Quasigroups and Related Systems **25** (2017), 291 − 306

Regularity of ternary semihypergroups

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Abstract. We study some properties of regular ternary semihypergroups, completely regular ternary semihypergroups, intra-regular ternary semihypergroups and characterize them by using various hyperideals of ternary semihypergroups.

1. Introduction and preliminaries

In 1965, Sioson [14] studied ideal theory in ternary semigroups. In [4, 5] Dudek et. al. studied the ideals in n-ary semigroups. In 1995, Dixit and Dewan [3] introduced and studied some properties of ideals and quasi-(bi-)ideals in ternary semigroups. Other important results on ternary semigroups are obtained in [12, 13, 16, 15].

Hyperstructure theory was introduced in 1934, when F. Marty [11] defined hypergroups based on the notion of hyperoperation, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Davvaz et al. in [2] considered a class of algebraic hypersystems which represent a generalization of semigroups, hypersemigroups and n-ary semigroups.

In this paper we extend the notion of regularity in ternary semihypergroups and we study some properties of regular ternary semihypergroups, completely regular ternary semihypergroups, intra-regular ternary semihypergroups and characterize them by using various hyperideals of ternary semihypergroups extending those for ternary semigroups.

Recall first the basic terms and definitions from the ternary semihypergroups theory.

Definition 1.1. A map $f : H \times H \times H \to \mathcal{P}^*(H)$ is called *ternary hyperoperation* on the set H, where H is a nonempty set and $\mathcal{P}^*(H)$ denotes the collection of all nonempty subsets of H.

A *ternary hypergroupoid* is called the pair (H, f) where f is a ternary hyperoperation on the set H.

2010 Mathematics Subject Classification: 20N20, 20N15, 20M17.
Keywords: ternary semihypergroup, left (right, lateral, bi-, quasi-) hyperideal, completely regular, intra-regular, regular, completely semiprime.

If A, B, C are nonempty subsets of H, then we define

$$f(A,B,C) = \bigcup_{a\in A, b\in B, c\in C} f(a,b,c).$$

A ternary hypergroupoid (H, f) is called a *ternary semihypergroup* if for all $a_1, a_2, \ldots, a_5 \in H$, we have

$$f(f(a_1,a_2,a_3),a_4,a_5) = f(a_1, f(a_2,a_3,a_4),a_5) = f(a_1,a_2,f(a_3,a_4,a_5)).$$

A nonempty subset T of H is called a *ternary subsemihypergroup* of H if and only if $f(T,T,T) \subseteq T$.

Definition 1.2. Let (H, f) be a ternary semihypergroup. Then H is called a *ternary hypergroup* if for all $a, b, c \in H$, there exist $x, y, z \in H$ such that:

$$c \in f(x,a,b) \cap f(a,y,b) \cap f(a,b,z).$$

Definition 1.3. Let (H, f) be a ternary hypergroupoid. Then

1. (H, f) is $(1,3)$-*commutative* if for all $a_1, a_2, a_3 \in H, f(a_1,a_2,a_3) = f(a_3,a_2,a_1)$;
2. (H, f) is $(2,3)$-*commutative* if for all $a_1, a_2, a_3 \in H, f(a_1,a_2,a_3) = f(a_1,a_3,a_2)$;
3. (H, f) is $(1,2)$-*commutative* if for all $a_1, a_2, a_3 \in H, f(a_1,a_2,a_3) = f(a_2,a_1,a_3)$;
4. (H, f) is *commutative* if for all $a_1, a_2, a_3 \in H$ and for all $\sigma \in \mathbb{S}_3, f(a_1,a_2,a_3) = f(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$.

Definition 1.4. A ternary semihypergroup (H, f) is said to have a *zero element* if there exists an element $0 \in H$ such that for all $a, b \in H, f(0,a,b) = f(a,0,b) = f(a,b,0) = \{0\}$. An element $e \in H$ is called *left* (*right*) *identity* element of H if for all $a \in H$, $f(a,e,e) = \{a\}(f(e,e,a) = \{a\})$. An element $e \in H$ is called an *identity* element of H if for all $a \in H$, $f(a,e,e) = f(e,e,a) = f(e,a,e) = \{a\}$.

Definition 1.5. Let (H, f) be a ternary semihypergroup. A nonempty subset I of a ternary semihypergroup H is called a *left* (*right, lateral*) *hyperideal* of H if

$$f(H,H,I) \subseteq I(f(I,H,H) \subseteq I, f(H,I,H) \subseteq I).$$

A nonempty subset I of H is called a *hyperideal* of H if it is a left, right and lateral hyperideal of H. A nonemtpy subset I of H is called *two-sided hyperideal* of H if it is a left and right hyperideal of H. A lateral hyperideal I of H is called a *proper lateral hyperideal* of H if $I \neq H$. A left hyperideal I of H is called *idempotent* if $f(I,I,I) = I$.

Example 1.6. Let $H = \{a, b, c, d, e, g\}$ and $f(x, y, z) = (x*y)*z$ for all $x, y, z \in H$, where $*$ is defined by the table:

$*$	a	b	c	d	e	g
a	a	$\{a, b\}$	c	$\{c, d\}$	e	$\{e, g\}$
b	b	b	d	d	g	g
c	c	$\{c, d\}$	c	$\{c, d\}$	c	$\{c, d\}$
d	d	d	d	d	d	d
e	e	$\{e, g\}$	c	$\{c, d\}$	e	$\{e, g\}$
g	g	g	d	d	g	g

Then (H, f) is a ternary semihypergroup. Clearly, $I_1 = \{c, d\}$, $I_2 = \{c, d, e, g\}$ and H are lateral hyperideals of H.

Let (H, f) be a ternary semihypergroup. It is clear that the intersection of all lateral hyperideals of a ternary subsemihypergroup T of H containing a nonempty subset A of T is the *lateral hyperideal of H generated by A*.

For every element $a \in H$, the left, right, lateral, two-sided and hyperideal generated by a are respectively given by

$$\begin{aligned}
\langle a\rangle_l &= \{a\} \cup f(H, H, a),\\
\langle a\rangle_r &= \{a\} \cup f(a, H, H),\\
\langle a\rangle_m &= \{a\} \cup f(H, a, H) \cup f(H, H, a, H, H),\\
\langle a\rangle_t &= \{a\} \cup f(H, H, a) \cup f(a, H, H) \cup f(H, H, a, H, H),\\
\langle a\rangle &= \{a\} \cup f(H, H, a) \cup f(a, H, H) \cup f(H, a, H) \cup f(H, H, a, H, H).
\end{aligned}$$

Definition 1.7. Let (H, f) be a ternary semihypergroup. A proper hyperideal P of H is called *prime hyperideal* of H if $f(A, B, C) \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ for any three hyperideals A, B, C of H.

A proper hyperideal P of H is said to be *strongly irreducible*, if for hyperideals T and K of H, $T \cap K \subseteq P$ implies that $T \subseteq P$ or $K \subseteq P$

A proper hyperideal A of a ternary semihypergroup H is called a *semiprime hyperideal* of H if $f(I, I, I) \subseteq A$ implies $I \subseteq A$ for any hyperideal I of H.

A proper hyperideal A of a ternary semihypergroup H is called *completely semiprime hyperideal* of H if $f(x, x, x) \subseteq A$ implies that $x \in A$ for any element $x \in A$.

Definition 1.8. A ternary subsemihypergroup B of a ternary semihypergroup H is called a *bi-hyperideal* of H if $f(B, H, B, H, B) \subseteq B$.

2. Regular ternary semihypergroups

Definition 2.1. A ternary semihypergroup H is said to be *regular* if for each $a \in H$, there exists an element $x \in H$ such that $a \in f(a, x, a)$.

A ternary semihypergroup H is called regular if all of its elements are regular.

It is clear that every ternary hypergroup is a regular ternary semihypergroup.

The ternary semihypergroup of the Example 1.6 is regular ternary semihypergroup.

We note that every left and right hyperideal of a regular ternary semihypergroup may not be a regular ternary semihypergroup; however, for a lateral hyperideal of a regular ternary semihypergroup, we have the following lemma:

Lemma 2.2. *Every lateral hyperideal of a regular ternary semihypergroup H is a regular ternary semihypergroup.*

Proof. Let L be a lateral hyperideal of a regular ternary semihypergroup H. Then for every $a \in L$, there exists $x \in H$ such that $a \in f(a, x, a)$. Now $a \in f(a, x, a) \subseteq f(a, x, f(a, x, a)) \subseteq f(a, f(x, a, x), a) \subseteq f(a, L, a)$. So there exists $b \in L$ such that $a \in f(a, b, a)$. This implies that L is a regular ternary semihypergroup. □

Obviously, every hyperideal of a regular ternary semihypergroup H is a regular ternary semihypergroup.

Theorem 2.3. *Let (H, f) be a ternary semihypergroup. Then the following statements are equivalent:*

(1) *H is regular.*

(2) *For any right hyperideal R, lateral hyperideal M and left hyperideal L of H, $f(R, M, L) = R \cap M \cap L$.*

(3) *For $a, b, c \in H$, $f(\langle a\rangle_r, \langle b\rangle_m, \langle c\rangle_l) = \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$.*

(4) *For $a \in H$, $f(\langle a\rangle_r, \langle a\rangle_m, \langle a\rangle_l) = \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l$.*

Proof. (1) $\Rightarrow$ (2). Let H be a regular ternary semihypergroup. Let R, M and L be a right hyperideal, a lateral hyperideal and a left hyperideal of H respectively. Then clearly, $f(R, M, L) \subseteq R \cap M \cap L$. Now for $a \in R \cap M \cap L$, we have $a \in f(a, x, a)$ for some $x \in H$. This implies that $a \in f(a, x, a) \subseteq f(f(a, x, a), x, f(a, x, a)) \subseteq f(R, M, L)$. Thus we have $R \cap M \cap L \subseteq f(R, M, L)$. So we find that $f(R, M, L) = R \cap M \cap L$.

Clearly, (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4).

It remains to show that (4) $\Rightarrow$ (1).

Let $a \in H$. Clearly, $a \in \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l = f(\langle a\rangle_r, \langle a\rangle_m, \langle a\rangle_l)$. Then we have, $a \in f(f(a, H, H) \cup \{a\}, f(H, a, H) \cup f(H, H, a, H, H) \cup \{a\}, f(H, H, a) \cup \{a\}) \subseteq f(a, H, a)$. So we find that $a \in f(a, H, a)$ and hence there exists an element $x \in H$ such that $a \in f(a, x, a)$. This implies that a is regular and hence H is regular. □

Corollary 2.4. *Let (H, f) be a ternary semihypergroup. Then the following statements are equivalent:*

(1) *H is regular.*

(2) *For any right hyperideal R and left hyperideal L of H, $f(R,H,L) = R \cap L$.*

(3) *For $a, b \in H$, $f(\langle a \rangle_r, H, \langle b \rangle_l) = \langle a \rangle_r \cap \langle b \rangle_l$.*

(4) *For $a \in H$, $f(\langle a \rangle_r, H, \langle a \rangle_l) = \langle a \rangle_r \cap \langle a \rangle_l$.*

Theorem 2.5. *A ternary semihypergroup H is regular if and only if every hyperideal of H is idempotent.*

Proof. Let H be a regular ternary semihypergroup and I be any hyperideal of H. Then $f(I,I,I) \subseteq f(H,H,I) \subseteq I$. Let $a \in I$. Then there exists $x \in H$ such that $a \in f(a,x,a) \subseteq f(a,x,f(a,x,a))$. Since I is a hyperideal and $a \in I$, $f(x,a,x) \subseteq I$. Thus $a \in f(a,x,a) \subseteq f(a,x,f(a,x,a)) \subseteq f(I,I,I)$. Consequently, $I \subseteq f(I,I,I)$ and hence $f(I,I,I) = I$, that is I is idempotent.

Conversely, suppose that every hyperideal of H is idempotent. Let A, B and C be three hyperideals of H. Then $f(A,B,C) \subseteq f(A,H,H) \subseteq A, f(A,B,C) \subseteq f(H,B,H) \subseteq B$ and $f(A,B,C) \subseteq f(H,H,C) \subseteq C$. This implies that $f(A,B,C) \subseteq A \cap B \cap C$. Also, $f(A \cap B \cap C, A \cap B \cap C, A \cap B \cap C) \subseteq f(A,B,C)$. Again, since $A \cap B \cap C$ is a hyperideal of H, $f(A \cap B \cap C, A \cap B \cap C, A \cap B \cap C) = A \cap B \cap C$. Thus $A \cap B \cap C \subseteq f(A,B,C)$ and hence $A \cap B \cap C = f(A,B,C)$. Therefore, by Theorem 2.3, H is a regular ternary semihypergroup. □

Theorem 2.6. *A commutative ternary semihypergroup H is regular if and only if every hyperideal of H is semiprime.*

Proof. Let H be a commutative regular ternary semihypergroup and I be any hyperideal of H such that $f(A,A,A) \subseteq I$ for any hyperideal A of H. From Theorem 2.3, it follows that $f(A,A,A) = A$. Consequently, $A \subseteq I$ and hence I is a semiprime hyperideal of H.

Conversely, suppose that every hyperideal of a commutative ternary semihypergroup H is semiprime. Let $a \in H$. Then $f(a,H,a)$ is a hyperideal of H. Now by hypothesis, $f(a,H,a)$ is a semiprime hyperideal of H. If $f(a,H,a) = H$, then we are done. Now suppose that $f(a,H,a) \neq H$. Then

$$\begin{aligned} f(\langle a \rangle, \langle a \rangle, \langle a \rangle) &= f(f(H,H,a) \cup f(a,H,H) \cup f(H,a,H) \cup \\ &\cup f(H,H,a,H,H) \cup \{a\}, f(H,H,a) \cup f(a,H,H) \cup \\ &\cup f(H,a,H) \cup f(H,H,a,H,H) \cup \{a\}, f(H,H,a) \cup \\ &\cup f(a,H,H) \cup f(H,a,H) \cup f(H,H,a,H,H) \cup \{a\}) \\ &\subseteq f(a,H,a) \end{aligned}$$

that is, $f(\langle a \rangle, \langle a \rangle, \langle a \rangle) \subseteq f(a,H,a)$. This implies that $\langle a \rangle \subseteq f(a,H,a)$, since $f(a,H,a)$ is a semiprime hyperideal of H. Consequently, $a \in f(a,x,a)$ for some $x \in H$ and hence H is a regular ternary semihypergroup. □

Let N be the *nuclear hyperideal* of a ternary semihypergroup (H,f), that is the intersection of all hyperideals in H, N_r the intersection of all right hyperideals in H, N_m the intersection of all lateral hyperideals of H, and N_l the intersection of all left hyperideals of H.

Theorem 2.7. *Let (H, f) be a ternary semihypergroup and let $N = N_r = N_m = N_l \neq \emptyset$. Then H is regular if and only if N is regular ternary semihypergroup.*

Proof. If H is regular, then clearly N is also regular as a hyperideal.

Conversely, suppose that N is a regular hyperideal of H, so that for any right hyperideal R, lateral hyperideal M, and left hyperideal L of H,

$$N \cup f(R, M, L) = R \cap M \cap L.$$

Since $f(N, N, N)$ is both a right and left hyperideal, then

$$f(R, M, L) \subseteq f(N, N, N) \subseteq N.$$

Whence $f(R, M, L) = R \cap M \cap L$. □

Corollary 2.8. *Let (H, f) be a ternary semihypergroup and let $N = N_r = N_m = N_l \neq \emptyset$. Then H is regular if and only if every hyperideal of H is regular.*

Proof. If H is regular, then N is a regular hyperideal. Hence any hyperideal I which necessary contains N is also a regular hyperideal.

Conversely, if every hyperideal of H is regular, then N is regular. Thus by the previous Theorem 2.7, H is regular ternary semihypergroup. □

Theorem 2.9. *Let (H, f) be a ternary semihypergroup and I a hyperideal of H. The following statements are equivalent:*

(1) *I is a regular hyperideal of H;*

(2) *For every $a \in H$, $I \cup f(\langle a \rangle_r, \langle a \rangle_m, \langle a \rangle_l) = I \cup (\langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l)$;*

(3) *For every $a \in H \backslash I$, either $a \in f(a, a_1, a, a_2, a)$ or $a \in f(a, b_1, b_2, a, b_3, b_4, a)$, for some $a_1, a_2, b_1, b_2, b_3, b_4 \in H$.*

Proof. (1) ⇒ (2). Suppose that I is a regular hyperideal. Then for each $a \in H$,

$$I \cup (\langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l) \subseteq \langle I \cup \langle a \rangle_r \rangle_r, \langle I \cup \langle a \rangle_m \rangle_m, \langle I \cup \langle a \rangle_l \rangle_l.$$

Moreover, since each of the three sets on the right side contains I, then we have

$$\begin{aligned} & I \cup (\langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l) \subseteq \langle I \cup \langle a \rangle_r \rangle_r \cap \langle I \cup \langle a \rangle_m \rangle_m \cap \langle I \cup \langle a \rangle_l \rangle_l \\ & = I \cup f(I \cup \langle a \rangle_r, I \cup \langle a \rangle_m, I \cup \langle a \rangle_l) \\ & = I \cup f(I, I \cup \langle a \rangle_m, I \cup \langle a \rangle_l) \cup f(\langle a \rangle_r, I, I \cup \langle a \rangle_l) \cup f(\langle a \rangle_r, \langle a \rangle_m, I) \cup f(\langle a \rangle_r, \langle a \rangle_m, \langle a \rangle_l) \\ & = I \cup f(\langle a \rangle_r, \langle a \rangle_m, \langle a \rangle_l) \subseteq I \cup (\langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l). \end{aligned}$$

(2) ⇒ (3). We note that

$$\begin{aligned}
\langle I \cup \langle a \rangle_r \rangle_r &= \langle I \cup \langle a \rangle_r \rangle_r \cap H \cap H = I \cup f(\langle I \cup \langle a \rangle_r \rangle_r, H, H) \\
&= I \cup f(I, H, H) \cup f(\langle a \rangle_r, H, H) \cup f(I, H, H, H, H) \cup \\
&\quad \cup f(\langle a \rangle_r, H, H, H, H) \\
&= I \cup f(I, H, H) \cup f(a, H, H, H) \cup f(a, H, H, H, H) \cup \\
&\quad \cup f(I, H, H, H, H) \cup \\
&\quad \cup f(a, H, H, H, H) \cup f(a, H, H, H, H, H, H) \\
&= I \cup f(I, H, H) \cup f(a, H, H) \cup f(a, H, H, H, H) \\
&= \langle I \cup f(a, H, H) \rangle_r = I \cup f(a, H, H).
\end{aligned}$$

In the same manner, we obtain

$$\begin{aligned}
\langle I \cup \langle a \rangle_m \rangle_m &= \langle I \cup f(H, a, H) \rangle_m = I \cup f(H, a, H) \cup f(H, H, a, H, H), \\
\langle I \cup \langle a \rangle_l \rangle_l &= \langle I \cup f(H, H, a) \rangle_l = I \cup f(H, H, a).
\end{aligned}$$

Then

$$\begin{aligned}
&\langle I \cup f(a, H, H) \rangle_r \cap \langle I \cup f(H, a, H) \rangle_m \cap \langle I \cup f(H, H, a) \rangle_l \\
&= I \cup f(\langle I \cup f(a, H, H) \rangle_m, \langle I \cup f(H, a, H) \rangle_m, \langle I \cup f(H, H, a) \rangle_l) \\
&= I \cup f(a, H, H, H, a, H, H, H, a) \cup f(a, H, H, H, H, a, H, H, H, H, a) \\
&= I \cup f(a, H, a, H, a) \cup f(a, H, H, a, H, H, a).
\end{aligned}$$

The result now follows.

(3) ⇒ (1). Let R be an arbitrary right hyperideal, M an arbitrary lateral hyperideal, L an arbitrary left hyperideal of H all containing I. Let us assume that I satisfies the condition (3). It is clear that,

$$I \cup f(R, M, L) \subseteq R \cap M \cap L.$$

Let $a \in R \cap M \cap L$. By (3), then $a \in I$ or $a \in f(f(a, a_1, a, a_2, a)$ or $a \in f(a, b_1, b_2, a, b_3, b_4, a)$ for some $a_1, a_2, b_1, b_2, b_3, b_4 \in H$. We note also that in the second and third cases we have:

$$\begin{aligned}
&a \in f(a, a_1, a, a_1, a, a_2, a, a_2, a) = f(f(a, a_1, a_2), f(a_1, a, a_2), f(a, a_2, a)), \\
&a \in f(a, b_1, b_2, a, b_1, b_2, a, b_3, b_4, a, b_3, b_4, a) = \\
&\quad = f(f(a, b_1, b_2), f(a, b_1, b_2), a, f(b_3, b_4, a), f(b_3, b_4, a)).
\end{aligned}$$

Hence in the last two cases we have

$$a \in f(f(a, x_2, x_3), f(y_1, a, y_3), f(z_1, z_2, a)),$$

for some $x_2, x_3, y_1, y_2, z_1, z_2 \in H$. Whence, in any case we have:

$$a \in I \cup f(R, M, L)$$

and therefore $I \cup f(R, M, L) = R \cap M \cap L$. □

Theorem 2.10. *Let (H, f) be a ternary semihypergroup and I a regular hyperideal of a H. Then, for any right hyperideal R, lateral hyperideal M, and left hyperideal L of H, if $f(R, M, L) \subseteq I$, then $R \cap M \cap L \subseteq I$.*

Proof. Suppose $f(R, M, L) \subseteq I$ and I is a regular hyperideal. Then

$$R \cap M \cap L \subseteq \langle I \cup R\rangle_r \cap \langle I \cup M\rangle_m \cap \langle I \cup L\rangle_l$$
$$I \cup f(\langle I \cup R\rangle_r, \langle I \cup M\rangle_m, \langle I \cup L\rangle_l) = I \cup f(I, \langle I \cup M\rangle_m, \langle I \cup L\rangle_l)$$
$$= f(R, I, \langle I \cup L\rangle_l) \cup f(R, M, I) \cup f(R, M, L) \subseteq I.$$ □

Corollary 2.11. *A regular and strongly irreducible hyperideal is always prime.*

Corollary 2.12. *Every regular hyperideal is prime.*

Definition 2.13. Let (H, f) be a ternary semihypergroup and Q a nonempty subset of H. Then Q is called a *quasi-hyperideal* of H if and only if

$$f(Q, H, H) \cap f(H, Q, H) \cap f(H, H, Q) \subseteq Q \text{ and}$$
$$f(Q, H, H) \cap f(H, H, Q, H, H) \cap f(H, H, Q) \subseteq Q.$$

Theorem 2.14. *Let (H, f) be a regular ternary semihypergroup and Q be a nonempty subset of H. Then Q is a quasi-hyperideal if and only if*

$$f(Q, H, Q, H, Q) \cap f(Q, H, H, Q, H, H, Q) \subseteq Q.$$

Proof. Let H be a regular ternary semihypergroup and Q be a quasi-hyperideal of H. Then

$$f(Q, H, Q, H, Q) \cap f(Q, H, H, Q, H, H, Q) \subseteq f(H, H, Q),\ f(Q, H, H), \text{ and}$$
$$f(H, Q, H) \cup f(H, H, Q, H, H)$$

and hence

$$f(Q, H, Q, H, Q) \cap f(Q, H, H, Q, H, H, Q) \subseteq$$
$$\subseteq f(H, H, Q) \cap (f(H, Q, H) \cup f(H, H, Q, H, H)) \cap f(Q, H, H) \subseteq Q.$$

Conversely, suppose that H is regular and

$$f(Q, H, Q, H, Q) \cap f(Q, H, H, Q, H, H, Q) \subseteq Q.$$

Then

$$f(Q, H, H) \cap (f(H, Q, H) \cup f(H, H, Q, H, H)) \cap f(H, H, Q)$$
$$= f(f(Q, H, H), f(H, Q, H) \cup f(H, H, Q, H, H), f(H, H, Q))$$
$$= f(f(Q, H, H), f(H, Q, H), f(H, H, Q)) \cup f(f(Q, H, H), f(H, H, Q, H, H),$$
$$f(H, H, Q)) \subseteq f(Q, H, Q, H, Q) \cup f(Q, H, H, Q, H, H, Q) \subseteq Q.$$ □

Theorem 2.15. *Let (H, f) be a regular ternary semihypergroup and Q_1, Q_2, Q_3 be three quasi-hyperideals of H. Then $f(Q_1, Q_2, Q_3)$ is a quasi-hyperideal.*

Proof.

$$f(f(Q_1,Q_2,Q_3),H,f(Q_1,Q_2,Q_3),H,f(Q_1,Q_2,Q_3)) \cup f(f(Q_1,Q_2,Q_3),H,H,$$
$$f(Q_1,Q_2,Q_3),H,H,f(Q_1,Q_2,Q_3))$$
$$= f(f(Q_1,f(Q_2,Q_3,H),Q_1,f(Q_2,Q_3,H),Q_1),Q_2,Q_3) \cup$$
$$\cup f(f(Q_1,f(Q_2,Q_3,H),H,Q_1,f(Q_2,Q_3,H),H,Q_1),Q_2,Q_3) \subseteq$$
$$\subseteq f(Q_1,Q_2,Q_3).$$ □

Corollary 2.16. *The family of all quasi-hyperideals of a regular ternary semihypergroup is a ternary semihypergroup.*

Theorem 2.17. *Let (H, f) be a ternary semihypergroup. If for every quasi-hyperideal Q of H, $f(Q, Q, Q) = Q$, then H is a regular ternary semihypergroup.*

Proof. Let R be a right hyperideal of H, L be a left hyperideal of H and M be a lateral hyperideal of H. By Theorem 2.2 [9], $R \cap M \cap L$ is a quasi-hyperideal of H. Then by hypothesis, we have

$$R \cap M \cap L = f(R \cap M \cap L, R \cap M \cap L, R \cap M \cap L) \subseteq f(R, M, L).$$

On the other hand, $f(R, M, L) \subseteq R \cap M \cap L$. Therefore we have $f(R, M, L) = R \cap M \cap L$. By Theorem 2.3(2), H is a regular ternary semihypergroup. □

Theorem 2.18. *Let (H, f) be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is regular;*

(2) *For every bi-hyperideal B of H, $f(B, H, B, H, B) = B$;*

(3) *For every quasi-hyperideal Q of H, $f(Q, H, Q, H, Q) = Q$.*

Proof. (1) ⇒ (2). Let us assume that H is regular and B be a bi-hyperideal of H. Let $b \in B$. From regularity of H, there exists $x \in H$, such that $b \in f(b, x, b)$. Thus, $B \subseteq f(B, H, B)$. We have $b \in f(b, x, b) \subseteq f(b, x, f(b, x, b)) \subseteq f(B, H, f(B, H, B)) = f(B, H, B, H, B)$. Therefore, $B \subseteq f(B, H, B, H, B)$. On the other hand, since B is a bi-hyperideal of H, we have $f(B, H, B, H, B) \subseteq B$. Thus, $f(B, H, B, H, B) = B$.

(2) ⇒ (3). It is clear by Lemma 4.2 [9] since every quasi-hyperideal is a bi-hyperideal.

(3) ⇒ (1). Let R be a right hyperideal of H, L be a left hyperideal of H and M be a lateral hyperideal of H. By Theorem 2.2 [9], $Q = R \cap M \cap L$ is a quasi-hyperideal of H. By (3) we have $f(Q, H, Q, H, Q) = Q$. Thus $R \cap M \cap L = Q = f(Q, H, Q, H, Q) =\subseteq f(R, H, M, H, L) \subseteq f(R, M, L)$. But $f(R, M, L) \subseteq R \cap M \cap L$. Therefore, since $f(R, M, L) = R \cap M \cap L$, by Theorem 2.3(2), H is a regular ternary semihypergroup. □

Corollary 2.19. *Let (H,f) be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is regular;*

(2) *For every bi-hyperideal B of H, $f(B,H,B) = B$;*

(3) *For every quasi-hyperideal Q of H, $f(Q,H,Q) = Q$.*

Theorem 2.20. *Let (H,f) be a ternary semihypergroup. If for every bi-hyperideal B of H, $f(B,B,B) = B$, then H is a regular ternary semihypergroup.*

Proof. The proof is a corollary of Theorem 2.17. □

Theorem 2.21. *Let (H,f) be a regular ternary semihypergroup. Then a ternary subsemihypergroup B of H is bi-hyperideal if and only if B is a quasi-hyperideal of H.*

Proof. Let H be a regular ternary semihypergroup and B a bi-hyperideal of H. By Theorem 2.3, we have $f(R \cap M \cap L) = f(R,M,L)$ for every right hyperideal R, lateral hyperideal M and left hyperideal L. Thus

$$\begin{aligned}
& f(B,H,H) \cap (f(H,B,H) \cup f(H,H,B,H,H)) \cap f(H,H,B) \\
= & f(f(B,H,H), f((f(H,B,H) \cup f(H,H,B,H,H)), f(H,H,B,H,H)) \\
= & f(B, f(H,H,H), B, f(H,H,H), B) \cup f(B, f(H,H,H), H, B, f(H,H,H), H, B) \\
\subseteq & f(B,H,B,H,B) \cup f(B,H,H,B,H,H,B) \\
\subseteq & B \cup f(B,H,B) = B \cup B = B.
\end{aligned}$$

Therefore, B is a quasi-hyperideal of H.

Conversely, let B be a quasi-hyperideal of H. Then, by Lemma 4.2 [9], B is a bi-hyperideal of H. □

Corollary 2.22. *Let (H,f) be a regular ternary semihypergroup. A ternary subsemihypergroup B of H is bi-hyperideal of H if and only if B is the intersection of a right hyperideal, a lateral hyperideal and a left hyperideal of H.*

Theorem 2.23. *Let (H,f) be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is regular;*

(2) *$M \cap B = f(B,M,B)$ for every lateral hyperideal M and for every bi-hyperideal B of H;*

(3) *$M \cap Q = f(Q,M,Q)$ for every lateral hyperideal M and for every bi-hyperideal Q of H.*

Proof. (1) $\Rightarrow$ (2). Let M be a lateral hyperideal of H and B a bi-hyperideal of H. We have $f(B,M,B) \subseteq f(H,M,H) \subseteq M$. By Corollary 2.19, we have $f(B,M,B) \subseteq f(B,H,B) = B$. Therefore, $f(B,M,B) \subseteq M \cap B$. Let $a \in M \cap B$. Since H is regular, there exists $h \in H$ such that $a \in f(a,h,a)$. We have $a \in f(a,h,a) \subseteq f(f(a,h,a),h,a) = f(a,f(h,a,h),a) \subseteq f(B,M,B)$. It follows that $M \cap B \subseteq f(B,M,B)$. Therefore $f(B,M,B) = M \cap B$.

(2) $\Rightarrow$ (3). It is clear since every quasi-hyperideal is a bi-hyperideal.

(3) $\Rightarrow$ (1). Let Q be a quasi-hyperideal of H. By (3) it follows that $Q = H \cap Q = f(Q,H,Q)$. By Corollary 2.19, it follows that H is a regular ternary semihypergroup. □

In the sequel, the following results hold. The proof of them is straightforward, so we omit it.

Theorem 2.24. *Let (H,f) be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is regular;*

(2) *$B \cap L \subseteq f(B,H,L)$ for every bi-hyperideal B of H and for every left hyperideal L;*

(3) *$Q \cap L \subseteq f(Q,H,L)$ for every quasi-hyperideal Q of H and for every left hyperideal L;*

(4) *$B \cap R \subseteq f(R,H,B)$ for every bi-hyperideal B of H and for every right hyperideal R;*

(5) *$Q \cap R \subseteq f(R,H,Q)$ for every quasi-hyperideal Q of H and for every right hyperideal R.*

Theorem 2.25. *Let (H,f) be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is regular;*

(2) *$B_1 \cap B_2 \subseteq f(B_1,H,B_2) \cap f(B_2,H,B_1)$ for every bi-hyperideals B_1, B_2 of H;*

(3) *$B \cap Q \subseteq f(B,H,Q) \cap f(Q,H,B)$ for every bi-hyperideal B and for every quasi-hyperideal Q of H;*

(4) *$B \cap L \subseteq f(B,H,L) \cap f(L,H,B)$ for every bi-hyperideal B of H and for every left hyperideal L;*

(5) *$Q \cap L \subseteq f(Q,H,L) \cap f(L,H,Q)$ for every quasi-hyperideal Q of H and for every left hyperideal L;*

(6) *$R \cap L \subseteq f(R,H,L) \cap f(L,H,R)$ for every right hyperideal R of H and for every left hyperideal L;*

(7) $B \cap R \subseteq f(R, H, B) \cap f(B, H, R)$ *for every bi-hyperideal* B *of* H *and for every right hyperideal* R*;*

(8) $Q \cap R \subseteq f(R, H, Q) \cap f(Q, H, R)$ *for every quasi-hyperideal* Q *of* H *and for every right hyperideal* R*.*

3. Completely regular and intra-regular ternary semihypergroups

Definition 3.1. Let (H, f) be a ternary semihypergroup. An element $a \in H$ is said to be *left* (resp. *right*) *regular* if there exists an element $x \in H$ such that $a \in f(x, a, a)$ (resp. $a \in f(a, a, x)$). An element $a \in H$ is said to be *completely regular* if it is left regular, right regular and regular.

If all the elements of a ternary semihypergroup H are left (resp. right, completely) regular, then H is called left (resp. right, completely) regular.

The ternary semihypergroup of the Example 1.6 is a completely regular ternary semihypergroup.

Theorem 3.2. *A ternary semihypergroup* (H, f) *is left (resp. right) regular if and only if every left (resp. right) hyperideal of* H *is completely semiprime.*

Proof. Let H be a left regular ternary semihypergroup and L be any left hyperideal of H. Suppose that $f(a, a, a) \subseteq L$ for $a \in H$. Since H is left regular, there exists an element $x \in H$ such that $a \in f(x, a, a) \subseteq f(x, f(x, a, a), a) \subseteq f(x, x, f(a, a, a)) \subseteq f(H, H, L) \subseteq L$. Thus L is completely semiprime.

Conversely, suppose that every left hyperideal of H is completely semiprime. Now for any $a \in H$, $f(H, a, a)$ is a left hyperideal of H. Then by hypothesis, $f(H, a, a)$ is a completely semiprime hyperideal of H. Now $f(a, a, a) \subseteq f(H, a, a)$. Since $f(H, a, a)$ is completely semiprime, it follows that $a \in f(H, a, a)$. So there exists an element $x \in H$ such that $a \in f(x, a, a)$. Consequently, a is left regular. Since a is arbitrary, it follows that H is left regular.

Similarly, it can be proved the theorem for the right regularity. □

Proposition 3.3. *A ternary semihypergroup* (H, f) *is completely regular if and only if* $a \in f(a, a, H, a, a)$ *for all* $a \in H$*.*

Proof. Suppose that H is a completely regular ternary semihypergroup. Let $a \in H$. Then, by the definition, we have that $a \in f(a, a, H)$ and $a \in f(H, a, a)$, that is $a \in f(a, a, H) \cap f(H, a, a)$. Since H is completely regular, there exists an element $x \in H$ such that $a \in f(a, x, a)$. So we have

$$a \in f(a, x, a) \subseteq f(f(a, a, H), x, f(H, a, a)) \subseteq$$
$$\subseteq f(a, a, f(H, x, H), a, a) \subseteq f(a, a, H, a, a).$$

Conversely, suppose that for any $a \in H$, $a \in f(a,a,H,a,a)$. Then

1. $a \in f(a,a,H,a,a) \subseteq f(a, f(a,H,a), a) \subseteq f(a,H,a)$, that is H is regular.
2. $a \in f(a,a,H,a,a) \subseteq f(f(a,a,H),a,a) \subseteq f(H,a,a)$, that is H is left regular.
3. $a \in f(a,a,H,a,a) \subseteq f(a,a,f(H,a,a)) \subseteq f(a,a,H)$, that is H is right regular. Therefore H is completely regular. □

Theorem 3.4. *A ternary semihypergroup (H,f) is completely regular if and only if every bi-hyperideal of H is completely semiprime.*

Proof. Suppose that H is completely regular ternary semihypergroup. Let B be any bi-hyperideal of H. Let $f(b,b,b) \subseteq B$ for $b \in B$. Since H is completely regular, from Proposition 3.3, it follows that $b \in f(b,b,H,b,b)$. This implies that there exists $x \in H$ such that

$$
\begin{aligned}
b \in f(b,b,x,b,b) &\subseteq f(b, f(b,b,x,b,b), x, f(b,b,x,b,b), b) = \\
&= f(b,b,b,f(x,b,b,x),b,f(b,b,x,b,b),x,b,b,b) \\
&= f(b,b,b,f(x,b,b,x),b,b,b,f(x,b,b,x),b,b,b) \subseteq f(B,H,B,H,B) \subseteq B.
\end{aligned}
$$

This shows that B is completely semiprime.

Conversely, suppose that every bi-hyperideal of H is completely semiprime. Since every left and right hyperideal of a ternary semihypergroup H is a bi-hyperideal of H, it follows that every left and right hyperideal of H is completely semiprime. Consequently, we have from Theorem 3.2 that H is both left and right regular.

Let $a \in H$. We consider $f(a,H,a)$. Let $x,y,z \in f(a,H,a)$ and $h_1, h_2 \in H$. Then for some $h_0, h_0^{'}, h_0^{''} \in H$ we have:

$$
\begin{aligned}
f(x,h_1,y,h_2,z) &\subseteq f(f(a,h_0,a), h_1, f(a,h_0^{'},a), h_2, f(a,h_0^{''},a)) \\
&\subseteq f(a, f(h_0,a,h_1,a,h_0^{'},a,h_2,a,h_0^{''}), a) \\
&\subseteq f(a,H,a).
\end{aligned}
$$

This implies that $f(f(a,H,a),H,f(a,H,a),H,f(a,H,a)) \subseteq f(a,H,a)$. That is, $f(a,H,a)$ is a bi-hyperideal of H. Since $f(a,a,a) \subseteq f(a,H,a)$ and $f(a,H,a)$ is completely semiprime, it follows that $a \in f(a,H,a)$, for all $a \in H$. That is H is regular. This completes the proof. □

Theorem 3.5. *If (H,f) is a completely regular ternary semihypergroup, then every bi-hyperideal of H is idempotent.*

Proof. Let H be a completely regular ternary semihypergroup and B be a bi-hyperideal of H. Since H is a completely regular ternary semihypergroup, it is also a regular ternary semihypergroup. Let $b \in B$. Then there exists $x \in H$ such that $b \in f(b,x,b)$. This implies that $b \in f(B,H,B)$ and hence $B \subseteq f(B,H,B)$. Also $f(B,H,B) \subseteq f(B,H,B,H,B) \subseteq B$. Thus we find that $B = f(B,H,B)$. Again, we have from Proposition 3.3 that $b \in f(b,b,H,b,b) \subseteq f(B,B,H,B,B)$.

This implies that $B \subseteq f(B,B,H,B,B) = f(B,f(B,H,B),B) = f(B,B,B) \subseteq B$. Consequently, $f(B,B,B) = B$. □

Definition 3.6. A ternary semihypergroup (H,f) is called *intra-regular* if for each element $a \in H$, there exist elements $x,y \in H$ such that $a \in f(x,a,a,a,y)$.

Theorem 3.7. [9, Theorem 6.4] *Let (H,f) be a ternary semihypergroup. Then the following statements are equivalent:*

(1) *H is intra-regular;*

(2) *For every left hyperideal L, lateral hyperideal M and right hyperideal R of H, $L \cap M \cap R \subseteq f(L,M,R)$.*

Proposition 3.8. *Let (H,f) be an intra-regular ternary semihypergroup. Then a non-empty subset I of H is a hyperideal of H if and only if I is a lateral hyperideal of H.*

Proof. Clearly, if I is a hyperideal of H, then I is a lateral hyperideal of H.

Conversely, let I be a lateral hyperideal of an intra-regular ternary semihypergroup. Let $a \in I$ and $s,t \in H$. Then $a \in H$ and hence there exist elements $x,y \in H$ such that $a \in f(x,a,a,a,y)$. Now $f(s,t,a) \subseteq f(s,t,f(x,a,a,a,y)) \subseteq f(H,I,H) \subseteq I$ and $f(a,s,t) \subseteq f(f(x,a,a,a,y),s,t) \subseteq f(H,I,H) \subseteq I$. This implies that I is both a left hyperideal and a right hyperideal of H. Consequently, I is an hyperideal of H. □

Lemma 3.9. *Every lateral hyperideal of an intra-regular ternary semihypergroup (H,f) is an intra-regular ternary semihypergroup.*

Proof. Let L be a lateral hyperideal of an intra-regular ternary semihypergroup H. Then for each $a \in L$, there exist $x,y \in H$ such that $a \in f(x,a,a,a,y)$. Now $a \in f(x,a,a,a,y) \subseteq f(x,f(x,a,a,a,y),f(x,a,a,a,y),f(x,a,a,a,y),y)$
$\subseteq f(f(x,x,a,a,a,y,y),f(a,a,a),f(y,x,a,a,a,y,y)) \subseteq f(L,f(a,a,a),L)$. This implies that there exist $u,v \in L$ such that $a \in f(u,f(a,a,a),v)$. Consequently, L is an intra-regular ternary semihypergroup. □

From the Proposition 3.8 we have the following corollary:

Corollary 3.10. *Every hyperideal of an intra-regular ternary semihypergroup H is an intra-regular ternary semihypergroup.*

Theorem 3.11. *Let I be a hyperideal of an intra-regular ternary semihypergroup H and J be a hyperideal of I. Then J is a hyperideal of the entire ternary semihypergroup H.*

Proof. It is sufficient to show that J is a lateral hyperideal of H. Let $a \in J \subseteq I$ and $s,t \in H$. Then $f(s,a,t) \subseteq I$. We have to show that $f(s,a,t) \subseteq J$. From Corollary 3.10, it follows that I is an intra-regular ternary semihypergroup. Thus there exist $u, v \in I$ such that $f(s,a,t) \subseteq f(u, f(s,a,t), f(s,a,t), f(s,a,t), v) \subseteq f(f(u,s,a,t,s), a, f(t,s,a,t,v)) \subseteq f(I,J,I) \subseteq J$. Consequently, J is a lateral hyperideal of H. □

Theorem 3.12. *A ternary semihypergroup (H, f) is intra-regular if and only if every hyperideal of H is completely semiprime.*

Proof. Let H be an intra-regular ternary semihypergroup and I be a hyperideal of H. Let $f(a,a,a) \subseteq I$ for $a \in H$. Since H is intra-regular, there exist $x, y \in H$ such that $a \in f(x, f(a,a,a), y) \subseteq I$. Consequently, I is completely semiprime.

Conversely, suppose that every hyperideal of H is completely semiprime. Let $a \in H$. Then $f(a,a,a) \subseteq \langle f(a,a,a)\rangle$. This implies that $a \in \langle f(a,a,a)\rangle$, since $\langle f(a,a,a)\rangle$ is completely semiprime.

Now $\langle f(a,a,a)\rangle = f(H,H,f(a,a,a)) \cup f(f(a,a,a),H,H) \cup f(H,f(a,a,a),H) \cup f(H,H,f(a,a,a),H,H) \cup f(a,a,a)$. So we have the following cases:

If $a \in f(H,H,f(a,a,a))$, then $f(a,a,a) \subseteq f(H,H,f(a,a,a),a,a)$. Hence $a \in f(H,H,H,H,f(a,a,a),a,a) \subseteq f(H,H,H,a,a,a,H) \subseteq f(H,f(a,a,a),H)$.

If $a \in f(f(a,a,a),H,H)$, then $f(a,a,a) \subseteq f(a,a,f(a,a,a),H,H)$. Hence $a \in f(a,a,f(a,a,a),H,H,H,H) \subseteq f(H,a,a,a,H,H,H) \subseteq f(H,f(a,a,a),H)$.

If $a \in f(H,f(a,a,a),H)$, then we are done.

If $a \in f(H,H,f(a,a,a),H,H)$, then $f(a,a,a) \subseteq f(a,H,H,f(a,a,a),H,H,a)$. Hence

$$\begin{aligned} a &\in f(H,H,a,H,H,f(a,a,a),H,H,a,H,H) \\ &\subseteq f(H,H,H,f(a,a,a),H,H,H) \subseteq f(H,f(a,a,a),H). \end{aligned}$$

If $a \in f(a,a,a)$, then

$$a \in f(a,a,a) \subseteq f(f(a,a,a),f(a,a,a),f(a,a,a)) \subseteq f(H,f(a,a,a),H).$$

So we find that in any case, H is intra-regular. □

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Received April 5, 2017

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