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On state equality algebras

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Abstract. We show that every state-morphism operator on an equality algebra is an internal state operator on it and prove that the converse is correct for the linearly ordered equality algebras under a special condition. Then we show that there is a one-to-one correspondening between congruence relations on a state-morphism (linearly ordered state) equality algebra and state-morphism (state) deductive systems on it. Moreover, we define the notion of homomorphism on equality algebras and we investigate the relation between state operators and state-morphism operators with equality-homomorphism. Finally, we characterize the simple and semisimple classes of state-morphism equality algebras.

1. Introduction

Equality algebras were introduced in [8] by Jenei, that the motivation cames from EQ-algebra [13]. State MV-algebras were introduced by Flaminio and Montagna as MV-algebras with internal states [6]. Di Nola and Dvurečenskij introduced the notion of state-morphism MV-algebra which is a stronger variation of a state MV-algebra [4]. State BCK-algebras and state-morphism BCK-algebras have been defined and studied by Borzooei, Dvurečenskij and Zahiri [2]. Recently, the state equality algebras and state-morphism equality algebras have been introduced in [3]. Now we prove that every state-morphism operator on an equality algebra is an internal state operator on it, and we prove the converse is true for a linearly ordered equality algebra under a special condition. Also, we remove the condition of [3, Th. 6.8] and [3, prop. 5.7(3)] and state them in general case. We introduce a deductive system on state (state-morphism) equality algebra and we investigate some related results. Then we show that for any linearly ordered sate (state-morphism) equality algebra (A,σ), there is a one-to-one correspondence between $Con(A,\sigma)$ and $IDS(A_\sigma)$ $(SDS(A_\sigma))$. We show that every internal state operator on an equality algebra is a state-morphism if it is equality-homomorphism. Finally, we study some classes of state-morphism equality algebras such as simple and semisimple state-morphism equality algebras.

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2. Preliminaries

In this section, we recall basic definitions and results relevant to equality algebra which will be used in the next sections.

Definition 2.1. (cf. [8]) An *equality algebra* is an algebra $(A, \wedge, \sim, 1)$ of type $(2, 2, 0)$ such that the following axioms are fulfilled for all $a, b, c \in A$:
(E_1) $(A; \wedge, 1)$ is a meet-semilattice with top element 1,
(E_2) $a \sim b = b \sim a$,
(E_3) $a \sim a = 1$,
(E_4) $a \sim 1 = a$,
(E_5) $a \leqslant b \leqslant c$ implies $a \sim c \leqslant b \sim c$ and $a \sim c \leqslant a \sim b$,
(E_6) $a \sim b \leqslant (a \wedge c) \sim (b \wedge c)$,
(E_7) $a \sim b \leqslant (a \sim c) \sim (b \sim c)$,
where $a \leqslant b$ iff $a \wedge b = b$.

Let $(A, \wedge, \sim, 1)$ be an equality algebra. A subset $D \subseteq A$ is called a *deductive system* of A if for all $a, b \in A$, (DS_1): $1 \in D$, (DS_2): $a \in D$ and $a \leqslant b$ implies $b \in D$, (DS_3): $a, a \sim b \in D$ implies $b \in D$.

A deductive system D of an equality algebra A is *proper* if $D \neq A$. The set of all deductive systems of A is denoted by DS(A). An equality algebra A is called *simple* if $DS(A) = \{\{1\}, A\}$. A non-empty subset S of an equality algebra $(A, \wedge, \sim, 1)$ which is closed under $\sim$ is called a *subalgebra* of A and the set of all subalgebras of A is denoted by $Sub(A)$. We know that $\sim$ is higher priority than the operation $\wedge$ (it means that first we calculate the operation $\wedge$ then apply the operation $\sim$). For simplify, some times we write $a \sim (a \wedge b) = a \sim a \wedge b$. The operations $\rightarrow$ (called *implication*) and $\leftrightarrow$ (called *equivalence*) on equality algebra A are defined as follows:

$$a \rightarrow b = a \sim (a \wedge b) \quad , \quad a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

If there exists *zero element* $0 \in A$ such that $0 \leqslant a$ (i.e, $0 \rightarrow a = 1$), for all $a \in A$, then A is called a *bounded equality algebra* and it is denoted by $(A, \wedge, \sim, 0, 1)$.

Proposition 2.2. (cf. [3, 8]) *Let $(A, \wedge, \sim, 1)$ be an equality algebra. Then the following hold for all $a, b, c \in A$:*
(E_8) $a \sim b \leqslant a \rightarrow b \leqslant a \leftrightarrow b$,
(E_9) $a \leqslant (a \sim b) \sim b$,
(E_{10}) $a \sim b = 1$ *iff* $a = b$,
(E_{11}) $a \rightarrow b = 1$ *iff* $a \leqslant b$,
(E_{12}) $a \rightarrow b = 1$ *and* $b \rightarrow a = 1$ *implies* $a = b$,
(E_{13}) $a \leqslant b \rightarrow a$,
(E_{14}) $a \leqslant (a \rightarrow b) \rightarrow b$,
(E_{15}) $a \rightarrow b \leqslant (b \rightarrow c) \rightarrow (a \rightarrow c)$,
(E_{16}) $a \leqslant b \rightarrow c$ *iff* $b \leqslant a \rightarrow c$,

(E_{17}) $a \to (b \to c) = b \to (a \to c)$,
(E_{18}) $x \leqslant y$ *implies* $y \to z \leqslant x \to z$,
(E_{19}) $x \leqslant y$ *implies* $z \to x \leqslant z \to y$,
(E_{20}) $b \leqslant a \sim a \wedge b$, $a \sim b \leqslant a \sim a \wedge b$,
(E_{21}) $a \leqslant (a \sim a \wedge b) \sim b$, $b \leqslant (a \sim a \wedge b) \sim b$,
(E_{22}) $((a \to b) \to b) \to b = a \to b$.

Proposition 2.3. (cf. [3, 5]) *Let* $(A, \wedge, \sim, 1)$ *be an equality algebra and* $D \in DS(A)$. *Then the following hold for all* $a, b \in A$:
(*i*) *if* $a, a \to b \in D$, *then* $b \in D$,
(*ii*) *if* $a, b \in D$, *then* $a \sim b \in D$ *and* $a \wedge b \in D$,
(*iii*) *if* A *is linearly ordered, then* $a \sim b \in D$ *iff* $a \leftrightarrow b \in D$ *iff* $b \to a, a \to b \in D$.

Proposition 2.4. (cf. [3]) *Every deductive system of an equality algebra* A *is a subalgebra of* A.

Proposition 2.5. (cf. [3, 9]) *Let* A *be an equality algebra and* $Con(A)$ *be the set of all congruence relations on* A. *Then the following hold:*
(*i*) *For any* $D \in DS(A)$, *the relation* θ_D *on* A *which is defined by* $(a, b) \in \theta_D \Leftrightarrow a \sim b \in D$, *is a congruence relation on* A.
(*ii*) *If* $\theta \in Con(A)$, *then* $[1]_\theta = \{a \in A : (a, 1) \in \theta\}$ *is a deductive system of* A.

For $D \in DS(A)$ and $\theta_D \in Con(A)$, we denote the set of all equivalence classes of θ_D by $A/D = \{a/D : a \in A\}$.

Theorem 2.6. (cf. [3, 9]) *Let* $(A, \wedge, \sim, 1)$ *be an equality algebra. Then there is a one-to-one correspondence between* $DS(A)$ *and* $Con(A)$.

Theorem 2.7. (cf. [3, 5]) *Let* $(A, \wedge, \sim, 1)$ *be an equality algebra and* $D \in DS(A)$. *Then* $(A/D, \wedge_D, \sim_D, 1_D)$ *is an equality algebra with the following operations:*

$$a/D \wedge_D b/D = (a \wedge b)/D \ , \qquad a/D \sim_D b/D = (a \sim b)/D.$$

In the following we recall definitions of internal state and state-morphism operators and their properties. For more details, see [3].

Definition 2.8. (cf. [3]) Let $(A, \wedge, \sim, 1)$ be an equality algebra. Then (A, σ) is called an *internal state equality algebra* if $\sigma : A \to A$ is a unary operator on A such that for all $a, b \in A$ the following conditions are satisfied:
(S_1) $\sigma(a) \leqslant \sigma(b)$, whenever $a \leqslant b$,
(S_2) $\sigma(a \sim a \wedge b) = \sigma((a \sim a \wedge b) \sim b) \sim \sigma(b)$,
(S_3) $\sigma(\sigma(a) \sim \sigma(b)) = \sigma(a) \sim \sigma(b)$,
(S_4) $\sigma(\sigma(a) \wedge \sigma(b)) = \sigma(a) \wedge \sigma(b)$.

In the following, we replace internal state equality algebra by state equality algebra.

For any state equality algebra (A, σ), the $Ker(\sigma)$ is defined as $\{a \in A | \sigma(a) = 1\}$. The state σ is called *faithful*, if $Ker(\sigma) = \{1\}$. The set of all internal states on an equality algebra A denote by $S(A)$. Clearly $S(A) \neq \emptyset$. In fact, the identity map 1_A is a faithful state on A. If A is linearly ordered, then $Id_A \in S(A)$.

Proposition 2.9. (cf. [3]) *Let $(A, \wedge, \sim, 1)$ be an state equality algebra. Then for all $a, b \in A$ the following hold:*
(1) $\sigma(1) = 1$,
(2) $\sigma(\sigma(a)) = \sigma(a)$,
(3) $\sigma(A) = \{a \in A : a = \sigma(a)\}$,
(4) $\sigma(A)$ *is a subalgebra of* A,
(5) $Ker(\sigma) \in DS(A)$,
(6) $Ker(\sigma)$ *is a subalgebra of* A,
(7) $Ker(\sigma) \cap \sigma(A) = \{1\}$.

Definition 2.10. (cf. [3])Let $(A, \wedge, \sim, 1)$ be an equality algebra. Then (A, σ) is called a *state-morphism equality algebra* if $\sigma : A \to A$ is a unary operator on A such that for all $a, b \in A$ the following conditions are satisfied:
(SM_1) $\sigma(a \sim b) = \sigma(a) \sim \sigma(b)$,
(SM_2) $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$,
(SM_3) $\sigma(\sigma(a)) = \sigma(a)$.

The set of all state-morphisms on an equality algebra A is denoted by $SM(A)$. Clearly $SM(A) \neq \emptyset$. Indeed, if A is an equality algebra, then the constant map $1_A(a) = 1$ and the identity map $Id_A(a) = a$ are state-morphism operators on A.

Proposition 2.11. (cf. [3]) *Let (A, σ) be a state-morphism equality algebra. Then the following hold:*
(1) $Ker(\sigma) \in DS(A)$,
(2) $Ker(\sigma) = \{\sigma(a) \sim a : a \in A\}$,
(3) *If* $Ker(\sigma) = \{1\}$, *then* $\sigma = Id_A$,
(4) *If A is a simple equality algebra, then* $SM(A) = \{1_A, Id_A\}$.

3. (State) deductive systems in equality algebras

In this section, by considering the notion of deductive system, we define the concept of state deductive system on state (state morphism) equality algebras then prove that the quotient algebra constructed with a state deductive system of a state-morphism (and linearly ordered state) equality algebra (A, σ) is a state-morphism (and state) equality algebra. Finally, we show that a deductive system on a state-morphism (and linearly ordered state) equality algebra define a congruence relation on (A, σ) and there is a one-to-one correspondence between $SDS(A_\sigma)$ $(IDS(A_\sigma))$ and $Con(A, \sigma)$.

Theorem 3.1. *Let X be a subset of an equality algebra A.*
(i) *The deductive system generated by X which is denoted by $\langle X \rangle$ is*

$$\langle X \rangle = \{a \in A \mid \exists n \in \mathbb{N} \text{ and } x_1, \ldots, x_n \in X \text{ st. } x_1 \to (x_2 \to \ldots(x_n \to a)\ldots) = 1\}$$

(*ii*) *If D is a deductive system of A and $S \subseteq A$, then*

$$\langle D \cup S\rangle = \{a \in A \mid \exists n \in \mathbb{N} \text{ and } s_1, \ldots, s_n \in S \text{ st. } s_1 \to (s_2 \to (...(s_n \to a)...)) \in D\}$$

Proof. It follows from [5, Prop. 4.3] and [11, Prop. 2.2.7]. □

For each x belonging to an equality algebra A, the deductive system generated by $\{x\}$ is called *principal deductive system*. Clearly,

$$\langle x\rangle = \{a \in A \mid x^n \to a = 1, \text{ for some } n \in \mathbb{N}\},$$

where $x^0 \to b = b$, $x^n \to b = x \to (x^{n-1} \to b)$.

Definition 3.2. A proper deductive system D of an equality algebra A is called
- *prime* if $a \sim a \wedge b \in D$ or $b \sim b \wedge a \in D$, for all $a, b \in A$,
- *maximal* if there is not any proper deductive system strictly containing D.

An equality algebra A is called *semisimple* if $Rad(A) = \bigcap_{D \in Max(A)} D = \{1\}$. The set of all prime (maximal) deductive systems of an equality algebra A is denoted by $Pr(A)(Max(A))$.

Proposition 3.3. *Any proper deductive system of a bounded equality algebra A is contained in a maximal deductive system of A.*

Proof. It is an immediate consequence of Zorn's Lemma. □

Example 3.4. (*i*). Let $A = \{0, a, b, 1\}$ be a poset with $0 < a, b < 1$. Then $(A, \wedge, \sim, 1)$ is an equality algebra with the operation $\sim$ on A, given as follows:

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

Then $DS(A) = \{\{1\}, \{a, 1\}, \{b, 1\}, A\}$, $Pr(A) = \{\{a, 1\}, \{b, 1\}\}$ and $Max(A) = \{\{a, 1\}, \{b, 1\}\}$. Also by Theorem 3.1, $\langle 0\rangle = A$, $\langle a\rangle = \{a, 1\}$, $\langle b\rangle = \{b, 1\}$ and $\langle 1\rangle = \{1\}$.

(*ii*). Let $B = \{0, b, 1\}$ be a chain such that $0 < b < 1$. Then $(B, \wedge, \sim, 1)$ is an equality algebra with the operation $\sim$ on B, given as follows:

$\sim$	0	b	1
0	1	b	0
b	b	1	b
1	0	b	1

Then $DS(B) = \{\{1\}, B\}$, $Pr(B) = \{1\}$ and $Max(B) = \{1\}$. By Theorem 3.1, $\langle 0\rangle = \langle b\rangle = B, \langle 1\rangle = \{1\}$.

(iii). Let $C = \{0, a, b, 1\}$ be a poset with $0 < a < b < 1$. Then $(C, \wedge, \sim, 1)$ is an equality algebra with the operation $\sim$ on C, given as follows:

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

Then $DS(C) = \{\{1\}, \{b, 1\}, A\}$, $Pr(C) = \{\{1\}, \{b, 1\}\}$ and $Max(C) = \{\{b, 1\}\}$. Also $\langle 0\rangle = \langle a\rangle = C$, $\langle b\rangle = \{b, 1\}$, $\langle 1\rangle = \{1\}$.

Proposition 3.5. *Let D be a proper deductive system of an equality algebra A. Then the following are equivalent:*

(i) *D is maximal.*

(ii) *For all $x \in A \setminus D$, $\langle D \cup \{x\}\rangle = A$.*

(iii) *For all $x \in A \setminus D$, $x^n \to a \in D$ for any $a \in A$.*

Proof. $(i) \Rightarrow (ii)$. If $x \in A \setminus D$, then $D \subset \langle D \cup \{x\}\rangle$. Since D is maximal, we get $\langle D \cup \{x\}\rangle = A$.

$(ii) \Rightarrow (i)$. Assume that F is a proper deductive system of A such that $D \subset F$. Hence there is $x \in F \setminus D$, and so by (ii), $\langle D \cup \{x\}\rangle = A$. Then $F = A$, that is a contradiction.

$(ii) \Leftrightarrow (iii)$. It is clearly by Theorem 3.1(ii). □

Proposition 3.6. *Let A be an equality algebra. The subalgebra S of A is a deductive system of A, if $a \in S$ and $b \in A\backslash S$ implies $a \wedge b \in A\backslash S$ and $a \sim b \in A\backslash S$.*

Proof. Let S be a subalgebra of A. Since $1 = a \sim a \in S$, thus (DS_1) satisfied. If $a \in S$ and $a \leqslant b$, then $a \wedge b = a \in S$. Assume that $b \notin S$. Since $a \in S$ and $b \in A \setminus S$ then a ∧ b ∈ A\S, which is a contradiction. Hence $b \in S$. Thus (DS_2) satisfied. Now, let $a, a \sim b \in S$, but $b \notin S$. Hence by assumption $a \sim b \in A \setminus S$, which is a contradiction. Thus $b \in S$. So (DS_3) is satisfied. □

Example 3.7. Let A be the equality algebra in Example 3.4(i). Then

$$Sub(A) = \{\{1\}, \{0, 1\}, \{a, 1\}, \{b, 1\}, A\}.$$

Clear that any member of $Sub(A)$ is a deductive system, except $\{0, 1\}$. It follows that Proposition 3.6 is not satisfied for subalgebra $\{0, 1\}$.

Proposition 3.8. *Let A be an equality algebra. Then the following hold.*

(i) *If A is linearly ordered and $a \in A$, then $A(a) = \{x \in A \mid a \leqslant x\}$ is a subalgebra of A.*

(ii) *If A is bounded, then $A_0 = \{a \in A \mid a \sim 0 = 0\}$ is a proper deductive system and subalgebra of A.*

Proof. (i). Let $a \in A$. Clearly, $A(a)$ is closed under $\wedge$. Put $x, y \in A(a)$. Since A is linearly ordered, we assume $x \leqslant y$. Now by (E_{13}) and (E_2) we get $a \leqslant x \leqslant y \to x = y \sim (y \wedge x) = y \sim x = x \sim y$. Hence $x \sim y \in A(a)$. For $y \leqslant x$, with a similar way, the result satisfies.

(ii). Since $1 \sim 0 = 0$, we get $1 \in A_0$. Let $a \in A_0$ and $a \leqslant b$. Then by (E_5), $b \sim 0 \leqslant a \sim 0 = 0$ and so $b \sim 0 = 0$. Thus $b \in A_0$. Now let $a, a \sim b \in A_0$. By (E_7), $b \sim 0 \leqslant (a \sim b) \sim (a \sim 0) = (a \sim b) \sim 0 = 0$. Hence $b \sim 0 = 0$ and so $b \in A_0$. Therefore, A_0 is a proper deductive system. Also, by Proposition 2.4, A_0 is a subalgebra of A. □

Proposition 3.9. *Let D be a proper deductive system of an equality algebra A. Then the following hold:*

(i) *D is prime iff A/D is a linearly ordered equality algebra,*

(ii) *if D is prime, then $\{F \in DS(A) \mid D \subseteq F\}$ is linearly ordered by inclusion.*

Proof. (i). For any $a, b \in A$, $a \sim a \wedge b \in D$ iff $(a \sim a \wedge b)/D = 1/D$ iff $a/D \sim_D a/D \wedge_D b/D = 1/D$ iff $a/D = a/D \wedge_D b/D$ iff $a/D \leqslant b/D$. By the similar way $b \sim b \wedge a \in D$ iff $b/D \leqslant a/D$. Hence D is prime iff A/D is a linearly ordered equality algebra.

(ii). Let $F, G \in \{F \in DS(A) \mid D \subseteq F\}$. If F and G are incomparable, then there exist $a \in F \setminus G$ and $b \in G \setminus F$. Since D is prime, by (i), A/D is linearly ordered. Then we can assume $a/D \leqslant b/D$, and so $a \sim a \wedge b \in D \subseteq F$. Since $a \in F$, by (DS_2), $a \wedge b \in F$ and since $a \wedge b \leqslant b$, by (DS_1) we get $b \in F$, which is a contradiction. Hence $F \subseteq G$ or $G \subseteq F$. □

Proposition 3.10. *Let A be an equality algebra. Then A is a linearly ordered iff each proper deductive systems of A are prime.*

Proof. Let A be a linearly ordered equality algebra. Then we have $a \leqslant b$ or $b \leqslant a$, for all $a, b \in A$. Thus for any proper $D \in DS(A)$, $a \sim a \wedge b = 1 \in D$ or $b \sim b \wedge a = 1 \in D$ and so D is prime. Conversely, by the assumption, $\{1\}$ is prime and so by Proposition 3.9, $A/\{1\} = A$ is a linearly ordered equality algebra. □

Corollary 3.11. *An equality algebra A is linearly ordered iff the set $DS(A)$ is linearly ordered by inclusion.*

Proof. It follows from Propositions 3.10 and 3.9(ii). □

Proposition 3.12. *Let A be an equality algebra. Then $D \in Max(A)$ iff A/D is simple.*

Proof. Let $D \in Max(A)$. If A/D is not simple, then there is $a \in A$ such that $\langle a/D \rangle \neq 1/D$. So $a \notin D$ and $D \subset \langle D \cup \{a\} \rangle$, which is a contradiction with the maximality of D. Hence A/D is simple. The converse is obvious. □

In the follows, we define the notion of state deductive system on state equality algebras.

Definition 3.13. Let (A,σ) be an state equality algebra. A deductive system D of A is called a *state deductive system* of A if $\sigma(D) \subseteq D$ (i.e., $a \in D$ implies $\sigma(a) \in D$). The set of all state deductive systems on state equality algebra (A,σ) is denoted by $IDS(A_\sigma)$. A proper state deductive system of (A,σ) is called a *maximal state deductive system* if there is no proper deductive system strictly containing it. The set of all maximal state deductive systems of (A,σ) is denoted by $IMax(A_\sigma)$. The intersection of all the maximal state deductive system of (A,σ) is denoted by $Rad(A,\sigma)$. Clearly, $Ker(\sigma)$ is a state deductive system of any state equality algebra.

Example 3.14. (i). Let A be the equality algebra in Example 3.4(i). Then $\sigma_1 : A \to A$ which is defined by $\sigma_1(0) = 0, \sigma_1(a) = 1, \sigma_1(b) = 0, \sigma_1(1) = 1$ is an state on A. We can check $\{b,1\} \in DS(A)$, but $\{b,1\} \notin IDS(A_{\sigma_1})$. Since $b \in \{b,1\}$ but $\sigma_1(b) = 0 \notin \{b,1\}$. Then $Rad(A) = \{1\}$ and $Rad(A,\sigma) = \{a,1\}$.

(ii). Let C be the equality algebra of Example 3.4(iii). Then $\sigma_1 : C \to C$ which is defined by $\sigma_1(0) = 0, \sigma_1(a) = a, \sigma_1(b) = a, \sigma_1(1) = 1$ is an state on C. We can check $\{b,1\} \in DS(C)$, but $\{b,1\} \notin IDS(C_{\sigma_1})$. Since $b \in \{b,1\}$ but $\sigma_1(b) = a \notin \{b,1\}$. Therefore $Rad(A) = \{b,1\}$ and $Rad(A,\sigma_1) = \{1\}$.

Example 3.15. (i). $\{1\}$ and A are state deductive systems of any state equality algebra (A,σ).

(ii). In any linearly ordered state equality algebra (A, Id_A), every $D \in DS(A)$ is a state deductive system of (A,σ). Then $Rad(A) = Rad(A,\sigma)$.

(iii). If C is the equality algebra in Example 3.4(iii). Then $\sigma : C \to C$ which is defined by $\sigma(0) = 0, \sigma(a) = a, \sigma(b) = 1, \sigma(1) = 1$ is an state on C. Then we can see that $D \in DS(C)$ iff $D \in IDS(C_\sigma)$, Since $x \in D$ follows $\sigma(x) \in D$. Then $Rad(A) = Rad(A,\sigma)$.

(iv). If A is the equality algebra of Example 3.4(i), then $\sigma : A \to A$ which is defined by $\sigma(0) = a, \sigma(a) = a, \sigma(b) = 1, \sigma(1) = 1$ is an state on A. Then we can see that $D \in DS(A)$ iff $D \in IDS(A_\sigma)$. Since $x \in D$ follows $\sigma(x) \in D$. Then $Rad(A) = Rad(A,\sigma)$.

Example 3.16. Let $(A, \wedge_A, \sim_A, 1_A)$ and $(B, \wedge_B, \sim_B, 1_B)$ be two equality algebras. Then $C = A \times B = \{(a,b) \in A \times B \mid a \in A, b \in B\}$ with operations $\wedge$, $\sim$, 1 as follows : $(a,b) \wedge (a^{'}, b^{'}) = (a \wedge_A a^{'}, b \wedge_B b^{'})$, $(a,b) \sim (a^{'}, b^{'}) = (a \sim_A a^{'}, b \sim_B b^{'})$, $1 = (1_A, 1_B)$, for all $(a,b), (a^{'}, b^{'}) \in C$, is an equality algebra.

Let $\sigma_1 : A \to A$ and $\sigma_2 : B \to B$ are states on A and B, respectively. Then $\sigma : C \to C$ which is defined by $\sigma(a,b) = (\sigma_1(a), \sigma_2(b))$ is an state on C, for all $(a,b) \in C$. Let $D_1 \in DS(A)$ and $D_2 \in DS(B)$. Then $D_1 \times D_2 \in DS(C)$ is a state deductive system of (C,σ) if for all $(a,b) \in D_1 \times D_2$ we get $\sigma(a,b) \in D_1 \times D_2$. Hence $D_1 \times D_2 \in IDS(C_\sigma)$ iff $D_1 \in IDS(A\sigma_1)$ and $D_2 \in IDS(B\sigma_2)$.

Proposition 3.17. *Let (A,σ) be an state equality algebra. Then*

(i) $\sigma(a \to b) \leqslant \sigma(a) \to \sigma(b)$, *for any* $a, b \in A$,

(ii) *if A is linearly ordered, then* $\sigma(a \sim b) \leqslant \sigma(a) \sim \sigma(b)$ *and* $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$.

Proof. (i). By (E_{21}) we have $a \leqslant (a \sim a \wedge b) \sim b$, so by (S_1), we get $\sigma(a) \leqslant \sigma((a \sim a \wedge b) \sim b)$. Now (E_{18}) follows $\sigma((a \sim a \wedge b) \sim b) \to \sigma(b) \leqslant \sigma(a) \to \sigma(b)$. Thus by (S_2), $\sigma(a \sim a \wedge b) = \sigma((a \sim a \wedge b) \sim b) \sim \sigma(b) \leqslant \sigma((a \sim a \wedge b) \sim b) \to \sigma(b)$. So $\sigma(a \to b) \leqslant \sigma(a) \to \sigma(b)$.

(ii). Since A is linearly ordered, assume that $a \leqslant b$. Then by $a \sim b \leqslant b \to a$ and (i), we get $\sigma(a \sim b) = \sigma(b \to a) \leqslant \sigma(b) \to \sigma(a) = \sigma(a) \sim \sigma(b)$. Moreover, if $a \leqslant b$ $(b \leqslant a)$ then by (S_1), $\sigma(a) \leqslant \sigma(b)$ $(\sigma(b) \leqslant \sigma(a))$. So $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$. □

Proposition 3.18. *Let (A, σ) be an state equality algebra and $S \subseteq A$. Then*

$$Fix(S) = \{a \in A \mid \sigma(a) \to s = s, \textit{for all } s \in S\}$$

is a state deductive system of (A, σ).

Proof. Obviously, $1 \in Fix(S)$. Let $a \in Fix(S)$ and $a \leqslant b$. Then $\sigma(a) \to s = s$. Hence by Definition 2.8(S_1) and (E_{18}), $\sigma(a) \leqslant \sigma(b)$ and so $\sigma(b) \to s \leqslant \sigma(a) \to s = s$, which implies that $\sigma(b) \to s = s$. Thus $b \in Fix(S)$. Let $a, a \sim b \in Fix(S)$. Then $\sigma(a) \to s = s$ and $\sigma(a \sim b) \to s = s$. Since $a \sim b \leqslant a \to b$, by Definition 2.8 and (E_{18}) we get $s \leqslant \sigma(a \to b) \to s \leqslant \sigma(a \sim b) \to s = s$. Hence $\sigma(a \to b) \to s = s$. Now by Proposition 3.17, we get $\sigma(a \to b) \leqslant \sigma(a) \to \sigma(b)$ and so $(\sigma(a) \to \sigma(b)) \to s = s$. Since $(\sigma(a) \to \sigma(b)) \to (\sigma(a) \to s) = s$ thus we have $s \leqslant \sigma(b) \to s \leqslant (\sigma(a) \to \sigma(b)) \to (\sigma(a) \to s) = s$, that follows $b \in Fix(S)$. Finally, let $a \in Fix(S)$. So $\sigma(a) \to s = s$. By $s \leqslant \sigma(\sigma(a)) \to s = \sigma(a) \to s = s$, we get $\sigma(a) \in Fix(S)$. Hence $Fix(S) \in IDS(A_\sigma)$. □

Definition 3.19. Let (A, σ) be an state equality algebra. If $S \subseteq A$, then $\langle\langle S \rangle\rangle$ is the *state deductive system generated by* S.

Proposition 3.20. *Let (A, σ) be an state equality algebra. If $D \in DS(A)$,*

$$\langle\langle D \rangle\rangle = \{a \in A \mid \exists n \in \mathbb{N}, \exists x_1, \ldots, x_n \in D \textit{ st. } \sigma(x_1) \to (...(\sigma(x_n) \to a)...) \in D\}.$$

Proof. Let
$S = \{a \in A \mid \exists n \in \mathbb{N}, \exists x_1, \ldots, x_n \in D \textit{ st. } \sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)) \in D\}$.

First, we show that $D \subseteq S$. For any $d \in D$, since $1 \in D$ and $\sigma(1) = 1 \in D$ we get $\sigma(1) \to d = 1 \to d = d \in D$ and so $d \in S$. Now we prove that S is a state deductive system of (A, σ). Since for all $x \in D, \sigma(x) \to 1 = 1 \in D$, by definition of S, $1 \in S$. Now, let $a \in S$ and $a \leqslant b$. Then there are $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n \in D$ such that $\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)) \in D$. Since $a \leqslant b$ and $D \in DS(A)$, from (E_{19}),

$$\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)) \leqslant \sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to b)...))$$

it follows that $\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to b)...)) \in D$. So $b \in S$. Finally, let $a, a \sim b \in S$. Then there are $m, n \in \mathbb{N}$, $x_1, x_2, ..., x_m \in D$ and $y_1, y_2, ..., y_n \in D$ such that

$$\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_m) \to a)...)) \in D$$

and $\sigma(y_1) \to (\sigma(y_2) \to (...(\sigma(y_n) \to (a \sim b))...)) \in D$. Since $a \sim b \leqslant a \to b$, we get $\sigma(y_1) \to (\sigma(y_2) \to (...(\sigma(y_n) \to (a \to b))...)) = Z \in D$. Now from (E_{19}) and (E_{17}), we have $\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_m) \to a)...)) \leqslant \sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_m) \to (\sigma(y_1) \to (\sigma(y_2) \to (...(\sigma(y_n) \to (Z \to b)))...))$. So

$$Z \to \sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_m) \to (\sigma(y_1) \to (\sigma(y_2) \to (...(\sigma(y_n) \to b))...)) \in D$$

and $Z \in D$. Hence by definition of S, $b \in S$. Thus S is a deductive system of A. Now, we prove that S is a state deductive system of A. For any $a \in S$, there are $x_1, x_2, ..., x_n \in D$ such that $\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)) = Y \in D$. Hence $Y \to (\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...))) = 1 \in D$ and $\sigma(Y \to (\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)))) = \sigma(1) = 1 \in D$. By using Propositions 3.17 and 2.9 (2), $\sigma(Y) \to (\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to \sigma(a))...))) = 1 \in D$. From $Y \in D$, by definition of S, $\sigma(a) \in S$. Finally we show that S is the smallest state deductive system of A containing D. Let $F \in IDS(A_\sigma)$ such that $D \subseteq F$. Assume $a \in S$, if $a = 1$, then $S \subseteq F$. Otherwise there are $x_1, x_2, ..., x_n \in D \subseteq F$ such that $\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)) \in D \subseteq F$. Since F is a state deductive system of A, thus $\sigma(x_1), \sigma(x_2), ..., \sigma(x_n) \in F$, so $a \in F$. Hence S is the smallest state deductive system of A containing D, that is $\langle\langle D\rangle\rangle = S$. □

Proposition 3.21. *Let D be a state deductive system of an state equality algebra (A, σ) and $x \in A$. Then*

$$\langle\langle D \cup \{x\}\rangle\rangle = \{a \in A \mid \sigma^m(x) \to (x^n \to a) \in D,\ \exists m, n \in \mathbb{N}\}.$$

A state deductive system M of a bounded state equality algebra is maximal iff for any $x \notin M$, there are $m, n \in \mathbb{N}$ such that $\sigma^m(x) \to (x^n \to 0) \in M$.

Proof. Set $S = \{a \in A \mid \sigma^m(x) \to (x^n \to a) \in D,\ \exists m, n \in \mathbb{N}\}$. First, we show that $\{D \cup \{x\}\} \subseteq S$. Let $y \in \{D \cup \{x\}\}$, if $y = x$ then $y \in S$. Otherwise $y \in D$, from $y \leqslant x \to y$ follows $x \to y \in D$. So $y \in S$. Now we prove that S is a state deductive system of (A, σ). Obviously, $1 \in S$. Let $a \in S$ and $a \leqslant b$. Then there are $m, n \in \mathbb{N}$ such that $\sigma^m(x) \to (x^n \to a) \in D$. By (E_{19}), $\sigma^m(x) \to (x^n \to b) \in D$. So $b \in S$. Now, let a and $a \sim b \in S$. Then there are $m, n, s, t \in \mathbb{N}$ such that $\sigma^m(x) \to (x^n \to a) \in D$ and $\sigma^s(x) \to (x^t \to (a \sim b)) \in D$. Since $a \sim b \leqslant a \to b$, thus $\sigma^m(x) \to (x^n \to (a \to b)) = Y \in D$. By routine proof we get $\sigma^m(x) \to (x^n \to a) \leqslant \sigma^m(x) \to (x^n \to (\sigma^s(x) \to (x^t \to (Y \to b))))$. Thus $\sigma^{m+s}(x) \to (x^{n+t} \to (Y \to b))) \in D$. On the other hand we have $Y \in D$ and so $b \in S$. Hence S is a deductive system of A. Moreover, for any $a \in S$ there are $m, n \in \mathbb{N}$ such that $\sigma^m(x) \to (x^n \to a) = Y \in D$. Then $Y \to (\sigma^m(x) \to (x^n \to a)) = 1 \in D$. By Propositions 2.9(1) and 3.17, we have $1 = \sigma(1) = \sigma(Y \to (\sigma^m(x) \to (x^n \to a)) \leqslant \sigma(Y) \to (\sigma\sigma^m(x) \to (\sigma\sigma^n(x) \to \sigma(a)))$. Since $Y \in D$ and D is state, we get $\sigma(Y) \in D$ and so by definition of S, $\sigma(a) \in S$. Hence S is a state deductive system of A, that is $S = \langle\langle D \cup \{x\}\rangle\rangle$. For proof of the second part, we assume that M is maximal and $x \notin M$. Then by maximality of M, $\langle\langle M \cup \{x\}\rangle\rangle = A$. Since A is bounded, we get $0 \in \langle\langle M \cup \{x\}\rangle\rangle$. Thus there are $m, n \in \mathbb{N}$ such that $\sigma^m(x) \to (x^n \to 0) \in M$. The converse is evident. □

Remark 3.22. Obviously, Propositions 3.20 and 3.21 hold for any state-morphism equality algebra, too.

Definition 3.23. Let (A, σ) be an state equality algebra and θ be a congruence relation on A. Then θ is called a *congruence relation* on (A, σ) if $(a, b) \in \theta$ implies $(\sigma(a), \sigma(b)) \in \theta$. The set of all congruence on (A, σ) denote by $Con(A, \sigma)$.

In the following, we show that if (A, σ) is a linearly ordered state equality algebra, there is a bijection between $IDS(A_\sigma)$ and $Con(A, \sigma)$.

Proposition 3.24. *Let (A, σ) be a linearly ordered state equality algebra. Then the following hold:*

(*i*) *if $D \in IDS(A_\sigma)$, then $\theta_D = \{(a, b) \in A \times A \mid a \sim b \in D\}$ is a congruence relation on (A, σ),*

(*ii*) *if $\theta \in Con(A, \sigma)$, then $[1]_\theta = \{a \in A \mid (a, 1) \in \theta\}$ is a state deductive system of (A, σ) (that is $[1]_\theta \in IDS(A_\sigma))$.*

Proof. (i). Let $D \in IDS(A_\sigma)$. By Proposition 2.5(i), θ_D is a congruence relation of A. Let $(a, b) \in \theta_D$. Then $a \sim b \in D$, by Definition 3.13, we get $\sigma(a \sim b) \in D$. Now since A is linearly ordered, so by Proposition 3.17, $\sigma(a) \sim \sigma(b) \in D$. Thus $(\sigma(a), \sigma(b)) \in \theta_D$. Hence θ_D is a congruence relation on (A, σ).

(ii) Let θ be a congruence relation on (A, σ). By Proposition 2.5(ii), $[1]_\theta$ is a deductive system of A. Let $a \in [1]_\theta$. Then $(a, 1) \in \theta$. Since $\theta \in Con(A, \sigma)$, thus $(\sigma(a), \sigma(1)) \in \theta$. From $\sigma(1) = 1$ follows $(\sigma(a), 1) \in \theta$ and so $\sigma(a) \in [1]_\theta$. Thus $[1]_\theta$ is a state deductive system of (A, σ). □

Theorem 3.25. *Let (A, σ) be a linearly ordered state equality algebra. Then there is a one-to-one correspondence between $IDS(A_\sigma)$ and $Con(A, \sigma)$.*

Proof. Define $f : Con(A, \sigma) \to IDS(A_\sigma)$ by $f(\theta) = [1]_\theta$. By Theorem 2.6 and Proposition 3.24, f is an one-to-one correspondence between $IDS(A_\sigma)$ and $Con(A, \sigma)$. Then the proof is complete. □

Theorem 3.26. *Let (A, σ) be a linearly ordered state equality algebra. If $D \in IDS(A_\sigma)$, then $\sigma' : A/D \to A/D$ is an state on A/D with $\sigma'(a/D) = \sigma(a)/D$.*

Proof. First, we show that σ' is well defined. Let $a/D = b/D$. Then $a \sim b \in D$ and so $\sigma(a \sim b) \in D$. By Proposition 3.17, $\sigma(a) \sim \sigma(b) \in D$ and so $\sigma(a)/D = \sigma(b)/D$. Hence $\sigma'(a/D) = \sigma'(b/D)$. Now we prove σ' is an state. For the proof of (S_1), let $a/D \leqslant b/D$. Then $a/D \sim (a/D \wedge b/D) = 1/D$ and so $a \sim (a \wedge b) \in D$. By Definition 3.13, we get $\sigma(a \sim (a \wedge b)) \in D$. Also, by Proposition 3.17, $\sigma(a) \sim \sigma(b) \wedge \sigma(b) \in D$. Thus $\sigma(a)/D \leqslant \sigma(b)/D$ and so $\sigma'(a/D) \leqslant \sigma'(b/D)$. For the proof of (S_2),

$$\begin{aligned}
\sigma'(a/D \sim a/D \wedge b/D) &= \sigma'((a \sim a \wedge b)/D) = \sigma(a \sim a \wedge b)/D \\
&= (\sigma((a \sim a \wedge b) \sim b) \sim \sigma(b))/D \\
&= \sigma((a \sim a \wedge b) \sim b)/D \sim \sigma(b)/D \\
&= \sigma'((a/D \sim a/D \wedge b/D) \sim b/D) \sim \sigma'(b/D).
\end{aligned}$$

For the proof of (S_3),

$$\begin{aligned}\sigma^{'}(\sigma^{'}(a/D)\sim\sigma^{'}(b/D)) &= \sigma^{'}(\sigma(a)/D\sim\sigma(b)/D)=\sigma^{'}((\sigma(a)\sim\sigma(b))/D)\\ &=(\sigma(\sigma(a)\sim\sigma(b)))/D=(\sigma(a)\sim\sigma(b))/D\\ &=\sigma(a)/D\sim\sigma(b)/D=\sigma^{'}(a/D)\sim\sigma^{'}(b/D).\end{aligned}$$

Also (S_4) satisfies since

$$\begin{aligned}\sigma^{'}(\sigma^{'}(a/D)\wedge\sigma^{'}(b/D)) &= \sigma^{'}(\sigma(a)/D\wedge\sigma(b)/D)=\sigma^{'}((\sigma(a)\wedge\sigma(b))/D)\\ &=\sigma(\sigma(a)\wedge\sigma(b))/D=(\sigma(a)\wedge\sigma(b))/D\\ &=\sigma(a)/D\wedge\sigma(b)/D=\sigma^{'}(a/D)\wedge\sigma^{'}(b/D).\end{aligned}$$

Finally (S_5) satisfies since

$$\sigma^{'}(\sigma^{'}(a/D))=\sigma^{'}(\sigma(a)/D)=\sigma(\sigma(a))/D=\sigma(a)/D=\sigma^{'}(a/D). \qquad \square$$

Note that in Proposition 3.26, $\sigma^{'}$ is faithful if $Ker(\sigma^{'})=\{x/D \mid \sigma^{'}(x/D)=1/D\}=\{1/D\}$ i.e., $Ker(\sigma^{'})=\{x/D \mid \sigma(x)\in D\}$.

Corollary 3.27. *Let (A,σ) be a linearly ordered state equality algebra. Then $\sigma^{'}:A/K\to A/K$ is an state on A/K such that $K=Ker(\sigma)$.*

Proof. Since $Ker(\sigma)$ is a state deductive system of (A,σ), so the result follows from Theorem 3.26. □

Definition 3.28. Let (A,σ) be a state-morphism equality algebra. A deductive system D of A is called the *state-morphism deductive system* of A if $\sigma(D)\subseteq D$, i.e., if $a\in D$ implies $\sigma(a)\in D$.

The set of all state-morphism deductive systems on a state-morphism equality algebra (A,σ) denote by $SDS(A_\sigma)$ and the set of all maximal state-morphism deductive systems of (A,σ) denote by $SMax(A_\sigma)$.

Remark 3.29. Clearly, by Theorem 4.6(i) and Definition 2.10, the above results proved for linearly ordered state equality algebra hold for state-morphism equality algebra.

Proposition 3.30. *Let (A,σ) be a state-morphism equality algebra and D be a deductive system of A. Then D is a prime state deductive system of (A,σ) iff $(A/D,\sigma^{'})$ is a linearly ordered state-morphism equality algebra.*

Proof. It follows by Proposition 3.9, Remark 3.29 and Theorem 3.26. □

Definition 3.31. Let (A,σ) be a state-morphism (an state) equality algebra. A subalgebra S of A is called *state subalgebra* if $a\in S$ implies $\sigma(a)\in S$.

Example 3.32. (i). If A is the equality algebra in Example 3.4(i), then σ_1 and $\sigma_2 : A \to A$ defined by $\sigma_1(0) = 0$, $\sigma_1(a) = 1, \sigma_1(b) = 0, \sigma_1(1) = 1$ and $\sigma_2(0) = a$, $\sigma_2(a) = a, \sigma_2(b) = 1, \sigma_2(1) = 1$ are state-morphisms on A. Also, $\{0, 1\}$ is a state subalgebra of (A, σ_1), which is not a state subalgebra of (A, σ_2), since $\sigma_2(0) = a \notin \{0, 1\}$.

(ii). Let C be an equality algebra. We know that $(C, 1_C)$ is a state-morphism equality algebra. Then every subalgebra of C is a state subalgebra of $(C, 1_C)$.

Remark 3.33. Let (A, σ) be a bounded state-morphism equality algebra. If A is linearly ordered, $a \in A$ and $a \leqslant \sigma(a)$, then by Proposition 3.8(i), $A(a)$ is a state subalgebra. Moreover, if $\sigma(0) = 0$, then by Proposition 3.8(ii), A_0 is a state deductive system. Since for any $a \in A_0$, $a \sim 0 = 0$. By Definition 2.10, $\sigma(a \sim 0) = \sigma(a) \sim \sigma(0) = \sigma(0)$, then we get $\sigma(a) \sim 0 = 0$. Thus $\sigma(a) \in A_0$.

Proposition 3.34. *Every state deductive system of an state equality algebra (A, σ) is a state subalgebra of (A, σ).*

Proof. By Proposition 2.4 and Definition 3.31, the proof is clear. □

4. Some properties of state equality algebra and state-morphism equality algebras

In the following, we state some properties of state equality algebra and state-morphism equality algebra. We proved every state-morphism operator on an equality algebra is a state operator on it and the converse is true for a linearly ordered equality algebra under a condition.

Proposition 4.1. *Let (A, σ) be a linearly ordered state equality algebra. The map $\sigma' : A/Ker(\sigma) \to A/Ker(\sigma)$ defined by $\sigma'(a/Ker(\sigma)) = \sigma(a)/Ker(\sigma)$, is a state on $A/Ker(\sigma)$, for any $a \in A$.*

Proof. First, we show that σ' is well defined. For this, let $K = Ker(\sigma)$ and $a/K = b/K$. Then $a \sim b \in K$ and so $\sigma(a \sim b) = 1$. Since A is linearly ordered, by Proposition 3.17, $\sigma(a) \sim \sigma(b) = 1$ and this conclude that $\sigma(a) = \sigma(b)$. Hence $\sigma'(a/K) = \sigma'(b/K)$. Now by Theorem 2.7, Definition 2.8 and Proposition 3.17, the proof is complete. □

Proposition 4.2. *Let (A, σ) be a state equality algebra and $Ker(\sigma)$ be prime. Then $\sigma(A)$ is linearly ordered.*

Proof. For all $a, b \in A$, $a \sim a \wedge b \in Ker(\sigma)$ or $b \sim b \wedge a \in Ker(\sigma)$. So $\sigma(a \sim a \wedge b) = 1$ or $\sigma(b \sim b \wedge a) = 1$. From $a \wedge b \leqslant a, b$ and Proposition 3.17, we get $\sigma(a) \sim \sigma(a \wedge b) = 1$ or $\sigma(b) \sim \sigma(b \wedge a) = 1$. Hence $\sigma(a) \leqslant \sigma(b)$ or $\sigma(b) \leqslant \sigma(a)$. Thus $\sigma(A)$ is linearly ordered. □

Example 4.3. Let A be the equality algebra in Example 3.4(i) and $\sigma = Id_A$. Then (A, σ) is a state equality algebra. But $Ker(\sigma) = \{1\}$ is not prime. Since by Proposition 4.2, $\sigma(A)$ is not linearly ordered ($\sigma(a) \nleq \sigma(b)$).

Proposition 4.4. *Let (A, σ) be a linearly ordered state equality algebra. Then the following statements are equivalent:*

(i) $\sigma(a \to b) = \sigma(a) \to \sigma(b)$,

(ii) $\sigma(a \sim b) = \sigma(a) \sim \sigma(b)$.

Proof. (i) $\Rightarrow$ (ii). Since A is linearly ordered, we can assume $a \leqslant b$. By (S_1), we get $\sigma(a) \leqslant \sigma(b)$ and so

$$\begin{aligned}\sigma(b \sim a) &= \sigma(b \sim b \wedge a) = \sigma(b \to a) = \sigma(b) \to \sigma(a)\\ &= \sigma(b) \sim \sigma(b) \wedge \sigma(a) = \sigma(b) \sim \sigma(a).\end{aligned}$$

For $b \leqslant a$ the proof is similarly.

(ii) $\Rightarrow$ (i). By Proposition 3.17,

$$\sigma(a \to b) = \sigma(a \sim a \wedge b) = \sigma(a) \sim \sigma(a \wedge b) = \sigma(a) \sim (\sigma(a) \wedge \sigma(b)) = \sigma(a) \to \sigma(b).$$

□

Proposition 4.5. *Let (A, σ) be a state equality algebra, σ be faithful and for any $a, b \in A$, $\sigma((a \sim a \wedge b) \sim b) = \sigma((b \sim b \wedge a) \sim a)$. Then*

(i) *$a < b$ implies $\sigma(a) < \sigma(b)$,*

(ii) *if A is linearly ordered, then $\sigma(a) = a$, for all $a \in A$.*

Proof. (i). Let $a < b$. By (S_1) we have $\sigma(a) \leqslant \sigma(b)$. Assume $\sigma(a) = \sigma(b)$. Then by (S_2) and assumption,

$$\begin{aligned}\sigma(a \sim b) &= \sigma(b \sim b \wedge a) = \sigma((b \sim b \wedge a) \sim a) \sim \sigma(a)\\ &= \sigma((a \sim a \wedge b) \sim b) \sim \sigma(a) = \sigma(b) \sim \sigma(a) = 1.\end{aligned}$$

So $a \sim b \in Ker(\sigma) = \{1\}$ and it follows $a = b$, which is a contradiction with $a < b$. Then $\sigma(a) < \sigma(b)$.

(ii). Let for all $a \in A$, $\sigma(a) \neq a$. Since A is linearly ordered, $\sigma(a) < a$ or $a < \sigma(a)$. By (i) we get $\sigma(\sigma(a)) < \sigma(a)$ or $\sigma(a) < \sigma(\sigma(a))$, which is a contradiction with (S_3). Hence $\sigma(a) = a$. □

Proposition 4.5, is not true for any state equality algebra. In Example 3.4(iii), with $\sigma_1 : C \to C$ defined by $\sigma_1(0) = 0, \sigma_1(a) = a, \sigma_1(b) = a, \sigma_1(1) = 1$, (C, σ_1) is a linearly ordered state equality algebra with $Ker(\sigma) = \{1\}$. But $\sigma(b) = a \neq b$, since $\sigma((a \sim a \wedge b) \sim b) \neq \sigma((b \sim b \wedge a) \sim a)$.

Theorem 4.6. *Let A be an equality algebra. Then*

(i) *any state-morphism on A is a state on A,*

(ii) *if (A, σ) is a linearly ordered state equality algebra in which for all $a, b \in A$ $\sigma((a \sim a \wedge b) \sim b) = \sigma((b \sim b \wedge a) \sim a)$, then σ is a state-morphism on A.*

Proof. (i) Let σ be a state-morphism operator on A. Clearly, (S_1) satisfies. Since $b \leqslant (a \to b) \to b)$, by (SM_1),

$$\begin{aligned}\sigma(a \sim a \wedge b) &= \sigma(a \to b) = \sigma(((a \to b) \to b) \to b)\\ &= \sigma(((a \sim a \wedge b) \sim b) \sim b) = \sigma((a \sim a \wedge b) \sim b) \sim \sigma(b).\end{aligned}$$

Thus (S_2) satisfies. Also (S_3) and (S_4) follow from $(SM_1) - (SM_3)$.

(ii). Let σ be a state operator on A and $a \leqslant b$. By (S_1) we have $\sigma(a) \leqslant \sigma(b)$. Then by (S_2),

$$\begin{aligned}\sigma(a \sim b) &= \sigma(b \sim b \wedge a) = \sigma((b \sim b \wedge a) \sim a) \sim \sigma(a)\\ &= \sigma((a \sim a \wedge b) \sim b) \sim \sigma(a) = \sigma(b) \sim \sigma(a) = \sigma(a) \sim \sigma(b).\end{aligned}$$

For $b \leqslant a$, with the similar proof, σ is a state-morphism operator on A. Finally, (SM_2) and (SM_3) follow from Propositions 3.17 and 2.9(2). □

Definition 4.7. (cf. [3]) Let A be an equality algebra and $a \in A$. Then

(i) A is called $(\sim_a)$-*involutive*, if for all $b \in A$, $((b \sim a) \sim a) = b$,

(ii) $x \in A$ is called a-*regular* if $(x \sim a) \sim a = x$,

(iii) A is called *involutive* if $A = Reg_a(A)$, for all $a \in A$, where $Reg_a(A)$ is the set of all a-regular elements of A.

Example 4.8. (1). Any equality algebra A is $(\sim_1)$-involutive and $A = Reg_1(A)$ (for all $b \in A, ((b \sim 1) \sim 1) = b$).
(2). Let A be the equality algebra in Example 3.4(i). Then A is $(\sim_a)$-involutive, for all a $\in$ A and $A = Reg_a(A)$.
(3). Let B be the equality algebra in Example 3.4(ii). Then B is $(\sim_0)$-involutive since $((0 \sim 0) \sim 0) = 0$, $((b \sim 0) \sim 0) = b$, $((1 \sim 0) \sim 0) = 1$. But B is not $(\sim_b)$-involutive, since $((0 \sim b) \sim b) = 1 \neq 0$.

Corollary 4.9. *Let A be a linearly ordered involutive equality algebra. Then σ is a state on A iff σ is a state-morphism on A.*

Proof. Since A is involutive, we get $(a \sim b) \sim b = a$, for all $a, b \in A$. Then by Theorem 4.6, the proof is complete. □

Example 4.10. Let C be the linearly ordered equality algebra of Example 3.4(iii). Then $\sigma_1, \sigma_2 : C \to C$ defined by $\sigma_1(0) = 1, \sigma_1(a) = 1, \sigma_1(b) = 1, \sigma_1(1) = 1$ and $\sigma_2(0) = 0, \sigma_2(a) = a, \sigma_2(b) = 1, \sigma_2(1) = 1$ are two state-morphisms on C. By Theorem 4.6, σ_1 and σ_2 are states on C. Moreover, $\sigma_3 : C \to C$ which is defined by $\sigma_3(0) = 0, \sigma_3(a) = a, \sigma_3(b) = a, \sigma_3(1) = 1$ is a state on C but it is not a state-morphism on C. Since $\sigma_3(a \sim b) \neq \sigma_3(a) \sim \sigma_3(b)$. Also, Theorem 4.6$(ii)$ is not satisfied, since $\sigma_3((b \sim b \wedge a) \sim a) = \sigma_3(1) = 1 \neq a = \sigma_3((a \sim a \wedge b) \sim b)$.

Proposition 4.11. *Let (A, σ) be a state-morphism equality algebra and $a \in A$. If $x \in Reg_a(A)$, then $\sigma(x) \in Reg_{\sigma(a)}(A)$.*

Proof. It is clearly by Definitions 4.7(ii) and 2.10. □

Proposition 4.12. *Let (A, σ) be a state-morphism equality algebra. Then $Ker(\sigma)$ is prime iff $\sigma(A)$ is linearly ordered.*

Proof. If $Ker(\sigma)$ is prime, then the proof is similar to the proof of Proposition 4.2. Conversely, assume that for all $a, b \in A$, $\sigma(a) \leqslant \sigma(b)$ or $\sigma(b) \leqslant \sigma(a)$. Let $a \sim a \wedge b \notin Ker(\sigma)$. Then $\sigma(a \sim a \wedge b) \neq 1$ and so by (SM_1) and (SM_2), $\sigma(a) \sim \sigma(a \wedge b) \neq 1$. Thus $\sigma(a) \nleq \sigma(b)$ and so by assumption, $\sigma(b) \leqslant \sigma(a)$. Hence $\sigma(b \sim b \wedge a) = 1$ and $b \sim b \wedge a \in Ker(\sigma)$. □

Proposition 4.13. *Let (A, σ) be a state-morphism equality algebra and $K = Ker(\sigma)$. Then*

(i) *$a/K \leqslant b/K$ iff $\sigma(a) \leqslant \sigma(b)$,*

(ii) *$a/K = b/K$ iff $\sigma(a) = \sigma(b)$.*

Proof. Applying Theorem 2.7 and Definition 2.10, we get

(i). $a/K \leqslant b/K$ iff $a/K = (a \wedge b)/K$ iff $(a \sim (a \wedge b))/K = 1/K$ iff $a \sim (a \wedge b) \in K$ iff $\sigma(a \sim (a \wedge b)) = 1$ iff $\sigma(a) \sim (\sigma(a) \wedge \sigma(b)) = 1$ iff $\sigma(a) \leqslant \sigma(b)$.

(ii). $a/K = b/K$ iff $(a \sim b)/K = 1/K$ iff $a \sim b \in K$ iff $\sigma(a \sim b) = 1$ iff $\sigma(a) \sim \sigma(b) = 1$ iff $\sigma(a) = \sigma(b)$. □

Proposition 4.14. *Let σ and μ be two state-morphisms on equality algebra A such that $Ker(\sigma) = Ker\mu$ and $Im\sigma = Im\mu$. Then $\sigma = \mu$.*

Proof. By Proposition 2.11, for all $a \in A$, $\sigma(a) \sim a \in Ker(\sigma) = Ker\mu$. Then $\mu(\sigma(a) \sim a) = 1$ and so we have $\mu(\sigma(a)) \sim \mu(a) = 1$. From $\sigma(a) \in Im\sigma = Im\mu$ follows $\mu(\sigma(a)) = \sigma(a)$. Hence $\sigma(a) \sim \mu(a) = 1$, that means $\sigma(a) = \mu(a)$. □

Theorem 4.15. *If (A, σ) is a state-morphism equality algebra, then*

$$A = \langle Ker(\sigma) \cup Im\sigma \rangle.$$

Proof. Obviously, $\langle Ker(\sigma) \cup Im\sigma \rangle \subseteq A$. Since $Ker(\sigma) \in Ds(A)$ and $Im\sigma \subseteq A$, thus by Theorem 3.1(ii), $\langle Ker(\sigma) \cup Im\sigma \rangle = \{a \in A \mid \sigma(a_1) \to (\sigma(a_2) \to (...(\sigma(a_n) \to a)...)) \in Ker(\sigma), \text{for some } a_1, ...a_n \in A\}$. Let a be an arbitrary element of A, by Proposition 2.11, $\sigma(a) \sim a \in Ker(\sigma)$. Since $\sigma(a) \sim a \leqslant \sigma(a) \to a$, then $\sigma(a) \to a \in Ker(\sigma)$ such that $\sigma(a) \in Im\sigma$ and so $a \in \langle Ker(\sigma) \cup Im\sigma \rangle$. Hence $A = \langle Ker(\sigma) \cup Im\sigma \rangle$. □

5. Equality-homomorphisms and their relation with the state-morphism operator

In this section, we define a homomorphism between two equality algebras and we state some related results. Then we prove that an state on an equality algebra, is a state-morphism if it is an equality-homomorphism.

Definition 5.1. *Let* $(A, \wedge, \sim, 1)$ *and* $(A^{'}, \wedge^{'}, \sim^{'}, 1^{'})$ *be two equality algebras. The map* $f : A \to A^{'}$ *is called an equality-homomorphism, if the following hold, for all* $a, b \in A$*:*
(H_1) $f(a \sim b) = f(a) \sim^{'} f(b)$,
(H_2) $f(a \wedge b) = f(a) \wedge^{'} f(b)$.

If $f : A \to A^{'}$ is a homomorphism of equality algebras, then f is called an *equality-endomorphism.* The set $Kerf = \{a \in A \mid f(a) = 1^{'}\}$ is called a *kernel* of f.
It is clear that every equality-homomorphism, is a BCK $\wedge$-semilattice homomorphism.

Proposition 5.2. *Let* $f : A \to A^{'}$ *be a bounded equality-homomorphism and* $f(0) = 0^{'}$*. Then:*
(i) $f(1) = 1^{'}$,
(ii) *f is monotone,*
(iii) $f(x \sim 0) = f(x) \sim^{'} 0^{'}$,
(iv) *$Kerf$ is a proper deductive system of A,*
(v) *Imf is a subalgebra of $A^{'}$,*
(vi) *f is injective iff* $Kerf = \{1\}$,
(vii) *if* $D^{'} \in DS(A^{'})$*, then* $f^{-1}(D^{'}) \in DS(A)$,
$(viii)$ *if f is surjective and* $Kerf \subseteq D \in DS(A)$*, then* $f(D) \in DS(A^{'})$.

Proof. The proofs of $(i) - (vi)$ are straightforward.
(vii). Assume that $D^{'} \in DS(A^{'})$. Since $f(1) = 1^{'} \in D^{'}$, thus $1 \in f^{-1}(D^{'})$. Let $a \in f^{-1}(D^{'})$ and $a \leqslant b$. Then $f(a) \in D^{'}$ and $f(a) \leqslant^{'} f(b)$. Thus $f(b) \in D^{'}$. Let $a, a \sim b \in f^{-1}(D^{'})$. Then $f(a \sim b) \in D^{'}$, by equality-homomorphism f, we get $f(a) \sim^{'} f(b) \in D^{'}$. So $f(a) \in D^{'}$ follows that $f(b) \in D^{'}$, thus $b \in f^{-1}(D^{'})$. Thus $f^{-1}(D^{'})$ is a deductive system of A.
$(viii)$. Since $1 \in D$, by (i), $1^{'} \in f(D)$. Let $a^{'}, b^{'} \in A^{'}$. If $a^{'} \in f(D)$ and $a^{'} \leqslant b^{'}$. Then there exists $a \in D$ such that $f(a) = a^{'}$. Since f is surjective, there exists $b \in A$ such that $b^{'} = f(b)$. So $f(a) \leqslant^{'} f(b)$ follows that $f(a) \to^{'} f(b) = 1$ and so $f(a \to b) = 1$, thus $a \to b \in Kerf \subseteq D$. Then $b \in D$, so $b^{'} = f(b) \in f(D)$. Let $a^{'}, a^{'} \sim^{'} b^{'} \in f(D)$. Then there are $a, z \in D$ such that $f(a) = a^{'}$ and $f(z) = a^{'} \sim^{'} b^{'}$. Since f is surjective, so there is $b \in A$, such that $f(b) = b^{'}$. So $f(z) = a^{'} \sim^{'} b^{'} = f(a) \sim^{'} f(b) = f(a \sim b)$. Thus $1 = f(z) \sim^{'} f(a \sim b) = f(z \sim (a \sim b))$. Then $z \sim (a \sim b) \in Kerf \subseteq D$ follows that $b \in D$ and so $b^{'} = f(b) \in f(D)$. Then $f(D)$ is a deductive system of $A^{'}$. □

Theorem 5.3. *If* $f : A \to A^{'}$ *is a surjective equality-homomorphism, then there is a bijective correspondence between* $\{D \mid D \in DS(A), Kerf \subseteq D\}$ *and* $DS(A^{'})$.

Proof. By Proposition 5.2(vii) and $(viii)$, $f : \{D \mid D \in DS(A), Kerf \subseteq D\} \to DS(A^{'})$ such that $D \longmapsto f(D)$ and $f^{-1} : Ds(A^{'}) \to \{D \mid D \in DS(A), Kerf \subseteq D\}$ such that $D^{'} \longmapsto f^{-1}(D^{'})$ are well defined functions. Now we will show $f(f^{-1}(D^{'})) = D^{'}$ and $f^{-1}(f(D)) = D$. Since f is surjective, then $f(f^{-1}(D^{'})) = D^{'}$. It is clear that $D \subseteq f^{-1}(f(D))$. Assume that $a \in f^{-1}(f(D))$ then $f(a) \in$

$f(D)$, so there is $x \in D$ such that $f(a) = f(x)$ then $f(a) \sim f(x) = 1$. By Definition 5.1, $f(a \sim x) = 1$ so $a \sim x \in Kerf \subseteq D$. since $x \in D$ we get $a \in D$, thus $f^{-1}(f(D)) \subseteq D$. So $f^{-1}(f(D)) = D$. □

Theorem 5.4. *Let A be an equality algebra. Then $f : A \to A$ is a state-morphism operator iff f is an equality-endomorphism with $f(a) \sim a \in Kerf$, for all $a \in A$.*

Proof. Let f be a state-morphism operator. Then by Definition 2.10, (H_1) and (H_2) satisfies. Also by (SM_3), we get, $1 = f(f(a)) \sim f(a) = f(f(a) \sim a)$. It follows $f(a) \sim a \in Kerf$.

Conversely, let f be an equality-endomorphism. By Definition 5.1, (SM_1) and (SM_2) satisfies. By the assumption, for all $a \in A, f(a) \sim a \in Kerf$. Thus $f(f(a) \sim a) = 1$. From (H_1), we get $1 = f(f(a) \sim a) = f(f(a)) \sim f(a)$ that it follows $f(f(a)) = f(a)$. So (SM_3) satisfies. □

Corollary 5.5. *If A is a simple equality algebra, then every equality-endomorphism $f : A \to A$ is a state-morphism operator, if $f = 1_A$ or $f = Id_A$.*

Proof. Assume f is an equality-endomorphism. Then By Theorem 5.4, f is a state-morphism operator if $f(a) \sim a \in Ker\sigma$ for any $a \in A$. Since A is simple so $Ker(\sigma) = \{1\}$ or $Ker(\sigma) = A$. Then $f = Id_A$ or $f = 1_A$. □

Example 5.6. Let A be the equality algebra as Example 3.4(i). Then $f : A \to A$ is an equality-endomorphism by define $f(0) = 0, f(a) = b, f(b) = a, f(1) = 1$. But f is not a state-morphism operator on A.

Lemma 5.7. *Let $f : A \to A$ be an endomorphism on equality algebra A and for all $a \in A, f(a) \sim a \in Kerf$. Then f is a state operator on A.*

Proof. By Theorems 5.4 and 4.6(i), f is an state on A. □

The converse of proposition 5.7 is not true. In Example 3.4(iii), $\sigma : C \to C$ which is defined by $\sigma(0) = 0, \sigma(a) = a, \sigma(b) = 1, \sigma(1) = 1$ is an state on the linearly ordered equality algebra C, but σ is not equality-endomorphism ($\sigma_2(a \sim b)$ $\neq \sigma_2(a) \sim \sigma_2(b)$).

Lemma 5.8. *Let σ be an state operator on a linearly ordered equality algebra A such that for all $a, b \in A$, $\sigma((a \sim a \wedge b) \sim b) = \sigma((b \sim b \wedge a) \sim a)$. Then σ is an equality-endomorphism with $\sigma^2 = \sigma$.*

Proof. By Theorems 4.6(ii) and 5.4, the proof is complete. □

Theorem 5.9. *Let $f : A \to A$ be an equality-endomorphism on an equality algebra A. Then the following are equivalent.*

(i) *f is a state operator on A.*

(ii) *f is a state-morphism operator on A.*

Proof. By Lemmas 5.7 and 5.8 and Theorem 5.4, the proof is clear. □

Theorem 5.10. *Let (A, σ) be a state-morphism equality algebra. Then,*

(i) *$\sigma(A)$ is a simple subalgebra of A iff $Ker(\sigma) \in SMax(A_\sigma)$,*

(ii) *(A, σ) is a simple state-morphism equality algebra iff A is a simple equality algebra,*

(iii) *if $\sigma(A)$ is a semisimple subalgebra of A, then the intersection of all maximal state-morphism deductive systems of (A, σ) is a subset of $Ker(\sigma)$.*

Proof. (i). Let (A, σ) be a state-morphism equality algebra. Then by Theorem 5.4, σ is an equality-endomorphism, which implies that $A/Ker(\sigma) \cong \sigma(A)$. Thus $Ker(\sigma) \in SMax(A_\sigma)$ iff $A/Ker(\sigma)$ is simple iff $\sigma(A)$ is simple.

(ii). Let (A, σ) be a simple state-morphism. Then $Ker(\sigma) \in SDS(A_\sigma)$ and so $Ker(\sigma) = \{1\}$ or $Ker(\sigma) = A$. Hence $\sigma = Id_A$ or $\sigma = 1_A$. In this case every deductive system of A is state. Thus $\{1\}$ and A are only deductive systems of A. Therefore, A is simple. Conversely, let A be a simple equality algebra. Then A has only two deductive systems, $\{1\}$ and A which they are state. Hence (A, σ) is a simple state-morphism equality algebra.

(iii). Let $\sigma(A)$ be a semisimple subalgebra of A. Then by Definition 3.2,

$$\bigcap_{I \in SMax(\sigma(A))} I = \{1\}.$$

Since $A/Ker(\sigma) \cong \sigma(A)$, then $A/Ker(\sigma)$ is a semisimple equality algebra. So $\cap\{D : Ker(\sigma) \subseteq D \in SMax(A)\} = 1/Ker\sigma$. Now we show that D is state. Let $D \in SMax(A_\sigma)$ and $Ker\sigma \subseteq D$. Then by Proposition 2.11, for all $a \in D$, $\sigma(a) \sim a \in Ker\sigma \subseteq D$. Therefore, $\sigma(a) \in D$. □

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