TCay(T)) = \infty$.*

(4) *$diam(TCay(R)) = \infty$, and $gr(TCay(R)) = 3$ if and only if $|T| \geqslant 3$, otherwise $gr(TCay(R)) = \infty$.*

3. The case that T is closed under addition

The following results and their proofs are analogous to some of the results in [7].

Theorem 3.1. *Let R be a commutative ring and T be a triangular subset of R^n which is not closed under addition. Then the following statements hold.*

(1) *$TCay(T)$ is connected and $diamTCay(T) = 2$.*

(2) *The graphs $TCay(T)$ and $TCay(R^n \setminus T)$ are not disjoint.*

(3) *If $TCay(R^n \setminus T)$ is connected, then so is $TCay(R^n)$.*

Proof. (1). Let $(x_1, \ldots, x_n) \in T$. Then $(x_1, \ldots, x_n)$ is adjacent to $(0, \ldots, 0)$. Thus $(x_1, \ldots, x_n) \sim (0, \ldots, 0) \sim (y_1, \ldots, y_n)$ is a path in $TCay(T)$ of length two between any two distinct vertices $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in T^*$. Moreover there are nonzero distinct vertices $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in T$ that are not adjacent, because U is not closed under addition. Therefore $\mathrm{diam}TCay(T) = 2$.

(2). Since U is not closed under addition, there are nonzero distinct vertices $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in T$ such that $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) \in R^n \setminus T$. We have $(x_1, \ldots, x_n) \in T$ is adjacent to $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) \in R^n \setminus T$ because

$$((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) - (y_1, \ldots, y_n) = (x_1, \ldots, x_n) \in T.$$

(3). This follows from (1) and (2). □

Theorem 3.2. *Let R be a commutative ring and T be a triangular subset of R^n that is not closed under addition. Then $TCay(R)$ is connected if and only if $\langle T \rangle = R^n$.*

Proof. Suppose that $TCay(R^n)$ is connected. Hence there is a path

$$(0, \ldots, 0) \sim (x_{1,1}, \ldots, x_{1,n}) \sim \cdots \sim (x_{k,1}, \ldots, x_{k,n}) \sim (1, \ldots, 1)$$

from $(0, \ldots, 0)$ to $(1, \ldots, 1)$ in $TCay(R^n)$. Now clearly we have

$$(x_{1,1}, \ldots, x_{1,n}), (x_{2,1} - x_{1,1}, \ldots, x_{2,n} - x_{1,n}), \ldots, (1 - x_{k,1}, \ldots, 1 - x_{k,n}) \in T.$$

Hence $(1, \ldots, 1)$ belongs to the set

$$\langle (x_{1,1}, \ldots, x_{1,n}), (x_{2,1} - x_{1,1}, \ldots, x_{2,n} + x_{1,n}), \ldots, (1 - x_{k,1}, \ldots, 1 - x_{k,n}) \rangle \subseteq \langle T \rangle.$$

Conversly, suppose that $\langle T\rangle = R^n$. We show that for each $(x_1, \ldots, x_n) \in T$, there exists a path in $TCay(R^n)$ from $(0, \ldots, 0)$ to $(x_1, \ldots, x_n)$. By assumption, there are elements $(x_{1,1}, \ldots, x_{1,n})$,$(x_{2,1}, \ldots, x_{2,n})$,$\ldots$,$(x_{k,1}, \ldots, x_{k,n}) \in T$ such that

$$(x_1, \ldots, x_n) = (x_{1,1}, \ldots, x_{1,n}) + \cdots + (x_{k,1}, \ldots, x_{k,n}).$$

Let $c_0 = (0, \ldots, 0)$ and $c_l = (x_{1,1}, \ldots, x_{1,n}) + \cdots + (x_{l,1}, \ldots, x_{l,n}))$ for every integer l with $1 \leqslant l \leqslant k$. Thus $c_l - c_{l-1} = (x_{l,1}, \ldots, x_{l,n})$ for each integer l with $1 \leqslant l \leqslant k$ and thus

$$(0, \ldots, 0) = c_0 \sim c_1 \sim \cdots \sim c_k = (x_1, \ldots, x_n)$$

is a path from $(0, \ldots, 0)$ to $(x_1, \ldots, x_n)$ in $TCay(R^n)$ of length at most k. Now, let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be in R^n. Then, by the preceding argument, there are paths from $(x_1, \ldots, x_n)$ to $(0, \ldots, 0)$ and $(0, \ldots, 0)$ to $(y_1, \ldots, y_n)$ in $TCay(R^n)$. Hence there is a path from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_n)$ in $TCay(R^n)$. Therefore $TCay(R^n)$ is connected. □

Theorem 3.3. *Let R be a commutative ring, T be a triangular subset of R^n which is not closed under addition such that $\langle T\rangle = R^n$. Let $k \geqslant 2$ be the least integer that $R = \langle (x_{1,1}, \ldots, x_{1,n}), \ldots, (x_{k,1}, \ldots, x_{k,n})\rangle$, for some distinct elements $(x_{1,1}, \ldots, x_{1,n}), \ldots, (x_{k,1}, \ldots, x_{k,n}) \in U$. Then $diam(TCay(R^n)) = k$.*

Proof. First, we show that any path from $(0, \ldots, 0)$ to $(1, \ldots, 1)$ has length at least l. Suppose that

$$(0, \ldots, 0) \sim (y_{1,1}, \ldots, y_{1,n}) \sim \cdots \sim (y_{l-1,1}, \ldots, y_{l-1,n}) \sim (1, \ldots, 1)$$

is a path from $(0, \ldots, 0)$ to $(1, \ldots, 1)$ in $TCay(R^n)$ of length l. Thus

$$(y_{1,1}, \ldots, y_{1,n}),(y_{2,1} - y_{1,1}, \ldots, y_{2,n} - y_{1,n}),\ (1 - y_{l-1,1}, \ldots, 1 - y_{l-1,n}) \in T.$$

Therefore $(1, \ldots, 1)$ belongs to

$$\langle (y_{1,1}, \ldots, y_{1,n}), (y_{2,1} - y_{1,1}, \ldots, y_{2,n} - y_{1,n}), (1 - y_{l-1,1}, \ldots, 1 - y_{l-1,n})\rangle \subseteq T.$$

Hence $l \geqslant k$. Now let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be distinct elements in R^n. We show that there is a path from $(a_1, \ldots, a_n)$ to $(b_1, \ldots, b_n)$ in $TCay(R^n)$ with length at most k. Let $(1, \ldots, 1) = (x_{1,1}, \ldots, x_{1,n}) + \cdots + (x_{k,1}, \ldots, x_{k,n})$, for some $(x_{1,1}, \ldots, x_{1,n}), \ldots, (x_{k,1}, \ldots, x_{k,n}) \in T$. Define $z_0 = (a_1, \ldots, a_n)$ and

$$z_l = (b_1 - a_1, \ldots, b_n - a_n)((x_{1,1}, \ldots, x_{1,n}) + \cdots + (x_{l,1}, \ldots, x_{l,n}))(a_1, \ldots, a_n)$$

for every integer l with $1 \leqslant l \leqslant k$. Then

$$z_{k+1} - z_k = (b_1 - a_1, \ldots, b_n - a_n)(b_{l+1,1}, \ldots, b_{l+1,n}) \in T$$

for every integer l with $0 \leqslant l \leqslant n - 1$. Thus

$$(a_1, \ldots, a_n) \sim z_1 \sim z_2 \sim \cdots \sim z_{k-1} \sim (b_1, \ldots, b_n)$$

is a path from $(a_1,\ldots,a_n)$ to $(b_1,\ldots,b_n)$ in $TCay(R^n)$ with length at most n. Specially, a shortest path between $(0,\ldots,0)$ and $(1,\ldots,1)$ in $TCay(R^n)$ has length at most k, and thus $\mathrm{diam}(TCay(R)) = k$. □

Corollary 3.4. *Let R be a commutative ring and T be a triangular subset of R^n which is not closed under addition and $TCay(R^n)$ is connected. Then the following statements hold.*

(1) *$diam(TCay(R^n)) = d((0,\ldots,0),(1,\ldots,1))$.*

(2) *If $diam(TCay(R^n)) = k$, then $diam(TCay(R^n \setminus T)) \geqslant m-2$.*

Proof. (1). This follows from Theorem 2.6.

(2). $\mathrm{diam}(TCay(R^n)) = d((0,\ldots,0),(1,\ldots,1))$, by (1). So, let

$$(0,\ldots,0) \sim (c_{1,1},\ldots,c_{1,n}) \sim \cdots \sim (c_{k-1,1},\ldots,c_{k-1,n}) \sim (1,\ldots,1)$$

be the shortest path from $(0,\ldots,0)$ to $(1,\ldots,1)$ in $TCay(R^n)$.

Clearly $(c_{1,1},\ldots,c_{1,n}) \in T^*$. If $(c_{i,1},\ldots,c_{i,n}) \in T^*$, for $2 \leqslant i \leqslant k-1$, then we can construct the path

$$(0,\ldots,0) \sim (c_{i,1},\ldots,c_{i,n}) \sim \cdots \sim (c_{k-1,1},\ldots,c_{k-1,n}) \sim (1,..,1)$$

from $(0,\ldots,0)$ to $(1,\ldots,1)$ in $TCay(R^n)$ which has length less than k, which is a contradication. Thus $(c_{i,1},\ldots,c_{i,n}) \in R^n \setminus T$, for $2 \leqslant i \leqslant k-1$. Hence

$$(c_{2,1},\ldots,c_{2,n}) \sim \cdots \sim (c_{k-1,1},\ldots{}_{k-1,n}) \sim (1,\ldots,1)$$

is the shortest path from $(c_{2,1},\ldots,c_{2,n})$ to $(1,\ldots,1)$ in $R^n \setminus T$ and it has length $k-2$. Thus $\mathrm{diam}(TCay(R^n \setminus T)) \geqslant m-2$. □

Now, for each $X \in T$, let i_X be a positive integer that the first nonzero component of X is in the i_X-th place. Also let

$$m := \min\{i_X \mid X \in U\}.$$

Lemma 3.5. *Let R be a commutative ring and T be a triangular subset of R^n which is not closed under addition. If $m \geqslant 2$, then*

$$gr(TCay(R^n \setminus T)) = gr(TCay(T)) = 3.$$

Proof. If $n \geqslant 3$, since $m \geqslant 2$, then exist $(0,\ldots,0,a,0) \in T$ such that $a \neq 0$. Hence

$$(0,\ldots,0,a,0),(0,\ldots,0,a),(0,\ldots,0)$$

are adjacent in T. Also

$$(1,\ldots,1,a,0),(1,\ldots,1,0,0),(1,\ldots,1,0,a)$$

are adjacent in $R^n \setminus T$.

If $n = 2$, since $m = 2$ and $R^n \neq T$, then exist $(a, 0)$ in T and (x, y) in $R^n \setminus T$ such that $a, x \neq 0$. Hence $(a, 0), (a, a), (0, 0) \in T$ that are adjacent. Also $(x, 0), (x, a), (x + a, 0) \in R^n \setminus T$ that are adjacent. Therefore $gr(TCay(R^n \setminus T)) = gr(TCay(T)) = 3$. □

Theorem 3.6. *Let R be a commutative ring and T be a triangular subset of R^n which is not closed under addition. If $m = 1$, then $gr(TCay(R^n \setminus T)) \leqslant 4$ and $gr(TCay(T)) \in \{3, 4, \infty\}$.*

Proof. Since T is traingular subset of R^n, then $T = \bigcup_{i\in\gamma} I_{i1} \times \ldots \times I_{in}$, where $I_{i1} \subseteq \ldots \subseteq I_{in}$ and I_{ij} are ideals of R, for $1 \leqslant j \leqslant n$ and $i \in \gamma$. Also T is not closed under addition and $m = 1$, therefore $i \geqslant 2$ and $T = \bigcup_{i\in\gamma}\{0\} \times \ldots \{0\} \times I_{in}$, which $I_{in} \neq \{0\}$.

Case 1: If $|I_{kn}| \geqslant 3$ for some $k \in \gamma$, then $gr(TCay(R^n \setminus T)) = gr(TCay(T)) = 3$.

Case 2: If $|I_{in}| \leqslant 2$ for every $i \in \gamma$, then $i \geqslant 2$, since T is not closed under addition. So, we have two subcases.

Case 2a: If exist nonzero element $(0, \ldots, 0, a), (0, \ldots, 0, b), (0, \ldots, 0, c) \in T$ such that $a, b, c \neq 0$ and $a + b = c$, then

$$(0, \ldots, 0), (0, \ldots, 0, a), (0, \ldots, 0, a + b), (0, \ldots, 0, b), (0, \ldots, 0)$$

is a cycle of length 4 in $TCay(T)$. Also

$$(1, \ldots, 1), (1, \ldots, 1, a), (1, \ldots, 1, a + b), (1, \ldots, 1, b), (1, \ldots, 1)$$

is a cycle of length 4 in $TCay(R\setminus T)$. Thus $gr(TCay(T)) = gr(TCay(R\setminus T)) = 3$.

Case 2b: If for every nonzero element $(0, \ldots, 0, x), (0, \ldots, 0, y) \in T$, then $(0, \ldots, 0, x + y) \notin T$. Since $i \geqslant 2$, then exist $(0, \ldots, 0, a), (0, \ldots, 0, b) \in T$, such that $a, b \neq 0$ and $a \neq b$. Now

$$(1, \ldots, 1, 0) \sim (1, \ldots, 1, a) \sim (1, \ldots, 1, a + b) \sim (1, \ldots, 1, b) \sim (1, \ldots, 1, 0)$$

is a cycle of length 4 in $TCay(R \setminus T)$, then $gr(TCay(R \setminus T)) \leqslant 4$. The graph $TCay(T)$ is isomorphic to $K_{1,i}$. Hence $gr(TCay(T)) = \infty$. □

4. Planarity

The graph G is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Theorem 4.1. *Let R be a commutative ring and T be a triangular subset of R^n which is closed under addition, then $TCay(R^n)$ is planar if and only if $|T| \leqslant 4$.*

Proof. Let $|T| = \alpha$ and $|R^n/T| = \beta$. Since T is closed under addition, then T is an ideal and by Theorem 2.2, $TCay(T)$ is a complete graph K_α and $TCay(R^n \setminus T)$ is the union of $\beta - 1$ disjoint K_α. Therefore $TCay(R^n)$ is planar if and only if $|T| \leqslant 4$. □

Theorem 4.2. *Let R be a commutative ring and T be a triangular subset of R^n which is not closed under addition and $m \leqslant n-1$, then $TCay(R^n)$ is not planar.*

Proof. Since T is not an ideal and $m \leqslant n-1$, then exist $(0,\ldots,0,a,0), (0,\ldots,0,b)$ in T where $a \neq b$ and $a, b \neq 0$. Then the vertices

$$(0,\ldots,0), (0,\ldots,0,a,0), (0,\ldots,0,a), (0,\ldots,0,a,a),$$

$$(0,\ldots,0,a+b,0), (0,\ldots,0,b), (0,\ldots,0,a+b), (0,\ldots,0,a+b,a)$$

forms a subdivision of K_5, hence $TCay(R^n)$ is not planar. □

Now, the only remaining case for investigating the planarity of $TCay(R^n)$, is the case that $m = n$. If T is not closed under addition, since T is a triangular subset of R^n, then $T = \bigcup_{i\in\gamma} I_{i1} \times \ldots \times I_{in}$, where $I_{i1} \subseteq \ldots \subseteq I_{in}$ and $i \geqslant 2$.

Theorem 4.3. *Let R be a commutative ring and T be a triangular subset of R^n which is not closed under addition and $m = n$ and $i \geqslant 4$, then $TCay(R^n)$ is not planar.*

Proof. Since $i \geqslant 4$, there exist ideals of R^n such that $\{0\} \times \ldots \times \{0\} \times \{x_1\}$, $\{0\} \times \ldots \times \{0\} \times \{x_2\}$, $\{0\} \times \ldots \times \{0\} \times \{x_3\}$ and $\{0\} \times \ldots \times \{0\} \times \{x_4\}$ where $x_1, x_2, x_3, x_4 \neq 0$.

CASE 1: If $x_r + x_p = x_q$, for $1 \leqslant r, p, q \leqslant 4$, then we may assume that $x_1 + x_2 = x_3$. Hence

$$(0,\ldots,0), (0,\ldots,0,x_1), (0,\ldots,0,x_2), (0,\ldots,0,x_3), (0,\ldots,0,x_4),$$

$$(0,\ldots,0,x_1+x_4), (0,\ldots,0,x_2+x_4), (0,\ldots,0,x_3+x_4)$$

forms a subdivision of K_5, and so $TCay(R^n)$ is not planar.

CASE 2: If $x_r + x_p \neq x_q$ for every $1 \leqslant r, p, q \leqslant 4$, then

$$(0,\ldots,0), (0,\ldots,0,x_1), (0,\ldots,0,x_2), (0,\ldots,0,x_3), (0,\ldots,0,x_2+x_3),$$

$$(0,\ldots,0,x_1+x_3), (0,\ldots,0,x_1+x_2), (0,\ldots,0,x_1+x_4), (0,\ldots,0,x_3+x_4)$$

$$(0,\ldots,0,x_2+x_3+x_4), (0,\ldots,0,x_1+x_2+x_4), (0,\ldots,0,x_1+x_2+x_3)$$

forms a subdivision of $K_{3,3}$, and so $TCay(R^n)$ is not planar. □

Theorem 4.4. *Let R be a commutative ring and T be a triangular subset of R^n which is not closed under addition and $m = n$ and $i = 3$, then $TCay(R^n)$ is planar if and only if $|T| = 4$.*

Proof. Since $i = 3$, then

$$T = (\{0\} \times \ldots \times \{0\} \times I_1) \cup (\{0\} \times \ldots \times \{0\} \times I_2) \cup (\{0\} \times \ldots \times \{0\} \times I_3)$$

where $|I_1|$,$|I_2|$ and $|I_3|$ are at least 2.

CASE 1: If $|T| > 4$, then there exists $|I_i| \geqslant 3$, for $1 \leqslant i \leqslant 3$. Hence, the elements $(0, \ldots, 0, a), (0, \ldots, 0, 2a), (0, \ldots, 0, b), (0, \ldots, 0, c)$ are belong T, where $a, 2a, b, c \neq 0$. Therefore

$$(0, \ldots, 0), (0, \ldots, 0, a), (0, \ldots, 0, 2a), (0, \ldots, 0, b), (0, \ldots, 0, b), (0, \ldots, 0, a + b),$$

$$(0, ..., 0, 2a+b), (0, ..., 0, 2a+c), (0, ..., 0, b+c), (0, ..., 0, a+b+c), (0, ..., 0, 2a+b+c)$$

forms a subdivision of $K_{3,3}$, and so $TCay(R^n)$ is not planar.

CASE 2: If $|T| = 4$, then $|I_1| = |I_2| = |I_3| = 2$. Since

$$T = (\{0\} \times \ldots \times \{0\} \times \{a\}) \cup (\{0\} \times \ldots \times \{0\} \times \{b\}) \cup (\{0\} \times \ldots \times \{0\} \times \{c\}).$$

Hence the graph $TCay(R^n)$ is the union of some copies of graph as Figure 1.

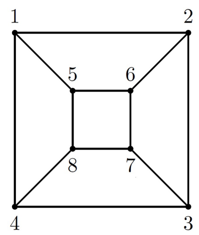

Figure 1.

The converse statement is clear. □

The proof of following lemma is similar to the proof of Lemma 4.1 in [3] and hence we omit it.

Lemma 4.5. *Let R be a commutative ring and T be a triangular subset of R^n which is not closed under addition, $m = n$ and $i = 2$. Then*

(1) *if T contains ideals P_1 and P_2 with $|P_1| \geqslant 4$, $|P_2| \geqslant 2$ and $|P_1 \cup P_2| \geqslant 5$, then $TCay(R^n)$ is not planar;*

(2) *If T contains ideals P_1 and P_2 with $|P_1|, |P_2| \geqslant 3$ and $|P_1 \cup P_2| \geqslant 5$, then $TCay(R^n)$ is not planar.*

Theorem 4.6. *Let R be a commutative ring and T be a triangular subset of R^n which is not closed under addition, $m = n$ and $i = 2$, then $TCay(R^n)$ is planar if and only if $|T| \leqslant 4$.*

Proof. Let $|T| \leqslant 4$.

CASE 1: $|T| = 4$, then T contains ideals P_1 and P_2 with $|P_1| = 3$, $|P_2| = 2$. We may assume that $P_1 = \{(0,\dots,0),(0,\dots,0,a),(0,\dots,0,2a)\}$ and $P_2 = \{(0,\dots,0),(0,\dots,0,b)\}$, where $a,b \neq 0$ and $a \neq b$. then $TCay(R^n)$ is the union of some copies of grpah as Figure 2. For every $(x_1,\dots,x_n) \in R^n$, we have

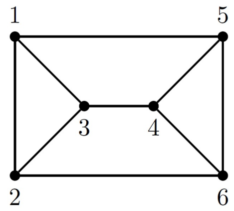

Figure 2.

$$x_1 = (x_1,\dots,x_n + a), \qquad x_2 = (x_1,\dots,x_n + 2a),$$
$$x_3 = (x_1,\dots,x_n), \qquad x_4 = (x_1,\dots,x_n + b),$$
$$x_5 = (x_1,\dots,x_n + a + b), \qquad x_6 = (x_1,\dots,x_n + 2a + b).$$

Therefore $TCay(R^n)$ is planar.

CASE 2: If $|T| = 4$, then T contains ideals P_1 and P_2 with $|P_1| = |P_2| = 2$ and hence the graph $TCay(R^n)$ is the union of some copies of C_4. Therefore $|T| = 4$ is planar.

The converse statement is a consequence of Theorem 4.5. □

Now we have the following corollary.

Corollary 4.7. *Let R be a commutative ring and T be a triangular subset of R^n, then $TCay(R^n)$ is planar if and only if following statment is hod:*

(1) *T is closed under addition and $|T| \leqslant 4$.*

(2) *T not closed under addition, $i = 3$ and $|T| = 4$.*

(3) *T not closed under addition, $i = 2$ and $|T| \leqslant 4$.*

References

[1] **M. Afkhami, M.R. Ahmadi, R. Jahani-Nezhad and K. Khashyarmanesh,** *Cayley graphs of ideals in a commutative ring*, Bull. Malays. Math. Sci. Soc. **37** (2014), 833 – 843.

[2] **M. Afkhami, Z. Barati, K. Khashyarmanesh and N. Paknejad,** *Cayley sum graphs of ideals of a commutative ring*, J. Aust. Math. Soc. **96** (2014), 289 – 302.

[3] **M. Afkhami, K.Hamidizadeh and K. Khashyarmanesh,** *On the generalization of Cayley graphs of commutative ring*, Beitrage Algebra Geom. **58** (2016), 395 – 404.

[4] **G. Alipour, S. Akbari,** *Some properties of a Cayley graph of a commutative ring,* Commun. Algebra **42** (2014) 1582 – 1593.

[5] **D.F. Anderson, A. Badawi,** *On the zero-divisor graph of a ring,* Commun. Algebra **36** (2008), 3073 – 3092.

[6] **D.F. Anderson, A. Badawi,** *The generelized total graph of a commutative ring,* J. Algebra Appl. **12** (2013), 1250212.

[7] **D.F. Anderson, A. Badawi,** *The total graph of a commutative ring,* J. Algebra. **320** (2008), 2706 – 2719.

[8] **D.F. Anderson, A. Badawi,** *The total graph of a commutative ring with out the zero element,* J. Algebra Appl. **11** (2012), 12500740.

[9] **J.A. Bondy, U.S.R. Murty,** *Graph Theory with applications,* American Elsevier, New York, 1976.

[10] **I. Kaplansky,** *Commutative Rings,* 1976.rev. ed., University of Chicago Press, Chicago, 1974

[11] **A.V. Kelarev,** *Directed graphs and nilpotent rings,* J. Austral. Math. Soc. **65** (1998), 326 – 332.

[12] **A.V. Kelarev,** *On undirected Cayley graphs,* Australasian J. Combinatorics **25** (2002), 73 – 78.

[13] **A.V. Kelarev,** *Ring Constructions and Applications,* World Scientific, River Edge, NJ, 2002.

[14] **A.V. Kelarev,** *Graph Algebras and Automata,* Marcel Dekker, New York, 2003.

[15] **A.V. Kelarev,** *Labelled Cayley graphs and minimal automata,* Australasian J. Combinatorics **30** (2004), 95 – 101.

[16] **A.V. Kelarev,** *On Cayley graphs of inverse semigroups,* Semigroup Forum **72** (2006), 411 – 418.

[17] **A.V. Kelarev, C.E. Praeger,** *On transitive Cayley graphs of groups and semigroups,* European J. Combinatorics **24** (2003), 59 – 72.

[18] **A.V. Kelarev, S.J. Quinn,** *A combinatorial property and power graphs of groups,* Contrib. General Algebra **12** (2000), 229 – 235.

[19] **A.V. Kelarev, S.J. Quinn,** *Directed graphs and combinatorial properties of semigroups,* J. Algebra **251** (2002), 16 – 26.

Received May 14, 2017

K. Hamidizadeh
Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran
E-mail: k.hamidizadeh@pnu.ac.ir

G. Aghababaei
Department of Mathematics, University of Applied Sciences and Technology, Tehran, Iran
E-mail: g-aghababaei@yahoo.com