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Note on the cyclic subgroup intersection graph of a finite group

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Abstract. The cyclic subgroup intersection graph of a finite group G, $\Gamma_{CSI}(G)$, is a simple graph with non-trivial cyclic subgroups as vertex set. Two cyclic subgroups are adjacent if and only if they have a non-trivial intersection. It is easy to see that $\Gamma_{CSI}(G)$ is a subgraph of the intersection graph was introduced by Csákány and Pollák many years ago. In this paper the main properties of this new graph is studied. The graph structure of the cyclic groups, dihedral groups, generalized quaternion groups and the group $Z_{p^\alpha} \times Z_{p^\beta}$ are completely determined.

1. Introduction

Throughout this paper all groups are assumed to be finite and graphs will be finite and simple. For notations not defined here, we refer the reader to [4, 7, 8]. The greatest common divisor and least common multiple of integers a and b are denoted by (a,b) and $[a,b]$, respectively. The number of positive divisors of an integer n is denoted by $d(n)$. Our calculations are done with the aid of GAP [2].

The intersection graph of a finite group G was introduced many years ago by Csákány any and Pollák [1]. The vertex set of this graph is all proper non-trivial subgroups of G and two vertices H and K are adjacent if and only if $H \cap K \neq 1$, where 1 denotes the trivial subgroup of G. In the mentioned paper, the authors proved that if G is abelian and there are two subgroups H and K in G such that there is no chain of subgroups which unites them, then G is the direct product of two simple cyclic groups. As a consequence, they proved that the diameter of this graph is at most 2, when G is an abelian group. The diameter of non-abelian, non-simple groups is at most 4. Some interesting open questions are also included in [1]. Zelinka [10], continued the study of this graph and conjectured that two finite Abelian groups with isomorphic intersection graphs are isomorphic.

Tamizh Chelvam and Sattanathan [9] continued the seminal paper of Csákány any and Pollák to introduce the subgroup intersection graph of a finite group G denoted by $\Gamma_{SI}(G)$. The vertex set of this graph is $G\backslash\{e\}$, and there is an edge

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between two distinct vertices x and y if and only if $\langle x\rangle \cap \langle y\rangle \neq 1$. As a consequence of a result in this paper, the subgroup intersection graph of a finite group G is complete if and only if G is a cyclic $p-$group or a generalized quaternion $2-$group. Moreover, the subgroup intersection graph of a finite abelian $p-$group is a union of complete graphs.

The cyclic subgroup intersection graph of G, $\Gamma_{CSI}(G)$, is another simple graph with proper non-trivial cyclic subgroups as vertex set. Two cyclic subgroups are adjacent if and only if they have a non-trivial intersection. It is easy to see that $\Gamma_{CSI}(G)$ is a subgraph of $\Gamma_{SI}(G)$.

Suppose Δ is a simple graph. Following Sabidussi [6], the $\Delta-$join of a family $\mathcal{F} = \{T_x \mid x \in V(\Delta)\}$ of simple graphs is another simple graph Γ with the following vertex and edge sets:

$$V(\Gamma) = \{(x, y) \mid x \in V(\Delta) \ \& \ y \in V(T_x)\},$$
$$E(\Gamma) = \{(x, y)(a, b) \mid xa \in E(\Delta) \ or \ x = a \ \& \ yb \in E(T_x)\}.$$

If $V(\Delta) = \{x_1, \ldots, x_n\}$ and $\mathcal{F} = \{T_1, \ldots, T_n\}$ then the $\Delta-$join of the family $\mathcal{F}$ is denoted by $\Delta[T_1, \ldots, T_n]$.

An independent set of a simple graph Γ is a subset of its vertices, no two of which are adjacent. The cardinality of an independent set in Γ of largest possible size is called the independence number of Γ. This number is denoted by $\alpha(\Gamma)$. We refer to the famous book of Harary [4] for our graph theory notations.

The aim of this paper is to investigate the main properties of the cyclic subgroup intersection graph. It is proved, among other things, that if $G = Z_{p^\alpha} \times Z_{p^\beta}$, where p is prime and α, β are two positive integers such that $\alpha \leqslant \beta$ then $\Gamma_{CSI}(G)$ is a union of the complete graphs $K_{(\beta-\alpha)p^\alpha + \frac{p^\alpha - 1}{p-1}}$ together with p copies of $K_{\frac{p^\alpha - 1}{p-1}}$, and $\Gamma_{SI}(G)$ is a union of the complete graphs $K_{p^{\alpha+\beta} - \frac{p^{2\alpha+1}+1}{p+1}}$ together with p copies of $K_{\frac{p^{2\alpha}-1}{p+1}}$.

2. Main results

Suppose G is a non-cyclic group and $A = \Gamma_{CSI}(G)$. For each $\langle a\rangle \in V(A)$, we define $T_{\langle a\rangle} = K_{\phi(|a|)}$, where ϕ denotes the Euler totient function. Then one can easily see that $\Gamma_{SI}(G)$ is an $A-$join of $\{T_{\langle x\rangle} \mid \langle x\rangle \in V(A)\}$.

Lemma 2.1. *Let G be a group of order n. Then $|V(\Gamma_{CSI}(G))| \geqslant d(n) - 2$ with equality if and only if G is cyclic.*

Proof. By [5], the number of cyclic subgroups of a group G of order n is at least $d(n)$ with equality if and only if $G \cong Z_n$, as desired. □

By Lemma 2.1, the cyclic subgroup intersection graph of a cyclic group of order p^{m+1} has exactly m vertices. This proves that for each positive integer m, there exists at least a group with an $m-$vertex cyclic subgroup intersection graph.

Example 2.2. Suppose $SmallGroup(n,i)$ denotes the $i-th$ group of order n in the small group library of GAP [2]. Define $G = SmallGroup(168,46) = (Z_7 \times A_4) : Z_2$, $H = SmallGroup(168,38) = (Z_{42} \times Z_2) : Z_2$ and $K = SmallGroup(168,42) = PSL(3,2)$. Then $\Gamma_{CSI}(G) \cong \Gamma_{CSI}(H) \cong \Gamma_{CSI}(K)$, but G, H and K are mutually non-isomorphic.

The previous example shows that if $\Gamma_{CSI}(G)$ and $\Gamma_{CSI}(H)$ are isomorphic then we cannot deduce that G and H are isomorphic, even in the case that one of these groups is simple.

Example 2.3. In this example the cyclic subgroup intersection graph of a dihedral group of order $2n$ will be computed. The dihedral group of order $2n$ can be presented as $D_{2n} = \langle x^n = y^2 = e, y^{-1}xy = x^{-1}\rangle$. Suppose $k_1, \ldots, k_{d(n)}$ are all divisors of n. Then

$$V(\Gamma_{CSI}(D_{2n})) = \{\langle a^{k_1}\rangle, \ldots, \langle a^{k_{d(n)-1}}\rangle, \langle b\rangle, \langle ab\rangle, \ldots, \langle a^{n-1}b\rangle\}.$$

It is easy to see that $\langle b\rangle, \langle ab\rangle, \ldots, \langle a^{n-1}b\rangle$ are pendant vertices of $\Gamma_{CSI}(D_{2n})$. Moreover, $\langle a^{k_i}\rangle$ and $\langle a^{k_j}\rangle$ are adjacent if and only if $[k_i, k_j] < n$.

Theorem 2.4. *The cyclic subgroup intersection graph of a finite group G is complete if and only if G is cyclic or a generalized quaternion $2-$group.*

Proof. It is well-known that a $p-$group G has a unique subgroup of order p if and only if G is cyclic or a generalized quaternion $2-$group. By this theorem, if G is cyclic or a generalized quaternion $2-$group then the intersection of non-trivial subgroups H and K contains the unique subgroup of G and so $H \cap K \neq 1$. This proves that $\Gamma_{CSI}(G)$ is complete. Conversely, if $\Gamma_{CSI}(G)$ is complete and p, q are two prime divisors of $|G|$ then there are elements a and b of orders p and q in G, respectively. Since $\langle a\rangle \cap \langle b\rangle = 1$,we lead to a contradiction. So, G is a $p-$group. Since $\Gamma_{CSI}(G)$ is complete, there is a unique subgroup of order p and by mentioned well-known result G is cyclic or a generalized quaternion $2-$group. □

Lemma 2.5. *Let G be a finite group. Then $\alpha(\Gamma_{CSI}(G))$ is the number of cyclic subgroups of a prime order.*

Proof. Suppose $\alpha = \alpha(\Gamma_{CSI}(G))$, $\{\langle a_1\rangle, \ldots, \langle a_k\rangle\}$ is the set of all cyclic subgroups of G of a prime order and $B = \{\langle b_1\rangle, \ldots, \langle b_\alpha\rangle\}$ is a given independent set of largest possible size for G. Since A is an independent set for $\Gamma_{CSI}(G)$, $k \leqslant \alpha$. Choose i, $1 \leqslant i \leqslant \alpha$, and element c_i of a prime order such that $\langle c_i\rangle \subseteq \langle b_i\rangle$. Since B is an independent set, $c_i \neq c_j$, when $i \neq j$. This shows that $\alpha \leqslant k$, proving the lemma. □

Suppose m and n are positive integers. Define:

$$I_{m,n} = \{(a,b,t) \in N^2 \times N_0 \mid a|m,\ b|n,\ 0 \leqslant t \leqslant (a, \frac{n}{b}) - 1\},$$

$$H_{a,b,t} = \{(ia + \frac{jta}{(a, \frac{n}{b})}, jb) \mid 0 \leqslant i \leqslant \frac{m}{a} - 1,\ 0 \leqslant j \leqslant \frac{n}{b} - 1\},\quad (a,b,t) \in I_{m,n}.$$

For the sake of completeness, we mention here a result in [3] which is crucial in our next result. If $ns = 0$ then we define $(mb, na, ns) = (mb, na)$.

Theorem 2.6. [3, Theorem 2] *Suppose* $s = \frac{ta}{(a, \frac{n}{b})}$. *Then,*

1. $H \leqslant Z_m \times Z_n$ *if and only if there exists* $(a, b, t) \in I_{m,n}$ *such that* $H = H_{a,b,t}$.

2. $H_{a,b,t}$ *is cyclic if and only if* $ab = (mb, na, ns)$.

3. *The number of cyclic subgroups in* $Z_m \times Z_n$ *is* $\sum_{a|m, b|n, (\frac{m}{a}, \frac{n}{b})=1} (a, b)$.

Theorem 2.7. *Suppose* $G = Z_{p^\alpha} \times Z_{p^\beta}$, *where* p *is prime and* α, β *are two positive integers such that* $\alpha \leqslant \beta$. *Then* $\Gamma_{CSI}(G)$ *is a union of the complete graphs* $K_{(\beta-\alpha)p^\alpha + \frac{p^\alpha - 1}{p-1}}$ *and* p *copies of* $K_{\frac{p^\alpha - 1}{p-1}}$.

Proof. By definition of $H_{a,b,t}$ and Theorem 2.6, it can easily see that for each d, $1 \leqslant d \leqslant p - 1$, the subgroups

$$H_{p^\alpha, p^{\beta-1}, d};$$
$$H_{p^\alpha, p^{\beta-k}, t}, 2 \leqslant k \leqslant \alpha, t = d, p + d, \ldots, (p-1)p + d;$$
$$H_{p^\alpha, p^{\beta-k'}, t}, 3 \leqslant k' \leqslant \alpha, t = p^2 + d, p^2 + p + d, \ldots, p^2 + (p^{k'-1} + p - 1)p + d,$$

are cyclic subgroups containing $(dp^{\alpha-1}, p^{\beta-1})$ which gives $p-1$ cliques isomorphic to $K_{\frac{p^\alpha-1}{p-1}}$. These complete subgraphs are denoted by $\Gamma_1, \ldots, \Gamma_{p-1}$. On the other hand, the cyclic subgroups $H_{p^k, p^\beta, 0}$, $0 \leqslant k \leqslant \alpha-1$ and $H_{p^{\alpha-l}, p^{\beta-k'}, t}$, $1 \leqslant k' \leqslant \alpha-1$, $1 \leqslant l \leq \alpha - k'$, $1 \leqslant t \leqslant p^{k'} - 1$, $t \not\equiv 0 (mod\ p)$ have a common element $(p^{\alpha-1}, 0)$ and so we will have another clique of size $\frac{p^\alpha-1}{p-1}$. Note that for each k', there are $(\alpha - k')(p^{k'} - p^{k'-1})$ cyclic subgroups $H_{p^{\alpha-l}, p^{\beta-k'}, t}$ that gives a clique of size $\frac{p^\alpha-1}{p-1}$. The complete subgraph induced by this clique is denoted by Γ_p. We now consider the cyclic subgroups $H_{p^\alpha, p^k, t}$, $0 \leqslant k \leqslant \beta - \alpha - 1$, $0 \leqslant t \leqslant p^\alpha - 1$ and the cyclic subgroups $H_{p^\alpha, p^{\beta-k}, t}$, $1 \leqslant k \leqslant \alpha$, $0 \leqslant t \leqslant p^k - 1$ and $t \equiv 0\ (mod\ p)$. These are $(\beta - \alpha)p^\alpha + \frac{p^\alpha-1}{p-1}$ cyclic subgroups containing element $(0, p^{\beta-1})$ which gives us a clique of order $(\beta-\alpha)p^\alpha + \frac{p^\alpha-1}{p-1}$. Define Γ_{p+1} to be the complete subgraph induced by the las clique. By Theorem 2.6(3), these are all cyclic subgroups of $Z_{p^\alpha} \times Z_{p^\beta}$ and we have to show that $\Gamma_{CSI}(Z_{p^\alpha} \times Z_{p^\beta})$ is the union of $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_{p+1}$.

To complete the proof, we will shows that there is no edge in $\Gamma_{CSI}(Z_{p^\alpha} \times Z_{p^\beta})$ connecting a vertex in Γ_i to a vertex in Γ_j, $i \neq j$. Suppose vertices $v_1 = \langle a_1 \rangle \in V(\Gamma_i)$ and $v_2 = \langle a_2 \rangle \in V(\Gamma_j)$ that are not adjacent in $\Gamma_{CSI}(G)$. We prove that there is no vertex $u_1 = \langle b_1 \rangle$ in Γ_i to be adjacent with a vertex $u_2 = \langle b_2 \rangle$ in Γ_j. If u_1 and u_2 are adjacent in $\Gamma_{CSI}(G)$ then b_1 and b_2 will be adjacent in $\Gamma_{SI}(G)$ and since $\Gamma_{SI}(G)$ is a union of complete graphs, a_1 and $a)2$ will be adjacent in $\Gamma_{SI}(G)$

and so v_1 and v_2 are adjacent in $\Gamma_{CSI}(G)$ which is impossible. To complete our argument, we consider the following cyclic subgroups:

$$H_{p^\alpha,p^{\beta-1},d} = \{(jdp^{\alpha-1}, jp^{\beta-1}) \mid 0 \leqslant j \leqslant p-1\};\ (1 \leqslant d \leqslant p-1),$$
$$H_{p^k,p^\beta,0} = \{(ip^k, 0) \mid 0 \leqslant i \leqslant p^{\alpha-k} - i\},$$
$$H_{p^\alpha,p^{\beta-1},0} = \{(0, jp^{\beta-1}) \mid 0 \leqslant j \leqslant p^\beta - 1\}.$$

We now prove that these vertices are not adjacent. Suppose $(jdp^{\alpha-1}, jp^{\beta-1}) \in H_{p^k,p^\beta,0} \cap H_{p^\alpha,p^{\beta-1},d}$. Then $jp^{\beta-1} = 0$, $0 \leqslant j \leqslant p-1$, and so $j = 0$. This shows that $(jdp^{\alpha-1}, jp^{\beta-1}) = (0,0)$. It is also clear that $H_{p^k,p^\beta,0} \cap H_{p^\alpha,p^{\beta-1},0} = \{(0,0)\}$. If $(jdp^{\alpha-1}, jp^{\beta-1}) \in H_{p^\alpha,p^{\beta-1},0}$, $1 \leqslant j, d \leqslant p-1$, then $j = 0$ and so $H_{p^\alpha,p^{\beta-1},0} \cap H_{p^\alpha,p^{\beta-1},d} = \{(0,0)\}$.

We now assume that $d' \neq d$. Choose a common element in two cyclic subgroups of the first type, say $(jdp^{\alpha-1}, jp^{\beta-1}) = (j'd'p^{\alpha-1}, j'p^{\beta-1})$. Then $j\beta^{p-1} \equiv j'\beta^{p-1}$, where $0 \leqslant j, j' \leqslant p-1$. Thus $j = j'$ and since $jdp^{\alpha-1} \equiv jd'p^{\alpha-1}$ $(mod\ p^\alpha)$. Therefore, $d = d'$ which completes our proof. □

Theorem 2.8. *Suppose $G = Z_{p^\alpha} \times Z_{p^\beta}$, where p is prime and α, β are two positive integers such that $\alpha \leqslant \beta$. Then $\Gamma_{SI}(G)$ is a union of the complete graphs $K_{p^{\alpha+\beta} - \frac{p^{2\alpha+1}+1}{p+1}}$ and p copies of $K_{\frac{p^{2\alpha}-1}{p+1}}$.*

Proof. By Theorem 2.7, the graph $\Gamma_{CSI}(Z_{p^\alpha} \times Z_{p^\beta})$ is a union of $p+1$ complete graph and by definition of Γ_{SI} and Γ_{CSI}, a given component of Γ_{SI} is constructed from a component of Γ_{CSI} by adding some vertices corresponding to generators of vertices in Γ_{CSI}. So the components of Γ_{SI} will also be a complete graph. Suppose $1 \leqslant d \leqslant p-1$. By the proof of Theorem 2.7, the vertices of $p-1$ components of Γ_{CSI} are as follows:

$$H_{p^\alpha,p^{\beta-1},d} = \{(jtp^{\alpha-1}, jp^{\beta-1});\ 0 \leqslant j \leqslant p-1\},$$
$$H_{p^\alpha,p^{\beta-k},t} = \{(jtp^{\alpha-k}, jp^{\beta-t}),\ 0 \leqslant j \leqslant p^k - 1\},$$
$$H_{p^\alpha,p^{\beta-k'},t'} = \{(jt'p^{\alpha-k'}, jp^{\beta-k'}),\ 0 \leqslant j \leqslant p^{k'} - 1\},$$

where $t \in A = \{d, p+d, \ldots, (p-1)p+d\}$, $2 \leqslant k \leqslant \alpha$ and
$t' \in B = \{p^2+d, \ldots, p^2+(p^{k'-1}-p-1)p+d\}$, $3 \leqslant k' \leqslant \alpha$.

On the other hand, $|H_{p^\alpha,p^{\beta-1},d}| = p$, $|H_{p^\alpha,p^{\beta-k},t}| = p^k$, $|A| = p$, $|H_{p^\alpha,p^{\beta-k'},t'}| = p^{k'}$, $|B| = p^{k'} - p$ and by considering the number of generators, we will have $p-1$ complete graph $K_{\frac{p^{2\alpha}-1}{p+1}}$.

By the proof of Theorem 2.7, the cyclic subgroups

$$H_{p^k,p^\beta,0} = \{(ip^k, 0),\ 0 \leqslant i \leqslant p^{\alpha-k} - 1\}, 0 \leqslant k \leqslant \alpha - 1,$$
$$H_{p^{\alpha-l},p^{\beta-k},t} = \{(ip^{\alpha-l} + jtp^{\alpha-l-k}, jp^{\beta-k}),\ 0 \leqslant i \leqslant p^l - 1,\ 0 \leqslant j \leqslant p^k - 1\}$$

constitutes a $\frac{p^\alpha-1}{p-1}-$vertex component of $\Gamma_{CSI}(Z_{p^\alpha}\times Z_{p^\beta})$. By an easy calculation, one can see that the number of generators of vertices are equal to $\frac{p^{2\alpha}-1}{p+1}$. We now consider the component $K_{(\beta-\alpha)p^\alpha+\frac{p^\alpha-1}{p-1}}$ of $\Gamma_{CSI}(G)$ with the following vertices:

$$H_{p^\alpha,p^k,t} = \{(jt, jp^{\beta-(\alpha+1)}) \mid 0 \leqslant j \leqslant p^{\beta-k}-1\}$$
$$H_{p^\alpha,p^{\beta-k'},t'} = \{(jt'p^{\alpha-k'}, jp^{\beta-k'}),\ 0 \leqslant j \leqslant p^{k'}-1\},$$

where $0 \leqslant k \leqslant \beta-\alpha+1,\ 0 \leqslant t \leqslant p^\alpha-1,\ 1 \leqslant k' \leqslant \alpha,\ 0 \leqslant t' \leqslant p^{k'}-1$ and $t' \equiv 0 \pmod p$. By counting the number of generators, a component isomorphic to $K_{p^{\alpha+\beta}-\frac{p^{2\alpha+1}+1}{p+1}}$ is obtained, as desired. □

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