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A note on semisymmetry

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Abstract. J.D.H. Smith showed how to replace homotopies between quasigroups by homomorphism between semisymmetric quasigroups. This is a semisymmetrization and it replaces a quasigroup by a semisymmetric structure defined on its Cartesian cube. The reason for a semisymmetrization is that homomorphisms behave more regularly than homotopies.

A thorough survey of properties of Smith's semisymmetrization is given in this paper. Also, new semisymmetrizations, which replace a quasigroup by semisymmetric structures defined on its Cartesian square are suggested.

1. Introduction

For a plausible category of quasigroups, it seems that homotopies between quasigroups, taken as morphisms, are better choice than homomorphisms (see [3] and [9]). However, homomorphisms are sometimes easier to work with. For example, isotopies (bijective homotopies) do not preserve units – every quasigroup is isotopic to a loop (quasigroup with a unit) but is not necessarily a loop itself. This note is about turning homotopies into homomorphisms.

Smith, [6], proved that there is an adjunction from the category of semisymmetric quasigroups with homomorphisms to the category of quasigroups with homotopies. Also, he proved in [6] that the latter category is isomorphic to a subcategory of the former category, and in [7], that every T algebra, for T being the monad defined by the above adjunction, is isomorphic to the image of a semisymmetric quasigroup under the comparison functor.

These results, especially the embedding of the category of quasigroups with homotopies into the category of semisymmetric quasigroups with homomorphisms, could be of interest to a working universal algebraist. Our intention is to make them more accessible to such a reader and to indicate a possible misusing. Also, we give a proof that the comparison functor is full, which completes the proof of monadicity of the adjunction.

At the end of the paper, we show that there is a more economical way to embed the category of quasigroups with homotopies into the category of semisymmetric quasigroups with homomorphisms. One could get an impression, due to [6], that for such an embedding it is necessary to have a semisymmetrization functor that

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is a right adjoint in an adjunction. If one is interested just in this embedding and not in reflectivity (see the end of Section 4), then this new semisymmetrization suits as any other.

We assume that the reader is familiar with the notions of category, functor and natural transformation. If not, we suggest to consult [5] for these notions. All other relevant notions from Category theory are introduced at the appropriate places in the text.

2. Quasigroups

We start by recapitulating a few basic facts about quasigroups.

One way to define a *quasigroup* is that it is a grupoid $(Q;\cdot)$ satisfying:

$$\forall ab \exists_1 x \ \ (x \cdot a = b) \qquad \text{and} \qquad \forall ab \exists_1 x \ \ (a \cdot x = b)$$

Uniqueness of the solution of the equation $x \cdot a = b$ ($a \cdot x = b$) enables one to define right (left) division operation $x = b/a$ ($x = a\backslash b$) which is also a quasigroup (short for: $(Q;/)$ is a quasigroup). We can define three more operations:

$$x * y = y \cdot x \qquad\qquad x /\!\!/ y = y/x \qquad\qquad x \backslash\!\!\backslash y = y \backslash x$$

dual to $\cdot, /, \backslash$ respectively. They are also quasigroups. The six operations $\cdot, /, \backslash, *, /\!\!/$ and $\backslash\!\!\backslash$ are *parastrophes* of $\cdot$ (and of each other).

A function $f : Q \to R$ between the base sets of quasigroups $(Q;\cdot)$ and $(R,\cdot)$ is a *homomorphism* iff:

$$f(x) \cdot f(y) = f(x \cdot y)$$

and *isomorphism* if f is a bijection as well.

A triple $\bar{f} = (f_1, f_2, f_3)$ of functions ($f_i : Q \to R$) is a *homotopy* iff:

$$f_1(x) \cdot f_2(y) = f_3(x \cdot y)$$

which implies (and is implied by any of):

$$f_3(x)/f_2(y) = f_1(x/y) \qquad\qquad f_2(x) /\!\!/ f_3(y) = f_1(x /\!\!/ y)$$
$$f_1(x)\backslash f_3(y) = f_2(x\backslash y) \qquad\qquad f_3(x) \backslash\!\!\backslash f_1(y) = f_2(x \backslash\!\!\backslash y)$$

If all three components of $\bar{f}$ are bijections, then $\bar{f}$ is an *isotopy*.

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We can also define a quasigroup as an algebra $(Q;\cdot,/,\backslash)$ with three binary operations: multiplication $(\cdot)$, right and left division. The axioms that a quasigroup satisfies are (xy is short for $(x \cdot y)$):

$$\begin{array}{ll} xy/y = x & x\backslash xy = y \\ (x/y)y = x & x(x\backslash y) = y \end{array} \qquad \text{(Q)}$$

For obvious reasons, such quasigroups are called *equational, primitive* or *equasigroups.*

Thus, we have the variety of all quasigroups. Another important variety is the variety of *semisymmetric* quasigroups, defined by one of the following five equivalent axioms (in addition to (Q)):

$$x \cdot yx = y \tag{2.1}$$

$$xy \cdot x = y \tag{2.2}$$

$$x/y = yx$$

$$x\backslash y = yx$$

$$x\backslash y = x/y$$

Smith, [6], defined a *semisymmetrization* of a quasigroup $\mathbb{Q} = (Q; \cdot, /, \backslash)$ as a one–operation quasigroup $\mathbb{Q}^{\Delta} = (Q^3; \circ)$ where the binary operation $\circ$ is defined by:

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (y_3/x_2, y_1\backslash x_3, x_1 y_2) \tag{2.3}$$

and proved that, for any quasigroup $\mathbb{Q}$, the semisymmetrization $\mathbb{Q}^{\Delta}$ of $\mathbb{Q}$ is a semisymmetric quasigroup.

3. Twisted quasigroups

For our purpose, there is a better way to define a quasigroup. In this definition the *twisted quasigroup* is an algebra $(Q; /\!\!/, \backslash\!\!\backslash, \cdot)$ satisfying appropriate paraphrasing of the above quasigroup axioms (Q):

$$y /\!\!/ xy = x \qquad xy \backslash\!\!\backslash x = y$$
$$(y /\!\!/ x)y = x \qquad x(y \backslash\!\!\backslash x) = y$$

We have the following symmetry result, lacking for quasigroups defined as $(Q; \cdot, /, \backslash)$.

Proposition 3.1. *An algebra* $(Q; /\!\!/, \backslash\!\!\backslash, \cdot)$ *is a* twisted quasigroup *iff* $(Q; \backslash\!\!\backslash, \cdot, /\!\!/)$ *is a* twisted quasigroup *iff* $(Q; \cdot, /\!\!/, \backslash\!\!\backslash)$ *is a* twisted quasigroup.

Analogously, we have the paraphrasing of axioms for *semisymmetric twisted semisymmetric quasigroups*: (2.1),(2.2) and

$$x /\!\!/ y = xy$$

$$x \backslash\!\!\backslash y = xy$$

$$x \backslash\!\!\backslash y = x /\!\!/ y$$

The last three identities we shorten to symbolic identities: $/\!\!/ = \cdot, \backslash\!\!\backslash = \cdot, \backslash\!\!\backslash = /\!\!/$.

There is also a result corresponding to Proposition 3.1:

Proposition 3.2. *An algebra* $(Q;/\!\!/,\backslash\!\!\backslash,\cdot)$ *is a* semisymmetric twisted quasigroup *iff* $(Q;\backslash\!\!\backslash,\cdot,/\!\!/)$ *is a* semisymmetric twisted quasigroup *iff* $(Q;\cdot,/\!\!/,\backslash\!\!\backslash)$ *is a* semisymmetric twisted quasigroup.

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Using twisted quasigroups we can see how a (twisted) semisymmetrization (defined below), which we call ∇, 'works'.

Let us start with three single–operation quasigroups $(Q;\cdot),(Q;/\!\!/)$ and $(Q;\backslash\!\!\backslash)$, where $/\!\!/$ and $\backslash\!\!\backslash$ are duals of appropriate division operations of $\cdot$. We can define direct (Cartesian) product $(Q;/\!\!/)\times(Q;\backslash\!\!\backslash)\times(Q;\cdot)$ and an operation $\otimes$ on Q^3 such that

$$(x_1,x_2,x_3)\otimes(y_1,y_2,y_3)=(x_1/\!\!/y_1,x_2\backslash\!\!\backslash y_2,x_3y_3) \tag{3.4}$$

defines multiplication in the direct product. Therefore $(Q^3;\otimes)$ is a quasigroup.

Define also a permutation $'\colon Q^3\to Q^3$ by $(x_1,x_2,x_3)'=(x_2,x_3,x_1)$. It follows that $(x_1,x_2,x_3)''=(x_3,x_1,x_2)$ and $(x_1,x_2,x_3)'''=(x_1,x_2,x_3)$. Define another operation $\nabla_3 : Q^3\times Q^3\to Q^3$ by $\bar{x}\nabla_3\bar{y}=\bar{x}'\otimes\bar{y}''$, where $\bar{u}=(u_1,u_2,u_3)$. The groupoid $(Q^3;\nabla_3)$ is also a quasigroup, so there are appropriate division operations of ∇_3 and their duals ∇_1 and ∇_2:

$$\bar{x}\nabla_3\bar{y}=\bar{z}\quad\text{iff}\quad\bar{y}\nabla_1\bar{z}=\bar{x}\quad\text{iff}\quad\bar{z}\nabla_2\bar{x}=\bar{y}.$$

Therefore $(Q^3;\nabla_1,\nabla_2,\nabla_3)$ is a twisted quasigroup.

Let us calculate ∇_1.

$$\begin{aligned}\bar{z}=(z_1,z_2,z_3)&=\bar{x}\nabla_3\bar{y}=(x_1,x_2,x_3)'\otimes(y_1,y_2,y_3)''\\&=(x_2,x_3,x_1)\otimes(y_3,y_1,y_2)=(x_2/\!\!/y_3,x_3\backslash\!\!\backslash y_1,x_1y_2).\end{aligned}$$

Therefore

$$\bar{x}=(y_2/\!\!/z_3,y_3\backslash\!\!\backslash z_1,y_1z_2)=(y_2,y_3,y_1)\otimes(z_3,z_1,z_2)=\bar{y}'\otimes\bar{z}''=\bar{y}\nabla_3\bar{z}$$

i.e. $\nabla_1=\nabla_3$ (and consequently $\nabla_2=\nabla_3$) hence $(Q^3;\nabla_1,\nabla_2,\nabla_3)$ is semisymmetric twisted quasigroup. So we recognize ∇_3 as a twisted analogue of Smith's $\circ$ (see identity (2.3)). Let us call $\mathbb{Q}^\nabla=(Q^3;\nabla_1,\nabla_2,\nabla_3)$ a *twisted semisymmetrization* of $\mathbb{Q}$.

For (f_1,f_2,f_3) being a homotopy from $\mathbb{Q}$ to $\mathbb{R}$, we also have:

$$\begin{aligned}(f_1\times f_2\times f_3)\,(\bar{x}\nabla_3\bar{y})&=(f_1\times f_2\times f_3)\,(\bar{x}'\otimes\bar{y}'')\\&=(f_1(x_2/\!\!/y_3),f_2(x_3\backslash\!\!\backslash y_1),f_3(x_1\cdot y_2))\\&=(f_2x_2/\!\!/f_3y_3,f_3x_3\backslash\!\!\backslash f_1y_1,f_1x_1\cdot f_2y_2)\\&=(f_2x_2,f_3x_3,f_1x_1)\otimes(f_3y_3,f_1y_1,f_2y_2)\\&=(f_1x_1,f_2x_2,f_3x_3)'\otimes(f_1x_1,f_2x_2,f_3x_3)''\\&=(f_1\times f_2\times f_3)\,(\bar{x})\nabla_3(f_1\times f_2\times f_3)\,(\bar{y}),\end{aligned}$$

so $f_1\times f_2\times f_3$ is a homomorphism.

4. The categories Qtp and P

This section follows the lines of [6] with some adjustments. The main novelty is a proof of [6, Corollary 5.3]. We try to keep to the notation introduced in [6]. However, we write functions and functors to the left of their arguments.

For a fixed, large enough universe U, a quasigroup $\mathbb{Q} = (Q; \cdot, /, \backslash)$ is *small* when Q belongs to U (see [5, I.2]). Let **Qtp** be the category with objects all small quasigroups $\mathbb{Q} = (Q; \cdot, /, \backslash)$ and arrows all homotopies. The identity homotopy on $\mathbb{Q}$ is the triple $(\mathbf{1}_Q, \mathbf{1}_Q, \mathbf{1}_Q)$, where $\mathbf{1}_Q$ is the identity function on Q, and the composition of homotopies $(f_1, f_2, f_3)\colon \mathbb{P} \to \mathbb{Q}$ and $(g_1, g_2, g_3)\colon \mathbb{Q} \to \mathbb{R}$ is the homotopy $(g_1 \circ f_1, g_2 \circ f_2, g_3 \circ f_3)\colon \mathbb{P} \to \mathbb{R}$.

Let **P** be the category with objects all small semisymmetric quasigroups and arrows all quasigroup homomorphisms. For every arrow $f\colon \mathbb{Q} \to \mathbb{R}$ of **P**, the triple (f, f, f) is a homotopy between $\mathbb{Q}$ and $\mathbb{R}$.

Let Σ be a functor from **P** to **Qtp**, which is identity on objects. Moreover, let Σf, for a homomorphism f, be the homotopy (f, f, f).

The category **P** is a full subcategory of the category **Q** with objects all small quasigroups and arrows all quasigroup homomorphisms. The functor Σ is just a restriction of a functor from **Q** to **Qtp**, which is defined in the same manner.

An *adjunction* is given by two functors, $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$, and two natural transformations, the *unit* $\eta\colon \mathbf{1_C} \dot{\to} GF$ and the *counit* $\varepsilon\colon FG \dot{\to} \mathbf{1_D}$, such that for every object C of $\mathbf{C}$ and every object D of $\mathbf{D}$

$$G\varepsilon_D \circ \eta_{GD} = \mathbf{1}_{GD}, \quad \text{and} \quad \varepsilon_{FC} \circ F\eta_C = \mathbf{1}_{FC}.$$

These two equalities are called *triangular identities*. The functor F is a *left adjoint* for the functor G, while G is a *right adjoint* for the functor F.

That $\Sigma : \mathbf{P} \to \mathbf{Qtp}$ has a right adjoint is shown as follows. Let $/\!\!/$ and $\backslash\!\!\backslash$ be defined as at the beginning of Section 3. For $\mathbb{Q}$ a quasigroup, let $\nabla_3 : Q^3 \times Q^3 \to Q^3$ be defined as in Section 3, i.e., for every $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$

$$\bar{x} \nabla_3 \bar{y} = (x_2 /\!\!/ y_3, x_3 \backslash\!\!\backslash y_1, x_1 \cdot y_2).$$

That $(Q^3; \nabla_3)$ is a semisymmetric quasigroup follows from the fact that the structure $(Q^3; \nabla_1, \nabla_2, \nabla_3)$ is a semisymmetric twisted quasigroup, which is shown in Section 3. The semisymmetric quasigroup $(Q^3; \nabla_3)$ is the semisymmetrization $\mathbb{Q}^{\Delta}$ of $\mathbb{Q}$ defined at the end of Section 2 (see (2.3)).

Let $\Delta\colon \mathbf{Qtp} \to \mathbf{P}$ be a functor, which maps a quasigroup $\mathbb{Q}$ to the semisymmetric quasigroup $(Q^3; \nabla_3)$. A homotopy (f_1, f_2, f_3) is mapped by Δ to the product $f_1 \times f_2 \times f_3$, which is a homomorphism as it is shown at the end of Section 3. By the functoriality of product, we have that Δ preserves identities and composition, and it is indeed a functor. A proof of the following proposition is given in [6, Theorem 5.2].

Proposition 4.1. *The functor Δ is a right adjoint for Σ.*

Moreover, every component of the counit of this adjunction is epi (i.e. right cancellable) and the semisymmetrization is one-one. This is sufficent for **Qtp** to be isomorphic to a subcategory of **P**. This is one way how to establish this fact using the previous proposition. However, if the goal was just to establish that **Qtp** is isomorphic to a subcategory of **P**, this adjunction is not necessary at all, which is shown below.

A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *faithful* when for every pair $f, g\colon A \to B$ of arrows of $\mathbf{C}$, $Ff = Fg$ implies $f = g$. An arrow $f\colon A \to B$ of $\mathbf{C}$ is *epi* when for every pair $g, h\colon B \to C$ of arrows of $\mathbf{C}$, the equality $g \circ f = h \circ f$ implies $g = h$. The following lemmas will help us to prove that **Qtp** is isomorphic to a subcategory of **P**.

Lemma 4.2. *The functor Δ is faithful.*

Proof. For homotopies (f_1, f_2, f_3) and (g_1, g_2, g_3) from $\mathbb{Q}$ to $\mathbb{R}$, if $f_1 \times f_2 \times f_3$ and $g_1 \times g_2 \times g_3$ are equal as homomorphisms from $\Delta\mathbb{Q}$ to $\Delta\mathbb{R}$ in **P**, then for every $i \in \{1, 2, 3\}$, $f_i = g_i$. Hence, these homotopies are equal in **Qtp**. □

Alternatively, by [5, IV.3, Theorem 1, Part (i)] (see also [2, Section 4, Proposition 4.1] for an elegant proof of a related result) one may establish that Δ is faithful by relying on Proposition 4.1. It suffices to prove that for every object $\mathbb{Q}$ of **Qtp**, the component $\varepsilon_{\mathbb{Q}}$ of the counit of the adjunction established in Proposition 4.1 is epi. The arrow $\varepsilon_{\mathbb{Q}}$ is defined as the triple (π_1, π_2, π_3), where $\pi_i\colon Q^3 \to Q$ is the ith projection. Let $g, h\colon \mathbb{Q} \to \mathbb{R}$ be a pair of arrows of **Qtp** such that $g \circ \varepsilon_{\mathbb{Q}} = h \circ \varepsilon_{\mathbb{Q}}$. This means that for every $i \in \{1, 2, 3\}$ we have that $g_i \circ \pi_i = h_i \circ \pi_i$. Hence, the function g_i is equal to the function h_i, since the function π_i is right cancellable. (However, the homotopy $\varepsilon_{\mathbb{Q}}$ need not have a right inverse in **Qtp**.)

Lemma 4.3. *If $(Q; \cdot, /, \backslash)$ and $(Q; \cdot', /', \backslash')$ are two different quasigroups, then there are $x, y \in Q$ such that*

$$x \cdot y \neq x \cdot' y.$$

Proof. Suppose that for every $x, y \in Q$, $x \cdot y = x \cdot' y$ holds. Then for every $z, t \in Q$ we have

$$z/t = ((z/t) \cdot' t)/'t = ((z/t) \cdot t)/'t) = z/'t.$$

Analogously, we prove that for every $u, v \in Q$, $u\backslash v = u\backslash' v$. Hence, $(Q; \cdot, /, \backslash)$ and $(Q; \cdot', /', \backslash')$ are the same, which contradicts the assumption. □

Lemma 4.4. *The functor Δ is one-one on objects.*

Proof. Suppose that $(Q; \cdot, /, \backslash)$ and $(Q'; \cdot', /', \backslash')$ are two different quasigroups. If Q and Q' are different sets, then $\Delta\mathbb{Q}$ and $\Delta\mathbb{Q}'$ are different. If $Q = Q'$, then, by Lemma 4.3, there are x and y in this set such that $x \cdot y \neq x \cdot' y$. Hence, the operations ∇_3 for $\Delta\mathbb{Q}$ and $\Delta\mathbb{Q}'$ differ when applied to (x, x, x) and (y, y, y). □

As a corollary of these two lemmas we have the following result.

Proposition 4.5. *The category* **Qtp** *is isomorphic to a subcategory of* **P***; namely, to its image under the functor* Δ.

As we have shown by the proof of Lemma 4.2, Proposition 4.5 is independent of Proposition 4.1. The adjunction, together with this embedding of **Qtp** in **P**, says that the category **P** reflects in **Qtp** in the following sense. A subcategory **A** of **B** is *reflective* in **B**, when the inclusion functor from **A** to **B** has a left adjoint called a *reflector* (see [5, IV.3]). The adjunction is called a *reflection* of **B** in **A**.

Propositions 4.1 and 4.5 say that **Qtp** may be considered as a reflective subcategory of **P**. The functor Σ is a reflector and the adjunction between Σ and Δ is a reflection of **P** in **Qtp**. However, this does not mean that the Δ-image of **Qtp** is an *iso-full* subcategory of **P**, i.e. that two quasigroups are isotopic in **Qtp** if and only if their semisymmetrizations are isomorphic in **P**. Im, Ko and Smith, [4, first paragraph in the introduction], refer to [6] for this iso-fullness. However, this is not considered at all in [6] and the question of fullness or iso-fullness of the image of **Qtp** in **P** remains open. The reader should be aware of this potential missusing of these results.

5. Monadicity of Δ

For $F : \mathbf{C} \to \mathbf{D}$ a left adjoint for $G : \mathbf{D} \to \mathbf{C}$, and η and ε, the unit and counit of this adjunction, a GF-algebra is a pair (C, h), where C is an object of **C** and $h : GFC \to C$ is an arrow of **C** such that the following equalities hold.

$$h \circ GFh = h \circ G\varepsilon_{FC}, \qquad h \circ \eta_C = \mathbf{1}_C.$$

A morphism of GF-algebras (C, h) and (C', h') is given by an arrow $f : C \to C'$ of **C** such that $f \circ h = h' \circ GFf$.

The category $\mathbf{C}^{GF}$ has GF-algebras as objects and morphisms of GF-algebras as arrows. The *comparison functor* $K : \mathbf{D} \to \mathbf{C}^{GF}$ is given by

$$KD = (GD, G\varepsilon_D), \qquad Kf = Gf.$$

In many cases the comparison functor is an isomorphism or an equivalence (i.e. there is a functor from $\mathbf{C}^{GF}$ to **D** such that both compositions with K are naturally isomorphic to the identity functors). The right adjoint of an adjunction or an adjunction are called *monadic* when the comparison functor is an isomorphism (see [5, VI.3], also [8, Section 4.2]). Some other authors (see [1, Section 3.3]) call an adjunction monadic (tripleable) when K is just an equivalence.

In the case of adjoint situation involving Σ and Δ, the comparison functor $K : \mathbf{Qtp} \to \mathbf{P}^{\Delta\Sigma}$ is just an equivalence. To prove this, by [5, IV.4, Theorem 1] it suffices to prove that K is full and faithful, and that every GF-algebra is isomorphic to $K\mathbb{Q}$ for some quasigroup $\mathbb{Q}$. The faithfulness of K follows from 4.2 since the arrow function K coincides with the arrow function Δ. That every GF-algebra is isomorphic to $K\mathbb{Q}$ for some quasigroup $\mathbb{Q}$ is proven in [7, Section 10, Theorem 33].

A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *full* when for every pair of objects C_1 and C_2 of $\mathbf{C}$ and every arrow $g\colon FC_1 \to FC_2$ of $\mathbf{D}$ there is an arrow $f : C_1 \to C_2$ of $\mathbf{C}$ such that $g = Ff$. It remains to prove that K is full. For this we use the following lemma.

Lemma 5.1. *Every arrow of $\mathbf{P}^{\Delta\Sigma}$ from $K\mathbb{Q}$ to $K\mathbb{R}$ is of the form $f_1 \times f_2 \times f_3$, for (f_1, f_2, f_3) a homotopy from $\mathbb{Q}$ to $\mathbb{R}$.*

Proof. For quasigroups $\mathbb{Q}$ and $\mathbb{R}$ we have that $K\mathbb{Q} = (\Delta\mathbb{Q}, \pi_1 \times \pi_2 \times \pi_3)$ and $K\mathbb{R} = (\Delta\mathbb{R}, \pi_1 \times \pi_2 \times \pi_3)$. So, let

$$f\colon (\Delta\mathbb{Q}, \pi_1 \times \pi_2 \times \pi_3) \to (\Delta\mathbb{R}, \pi_1 \times \pi_2 \times \pi_3)$$

be an arrow of $\mathbf{P}^{\Delta\Sigma}$. Since f is a morphism of $\Delta\Sigma$-algebras, we have that

$$f \circ (\pi_1 \times \pi_2 \times \pi_3) = (\pi_1 \times \pi_2 \times \pi_3) \circ (f \times f \times f)$$

as functions from $(Q^3)^3$ to R^3.

For $i \in \{1, 2, 3\}$ and $u \in Q$, let $f_i(u) = \pi_i(f(u, u, u))$. Moreover, let (x, y, z) be an arbitrary element of Q^3. Apply the both sides of the above equality to $((x, x, x), (y, y, y), (z, z, z)) \in (Q^3)^3$ in order to obtain

$$f(x, y, z) = (\pi_1(f(x, x, x)), \pi_2(f(y, y, y)), \pi_3(f(z, z, z))) = (f_1(x), f_2(y), f_3(z)).$$

Hence, $f = f_1 \times f_2 \times f_3$ and since it is a homomorphism from $\Delta\mathbb{Q}$ to $\Delta\mathbb{R}$, we have for every $\bar{x}, \bar{y} \in Q^3$

$$(f_1 \times f_2 \times f_3)(\bar{x})\ \nabla_3\ (f_1 \times f_2 \times f_3)(\bar{y}) = (f_1 \times f_2 \times f_3)(\bar{x}\ \nabla_3\ \bar{y}).$$

By restricting this equality to the third component, we obtain $f_1(x_1) \cdot f_2(y_2) = f_3(x_1 \cdot y_2)$, and hence (f_1, f_2, f_3) is a homotopy from $\mathbb{Q}$ to $\mathbb{R}$. □

6. A new semisymmetrization

Definition 6.1. An algebra $(Q; /\!\!/, \backslash\!\!\backslash)$ is a *biquasigroup* iff $/\!\!/ (\backslash\!\!\backslash)$ is the dual of the right (left) division operation of a quasigroup operation $\cdot$ on Q.

A biquasigroup is *semisymmetric* iff $\backslash\!\!\backslash = /\!\!/$.

Proposition 6.2. *An algebra $(Q; /\!\!/, \backslash\!\!\backslash)$ is a biquasigroup iff $(Q; \backslash\!\!\backslash, \cdot)$ is a biquasigroup iff $(Q; \cdot, \backslash\!\!\backslash)$ is a biquasigroup.*

Proposition 6.3. *An algebra $(Q; /\!\!/, \backslash\!\!\backslash)$ is a semisymmetric biquasigroup iff $(Q; \backslash\!\!\backslash, \cdot)$ is a semisymmetric biquasigroup iff $(Q; \cdot, \backslash\!\!\backslash)$ is a semisymmetric biquasigroup.*

Let us start with three single–operation quasigroups $(Q; \cdot), (Q; /\!\!/)$ and $(Q; \backslash\!\!\backslash)$, where $/\!\!/$ and $\backslash\!\!\backslash$ are duals of appropriate division operations of $\cdot$. We can define direct (Cartesian) product $(Q; /\!\!/) \times (Q; \backslash\!\!\backslash)$ and an operation $\otimes$ on Q^2 such that

$$(x_1, x_2) \otimes (y_1, y_2) = (x_1 /\!\!/ y_1, x_2 \backslash\!\!\backslash y_2)$$

defines multiplication in the direct product. Therefore $(Q^2;\otimes)$ is a quasigroup.

Define also a permutation $' : Q^2 \to Q^2$ by $(x_1,x_2)' = (x_2,x_1)$. Define another operation $\nabla : Q^2\times Q^2 \to Q^2$ by $\hat{x}\nabla\hat{y} = R_{\hat{y}}(\hat{x}')\otimes L_{\hat{x}}(\hat{y}')$, where $\hat{u}$ is (u_1,u_2), $L_{\hat{x}}(\hat{y}) = (x_1\cdot y_1, y_2)$ and $R_{\hat{y}}(\hat{x}) = (x_1, x_2\cdot y_2)$. The groupoid $(Q^2;\nabla)$ is also a quasigroup, moreover a semisymmetric one. Therefore $(Q^2;\nabla,\nabla)$ is a semisymmetric biquasigroup.

Let us define:

$$x \text{①} y = x/\!\!/y \qquad\qquad x \text{②} y = x\backslash\!\!\backslash y \qquad\qquad x \text{③} y = x\cdot y$$

Then the definition of ∇_{12}, which we abbreviate just by ∇, is:

$$(x_1,x_2)\nabla_{12}(y_1,y_2) = (x_2 \text{①} (x_1 \text{③} y_2), (x_1 \text{③} y_2) \text{②} y_1).$$

There are two more alternative semisymmetrizations with corresponding definitions in $(Q^2;\backslash\!\!\backslash,\cdot)$ (respectively $(Q^2;\cdot,/\!\!/)$):

$$(x_1,x_2)\nabla_{23}(y_1,y_2) = (x_2 \text{②} (x_1 \text{①} y_2), (x_1 \text{①} y_2) \text{③} y_1)$$

$$(x_1,x_2)\nabla_{31}(y_1,y_2) = (x_2 \text{③} (x_1 \text{②} y_2), (x_1 \text{②} y_2) \text{①} y_2).$$

The indexing of operations is used to emphasize the symmetry.

In this section we introduce a new semisymmetrization functor from **Qtp** to **P**. This leads to another subcategory of **P** isomorphic to **Qtp**. We start with an auxiliary result.

Lemma 6.4. *The third component f_3 of a homotopy is determined by the first two components f_1 and f_2.*

Proof. Let $\mathbb{Q}$ be a quasigroup. For every element $x\in Q$ there are $y,z\in Q$ such that $x = y\cdot z$ (e.g. $x = y\cdot(y\backslash x)$). Hence, $f_3(x) = f_1(y)\cdot f_2(z)$. ☐

Let $\Gamma\colon \mathbf{Qtp}\to\mathbf{P}$ be a functor defined on objects so that $\Gamma\mathbb{Q}$ is a semisymmetric quasigroup $(Q^2;\nabla)$ whose elements are pairs (x_1,x_2), abbreviated by $\hat{x}$, and ∇ is defined so that

$$(x_1,x_2)\nabla(y_1,y_2) = (x_2/\!\!/(x_1\cdot y_2), (x_1\cdot y_2)\backslash\!\!\backslash y_1).$$

(It is straightforward to check that $(\hat{y}\nabla\hat{x})\nabla\hat{y} = \hat{y}\nabla(\hat{x}\nabla\hat{y}) = \hat{x}$, hence $\Gamma\mathbb{Q}$ is a semisymmetric quasigroup.)

A homotopy (f_1,f_2,f_3) is mapped by Γ to the product $f_1\times f_2$, which is a homomorphism:

$$\begin{aligned}(f_1\times f_2)(\hat{x})\ \nabla\ (f_1\times f_2)(\hat{y}) &= (f_2(x_2)/\!\!/(f_1(x_1)\cdot f_2(y_2)), (f_1(x_1)\cdot f_2(y_2))\backslash\!\!\backslash f_1(y_1))\\ &= (f_1(x_2/\!\!/(x_1\cdot y_2)), f_2((x_1\cdot y_2)\backslash\!\!\backslash y_1))\\ &= (f_1\times f_2)(\hat{x}\nabla\hat{y}).\end{aligned}$$

By the functoriality of product, we have that Γ preserves identities and composition, and it is indeed a functor.

The functor Γ is not a right adjoint for Σ since a right adjoint is unique up to isomorphism and $\Gamma\mathbb{Q}$ is not isomorphic to $\Delta\mathbb{Q}$ for every object $\mathbb{Q}$ of **Qtp**. However, this adjunction is not necessary for the faithfulness of Γ.

Lemma 6.5. *The functor Γ is faithful.*

Proof. We proceed as in the second proof of Lemma 4.2. If (f_1, f_2, f_3) and (g_1, g_2, g_3) are two homotopies from $\mathbb{Q}$ to $\mathbb{R}$, then $\Gamma(f_1, f_2, f_3) = \Gamma(g_1, g_2, g_3)$ means that $f_1 \times f_2 = g_1 \times g_2$. Hence, $f_1 = g_1$ and $f_2 = g_2$, and by Lemma 6.4, $f_3 = g_3$. $\square$

The functor Γ, as defined, is not one-one on objects. For example,

$$(\{0,1\}, +, +, +) \quad \text{and} \quad (\{0,1\}, \oplus, \oplus, \oplus),$$

where $+$ is addition mod 2 and $x \oplus y = x + y + 1$, are mapped by Γ to the same object of **P**. To remedy this matter, one may redefine Γ so that

$$\Gamma\mathbb{Q} = (Q^2 \times \{\mathbb{Q}\}, \nabla),$$

where $\mathbb{Q}$, as the third component of every element, guarantees that Γ is one-one on objects. The operation ∇ is defined as above, just neglecting the third component. Hence, **Qtp** may be considered as another subcategory of **P**.

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