Behrends-Humble simple maps are regular

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Abstract. We consider simple binary operations in the sense of Behrends and Humble. We prove that a groupoid (magma) with such a map is regular. As a consequence, a division groupoid with simple binary operation is a quasigroup.

Let G be a groupoid (magma) with binary operation φ. The map φ induces maps $\varphi_n : G^{n+1} \to G^n$ by

$$\varphi_n(s_0, s_1, \ldots, s_n) = \big(\varphi(s_0, s_1), \varphi(s_1, s_2), \ldots, \varphi(s_{n-1}, s_n)\big).$$

Let Φ_n be the composition $\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n$. For an integer $k \geqslant 2$, we say that φ is *k-simple* if $\Phi_k(s_0, \ldots, s_k) = \varphi(s_0, s_k)$ for all $s_0, \ldots, s_k \in G$ and that φ is *simple* if it is k-simple for some k.

Simple maps were first studied by Ehrhard Behrends and Steve Humble [1]. Michael Jones, Brittany Shelton, and the author recently proved that any groupoid with simple binary operation is medial [4]. In this note, we establish that any groupoid with simple binary operation is regular and show that any division groupoid with simple binary operation is a quasigroup. We also offer remarks on cancellation groupoids and 2-simple maps.

Theorem 1. *If G is a groupoid with simple binary operation φ, then G is regular.*

Proof. Suppose that φ is n-simple for some integer n, $a, b \in G$ and $\varphi(a, x) = \varphi(b, x)$ for some $x \in G$. Given any other $y \in G$, we find that

$$\begin{aligned}\varphi(a, y) &= \Phi_n(a, x, \ldots, x, y) = \Phi_{n-1}\big(\varphi(a, x), \varphi(x, x), \ldots, \varphi(x, x), \varphi(x, y)\big)\\ &= \Phi_{n-1}\big(\varphi(b, x), \varphi(x, x), \ldots, \varphi(x, x), \varphi(x, y)\big) = \Phi_n(b, x, \ldots, x, y) = \varphi(b, y).\end{aligned}$$

Similarly, $\varphi(x, a) = \varphi(x, b)$ implies $\varphi(y, a) = \varphi(y, b)$ for all $y \in G$. □

Theorem 2. *A division groupoid with simple binary operation is a quasigroup.*

Proof. If $(G, \cdot)$ is a division groupoid with simple binary operation, it is medial [4] and regular. Thus there exists a binary operation $+$ on G such that $(G, +)$ is an Abelian group and there exist commuting surjective endomorphisms f and g of $(G, +)$ and an element $c \in G$ such that $xy = f(x) + g(y) + c$ for all $x, y \in G$ [2].

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Let 0 be the identity element of $(G,+)$. For $q_0,\dots,q_n \in Q$, by simplicity,

$$\begin{aligned} f(q_0) + g(q_n) + c = \\ & f^n(q_0) + \binom{n}{1} f^{n-1} g(q_1) + \cdots + \binom{n}{n-1} f g^{n-1}(q_{n-1}) + g^n(q_n) \\ & \qquad + \big((f+g) + (f+g)^2 + \cdots + (f+g)^{n-1}\big)(c) + c. \end{aligned}$$

If we let the q_i all be 0, we find $\big((f+g)+(f+g)^2+\cdots+(f+g)^{n-1}\big)(c) = 0$. Next, letting all of the q_i except q_0 or q_n be 0, we find $g = g^n$ and $f = f^n$. Since f and g are also surjective, they must be automorphsims. By the Bruck-Murdoch-Toyoda Theorem [3], (G,φ) is a quasigroup. □

Theorem 3. *If (G,φ) is a groupoid and φ is 2-simple, then (G,φ) is a semigroup.*

Proof. If φ is 2-simple, (G,φ) is medial [4]. Then, for any $a,b,c \in G$,

$$\begin{aligned} \varphi\big(a,\varphi(b,c)\big) &= \Phi_2\big(a,\varphi(b,b),\varphi(b,c)\big) = \varphi\big(\varphi(a,\varphi(b,b)),\varphi(b,c)\big) \\ &= \varphi\big(\varphi(a,b),\varphi(\varphi(b,b),c)\big) = \Phi_2\big(\varphi(a,b),\varphi(b,b),c\big) = \varphi\big(\varphi(a,b),c\big), \end{aligned}$$

so that φ is associative. □

If G is a cancellation groupoid with simple binary operation, then G is medial [4]. As a result, there exists a medial quasigroup $(Q,\cdot)$ such that G is a dense subgroupoid of Q; moreover, G and Q satisfy the same identities [5, 2]. In particular, Q has simple binary operation. Let n be a positive integer such that the operation of $(Q,\cdot)$ is n-simple. Define Φ_n as above using $\varphi(x,y) = xy$. If $x,y \in G$, $q \in Q$, and $xq \in G$, then $yq \in G$, since $yq = \Phi_n(y,x,\dots,x,q) = \Phi_{n-1}(yx,xx,\dots,xx,xq)$ and $yx,xx,xq \in G$. Can $G \neq Q$?

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