Quasigroups and Related Systems **26** (2018), 41 − 44

Pseudoisomorphisms of quasigroups

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Abstract. A simple method of construction of autotopies of quasigroups is presented.

1. In this short note all considered quasigroups are defined on a finite set Q, α, β, γ are permutations of the set Q. The composition of permutations is defined as $\alpha\beta(x) = \alpha(\beta(x))$. Let $(Q, \cdot)$ and $(Q, \circ)$ be two quasigroups. Their permutations $L_a(x) = a \cdot x$, $R_a(x) = x \cdot a$, $M_a(x) : x \cdot M_a(x) = a$ and $L^{\circ}_a(x) = a \circ x$, $R^{\circ}_a(x) = x \circ a$, $M^{\circ}_a(x) : x \circ M^{\circ}_a(x) = a$ are called *left*, *right* and *middle translations* of the corresponding quasigroup. Quasigroups $A = (Q, \cdot)$ and $B = (Q, \circ)$ are *isotopic* if there exists triplet $T = (\alpha, \beta, \gamma)$, called an *isotopism*, such that

$$\gamma(x \cdot y) = \alpha(x) \circ \beta(y), \tag{1}$$

or equivalently

$$\gamma^{-1}(x \circ y) = \alpha^{-1}(x) \cdot \beta^{-1}(y), \tag{2}$$

for all $x, y \in Q$. In this case, we will write $A \sim B$. Obviously $A \sim A$. In this case we say that $T = (\alpha, \beta, \gamma)$ is an *autotopism.*

Left, right and middle translations play an important role in the investigation of isotopies of quasigroups (see for example [2], [3] and [4]). They also are used in the construction of prolongations and contractions of Latin squares (see [5] and [6]). Below we present a simple method of construction of autotopies of quasigroups based on such translations.

2. An isotopy of the form $T_l = (\alpha, \beta, \alpha)$ is called a *left pseudoisomorphism.* An isotopy $T_r = (\alpha, \beta, \beta)$ is called a *right pseudoisomorphism*, and an isotopy $T_m = (\alpha, \alpha, \gamma)$ is called a *middle pseudoisomorphism.*

As a simple consequence of (1) we obtain

Lemma 1. *Two quasigroups $(Q, \cdot)$ and $(Q, \circ)$ are isotopic if and only if at least one of the following identities*

$$L_i = \gamma^{-1} L^{\circ}_{\alpha(i)} \beta, \quad R_i = \gamma^{-1} R^{\circ}_{\beta(i)} \alpha, \quad M_i = \beta^{-1} M^{\circ}_{\gamma(i)} \alpha$$

is satisfied for some permutations α, β, γ of Q.

2010 Mathematics Subject Classification: 20N05
Keywords: Quasigroup, autotopism, pseudoisomorphism.

Corollary 1. *The triplet $T = (\alpha, \beta, \gamma)$ is an autotopism of a quasigroup $(Q, \cdot)$ if and only if at least one of the following identities*

$$L_i = \gamma^{-1} L_{\alpha(i)} \beta, \quad R_i = \gamma^{-1} R_{\beta(i)} \alpha, \quad M_i = \beta^{-1} M_{\gamma(i)} \alpha$$

is satisfied for some permutations α, β, γ of Q.

By a *left b-adjoint quasigroup* of a quasigroup $(Q, \cdot)$ we mean a quasigroup $(Q, *)$ with the operation $x * y = L_b^{-1}(x \cdot y)$, where $b \in Q$ is fixed. A *right b-adjoint quasigroup* is defined analogously. A a *middle b-adjoint quasigroup* of a quasigroup $(Q, \cdot)$ is defined as a quasigroup $(Q, *)$ with the operation
$x * M_b^{-1}(y) = x \cdot y$.

Theorem 1. *Two quasigroups are isotopic if and only if its left b-adjoint quasigroups are right pseudoisomorphic.*

Proof. Let quasigroups $(Q, \cdot)$ and $(Q, \circ)$ be isotopic and let this isotopism has the form (1). If $(Q, *)$ and $(Q, \diamond)$ are left b-adjoint quasigroups of quasigroups $(Q, \cdot)$ and $(Q, \circ)$, respectively, then $x * y = L_b^{-1}(x \cdot y)$ and $x \diamond y = (L_b^\circ)^{-1}(x \circ y)$. Thus, $L_x^\diamond = (L_b^\circ)^{-1} L_x^\circ$. This, by Lemma 1, gives

$$L_x^* = L_b^{-1} L_x = (\gamma^{-1} L_{\alpha(b)}^\circ \beta)^{-1} \gamma^{-1} L_{\alpha(x)}^\circ \beta = \beta^{-1} (L_{\alpha(b)}^\circ)^{-1} L_{\alpha(x)}^\circ \beta = \beta^{-1} L_{\alpha(x)}^\diamond \beta.$$

Hence $\beta(x * y) = \alpha(x) \diamond \beta(y)$.

The converse statement is obvious. □

Corollary 2. *If quasigroups $(Q, \cdot)$ and $(Q, \circ)$ are isotopic, then, for every $b \in Q$, its left b-adjoint quasigroups are right pseudoisomorphic. Conversely, if for some $b \in Q$ left b-adjoint quasigroups of quasigroups $(Q, \cdot)$ and $(Q, \circ)$ are pseudoisomorphic, then $(Q, \cdot)$ and $(Q, \circ)$ are isotopic.*

In a similar way we can prove the following two theorems.

Theorem 2. *Two quasigroups are isotopic if and only if its right b-adjoint quasigroups are left pseudoisomorphic.*

Theorem 3. *Two quasigroups are isotopic if and only if its middle b-adjoint quasigroups are middle pseudoisomorphic.*

3. As it is well known, any permutation φ of the set $\{1, 2, \ldots, n\}$ can be decomposed into $r \leqslant n$ cycles of the length $k_1, k_2, \ldots, k_r$ with $k_1 + k_2 + \ldots + k_r = n$. We denote this fact by $C(\varphi) = \{k_1, k_2, \ldots, k_r\}$. $C(\varphi)$ is called a *cyclic type* of φ.

Two permutations $\varphi, \psi \in S_n$ are *conjugate* if there exists a permutation $\rho \in S_n$ such that $\rho \varphi \rho^{-1} = \psi$. Two permutations are conjugate if and only if they have the same cyclic type (cf. [7]).

Let $(Q, \circ)$, where $Q = \{1, 2, \ldots, n\}$ be a quasigroup and let

$$C(L^\circ) = \{C(L_1^\circ), C(L_2^\circ), \ldots, C(L_n^\circ)\},$$

$$C(R^\circ) = \{C(R_1^\circ), C(R_2^\circ), \ldots, C(R_n^\circ)\},$$
$$C(M^\circ) = \{C(M_1^\circ), C(M_2^\circ), \ldots, C(M_n^\circ)\}.$$

Then as a consequence of the above results we obtain

Theorem 4. *If quasigroups* $(Q,\cdot)$ *and* $(Q,\circ)$ *are isotopic, then* $C(L^*) = C(L^\diamond)$, $C(R^*) = C(R^\diamond)$, $C(M^*) = C(M^\diamond)$ *for all its left (right, middle) b-adjoint quasigroups* $(Q,*)$ *and* $(Q,\diamond)$.

The converse statement is not true.

Example 1. Consider two loops defined by the following tables.

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	4	5	6	1
3	3	4	5	6	1	2
4	4	5	6	1	2	3
5	5	6	1	2	3	4
6	6	1	2	3	4	5

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	4	5	6	1	2
4	4	3	6	5	2	1
5	5	6	1	2	4	3
6	6	5	2	1	3	4

The first loop is a group isomorphic to the cyclic group $\mathbb{Z}_6$; the second is not a group because $(5\circ 4)\circ 3 \neq 5\circ(4\circ 3)$. So, by the Albert's theorem, they are not isotopic, but, as it is not difficult to see, $C(L^*) = C(L^\diamond)$, $C(R^*) = C(R^\diamond)$, $C(M^*) = C(M^\diamond)$ for all its b-adjoint quasigroups $(Q,*)$ and $(Q,\diamond)$.

4. Basing on the above results we can find autotopies of quasigroups for which left, right or middle translations have different cyclic types.

For simplicity permutations will be written in the form of cycles and cycles will be separated by points, e.g.

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix} = (132.45.6.).$$

Example 2. Let $(Q,\cdot)$ be a quasigroup defined by the following table:

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	8	6	5	4	7	1	3
3	3	7	8	1	2	4	5	6
4	4	5	2	8	3	1	6	7
5	5	6	7	2	8	3	4	1
6	6	3	4	7	1	5	8	2
7	7	4	1	3	6	8	2	5
8	8	1	5	6	7	2	3	4

This quasigroup has the following left and right translations:

	left translations	*cyclic type*
L_1	(1.2.3.4.5.6.7.8.)	{1,1,1,1,1,1,1,1}
L_2	(128367.45.)	$\{2,6\}$
L_3	(13864.275.)	$\{3,5\}$
L_4	(14876.253.)	$\{3,5\}$
L_5	(158.26374.)	$\{3,5\}$
L_6	(165.23478.)	$\{3,5\}$
L_7	(17243.568.)	$\{3,5\}$
L_8	(18462.357.)	$\{3,5\}$

	right translations	*cyclic type*
R_1	(1.2.3.4.5.6.7.8.)	{1,1,1,1,1,1,1,1}
R_2	(128.37456.)	$\{3,5\}$
R_3	(13857.264.)	$\{3,5\}$
R_4	(148673.25.)	$\{2,6\}$
R_5	(15876.243.)	$\{3,5\}$
R_6	(16534.278.)	$\{3,5\}$
R_7	(172.35468.)	$\{3,5\}$
R_8	(18475.236.)	$\{3,5\}$

If this quasigroup has an autotopy of the form $\beta(x \cdot y) = \alpha(x) \cdot \beta(y)$, then by Corollary 2 we have $L_i = \beta^{-1}L_{\alpha(i)}\beta$ for every $i \in Q$. Since L_i and $L_{\alpha(i)}$, as a conjugate permutations, have the same cyclic type, in the case $i = 2$ must be $L_2 = L_{\alpha(2)}$, so $\alpha(2) = 2$. Thus $L_2 = \beta^{-1}L_2\beta$, i.e., $\beta L_2 = L_2\beta$. The last equation is satisfied by $\beta = (128367.5.4.)$. Now, using $\beta L_i = L_{\alpha(i)}\beta$, we see that $\alpha = (1.2.354786.)$. This shows that our quasigroups has an autotopy (α, β, β). Since autotopies form a group (cf. [1]), $(\alpha^k, \beta^k, \beta^k)$ also are autotopies for each natural k.

An autotopy of the form (σ, ρ, σ) is induced by right translations. Indeed, from the fact that $R_i = \sigma^{-1}R_{\rho(i)}\sigma$ have the same cyclic type we obtain $R_4 = R_{\rho(4)}$. So, $\rho(4) = 4$. Hence, analogously, as in the previous part, we calculate $\sigma = (137684.2.5.)$ and $\rho = (1.4.257863.)$. Obviously $(\sigma^t, \rho^t, \sigma^t)$ also is an autotopy for each natural t. Moreover, composition of $(\alpha^k, \beta^k, \beta^k)$ and $(\sigma^t, \rho^t, \sigma^t)$ is an autotopy too. For example, composition of (α, β, β) and (σ, ρ, σ) gives an autotopy $(\alpha_1, \beta_1, \gamma_1)$ with $\alpha_1 = \alpha\sigma = (154.387.2.6.)$, $\beta_1 = \beta\rho = (125.387.4.6.)$, $\gamma_1 = \beta\sigma = (163.284.5.7.)$.

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Received February 10, 2018

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