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On the lattice of congruences on completely regular semirings

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Abstract. A semiring S is called completely regular if it is the disjunctive union of its subrings. If S is a completely regular semiring, then the Green's relation $\mathcal{H}^+$ is a congruence on S and $S/\mathcal{H}^+$ is an idempotent semiring. Let $\mathcal{V}$ be a variety of idempotent semirings. Here we characterize the lattice $C(S)$ of all congruences on S when S is completely regular and $S/\mathcal{H}^+ \in \mathcal{V}$. The lattice $C(S)$ can be embedded into the product of the lattice $\mathcal{V}(S)$ of all $\mathcal{V}$-congruences on S and the lattice $M(S)$ of all additive idempotent-separating congruences on S if and only if S is τ-modular completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$.

1. Introduction

A semigroup is called completely regular if it is the (disjunctive) union of its subgroups. Completely regular semigroups were introduced in [3] by A. H. Clifford, though he used the terminology 'semigroups admitting relative inverses' to refer to such semigroups. Such semigroups have been studied extensively. For an account of the theory of completely regular semigroups, we refer to the book [13].

A *semiring* is a $(2,2)$ algebra $(S,+,\cdot)$ such that both the additive reduct $(S,+)$ and the multiplicative reduct $(S,\cdot)$ are semigroups and connected by the distributive laws

$$x(y+z) = xy + xz,\ (x+y)z = xz + yz.$$

An element $e \in S$ is called an *additive idempotent* if $e + e = e$. The set of all additive idempotents of S is denoted by $E^+(S)$. A semiring S is called *additive regular* if the additive reduct $(S,+)$ is a regular semigroup. By an *idempotent semiring* we mean a semiring S such that both the additive reduct $(S,+)$ and $(S,\cdot)$ are bands. If moreover, the reduct $(S,+)$ is commutative then S is called a *b-lattice*. Also we refer to [5] for the undefined terms and notions in semirings and [6] for background on semigroups.

Let us denote the Green's relations $\mathcal{L},\mathcal{R},\mathcal{D}$ and $\mathcal{H}$ on the additive reduct $(S,+)$ by $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{D}^+$ and $\mathcal{H}^+$, respectively. Also we denote the $\mathcal{L}^+,\mathcal{R}^+,\mathcal{D}^+$ and $\mathcal{H}^+$ classes of $x \in S$ by L^+_x, R^+_x, D^+_x and H^+_x, respectively. A semiring S is called an *idempotent semiring* (*b-lattice, distributive lattice*) *of rings* if there is a congruence

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ρ on S such that the quotient semiring S/ρ is an idempotent semiring (b-lattice, distributive lattice) and each ρ-class is a ring.

Rings and distributive lattices both are semirings with commutative regular addition. So, it is interesting to consider the semirings which are subdirect products of rings and distributive lattices. Such semirings were studied by Bandelt and Petrich [1]. The study was continued by Ghosh [4] and he characterized the Clifford semirings, equivalently, the semirings which are strong distributive lattices of rings. Pastijn and Guo [12] proved that the semirings which are disjoint unions of rings form a variety and they established various structure theorems for such semirings. They proved that if S is a disjoint union of its subrings then $\mathcal{H}^+$ is an idempotent semiring congruence on S. The term 'completely regular semiring' was first used by Sen, Maity and Shum [16] to mean the semirings which are disjoint union of skew-rings (rings without commutativity of addition).

In [2], we establish several equivalent characterizations for the semirings which are the disjunctive unions of rings. Let $(S, +, \cdot)$ be the disjunctive union of its subrings. Then the additive reduct $(S, +)$ is the disjunctive union of its subgroups. For every $x \in S$, denote the zero in the subgroup $(H_x^+, +)$ of $(S, +)$ by x^o and the unique inverse of x in H_x^+ by x'. Then $x^o = x + x' = x' + x$. Hence S can be treated as an algebra $(S, +, \cdot, ')$ of type (2, 2, 1), where the reduct $(S, +, \cdot)$ is a semiring and the reduct $(S, +, ')$ is a completely regular semigroup. The following result is useful.

Lemma 1.1. [2] *Let S be a semiring. Then the following conditions are equivalent:*

(i) *S is the (disjunctive) union of its subrings;*

(ii) *for every $x, y \in S$ there exists unique $x' \in S$ such that*

$$x = x + x' + x,\ x + x' = x' + x,\ (x')' = x,\ x + y^o + x^o + y = x^o + y + x + y^o$$

and $xx^o = x^o$, *where* $x^o = x + x'$;

(iii) *$\mathcal{H}^+$ is an idempotent semiring congruence on S and each $\mathcal{H}^+$-class is a ring;*

(iv) *S is an idempotent semiring of rings.*

Definition 1.2. A semiring S is called *completely regular* if it satisfies either of the four equivalent conditions in Lemma 1.1.

Throughout the rest of this article, unless otherwise stated, S stands for a completely regular semiring.

It follows from a result of Kapp and Schneider [8] that the lattice $C(S)$ of all congruences on a semigroup S can be embedded in the product of certain sublattices if the semigroup S is completely simple. The problem of embedding the lattice $C(S)$ in a product of sublattices, when S is an arbitrary band of groups, was characterized by C. Spitznagel [17]. The principal tool used in these two texts is the τ-relation introduced by Reilly and Scheiblich [14]. In this last article, this relation is marked with θ ([14], Theorem 3.4). Also there are many other articles devoted to these directions [6], [7], [11].

The set of all congruences on a semiring S is a complete lattice, which we denote by $C(S)$. A sublattice L of $C(S)$ is called a modular sublattice if the lattice L is modular. It is well known that if the congruences in L commute then L is modular. The trace of a congruence ρ on a completely regular semiring S is defined by:

$$\text{tr } \rho = \rho \cap (E^+(S) \times E^+(S)).$$

Define a relation τ on the lattice $C(S)$ by: for $\rho, \sigma \in C(S)$,

$$\rho \tau \sigma \text{ if tr } \rho = \text{tr } \sigma.$$

In Section 2, we characterize completely regular semirings S in terms of the maximum additive idempotent-separating congruence on S. In Section 3, we show that each τ-class in the lattice $C(S)$ of all congruences on an additive regular semiring S contains at most one $\mathcal{V}$-congruence on S, where $\mathcal{V}$ is a variety of idempotent semirings. We also have a necessary and sufficient condition for the greatest element of each τ-class to be a $\mathcal{V}$-congruence. In Section 4, we prove that the lattice $C(S)$ of all congruences on a τ-modular completely regular semiring S can be embedded in a certain product lattice.

Now let us fix the following notations:

$C(S)$:	the lattice of all congruences on S;
$M(S)$:	the lattice of all additive idempotent-separating congruences on S;
$D^+(S)$:	the lattice of all congruences on S that are contained in $\mathcal{D}^+$;
$\mathcal{V}(S)$:	the lattice of all $\mathcal{V}$-congruences on S;
$\rho_{\mathcal{V}}$:	the minimum $\mathcal{V}$-congruence on S;
β:	the minimum idempotent semiring congruence on S;
δ:	the minimum b-lattice congruence on S;
η:	the minimum distributive lattice congruence on S;
μ:	the maximum additive idempotent-separating congruence on S.

2. Additive idempotent separating congruences

A congruence ρ on S is called *additive idempotent-separating* if each ρ-class contains at most one additive idempotent, i.e., for every $e, f \in E^+(S)$, $e\rho f$ implies $e = f$. In this section, we characterize a completely regular semiring S by the maximum additive idempotent-separating congruence μ on itself.

In [10], Lallement proved that on a regular semigroup S, a congruence ρ is idempotent separating if and only if $\rho \subseteq \mathcal{H}$ on S. Since S is a completely regular semiring, the additive reduct $(S, +)$ is a regular semigroup, and so it follows that $\mu \subseteq \mathcal{H}^+$.

Now we have the following result.

Theorem 2.1. *Let S be an additive regular semiring and $\mathcal{V}$ be a variety of idempotent semirings. Then the following statements are equivalent:*

(*i*) *S is a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$;*

(*ii*) *$\mu = \mathcal{H}^+ = \rho_{\mathcal{V}}$ and $x + y^o + x^o + y = x^o + y + x + y^o$ for every $x, y \in S$;*

(*iii*) *$\mu = \rho_{\mathcal{V}}$ and $x + y^o + x^o + y = x^o + y + x + y^o$ for every $x, y \in S$.*

Proof. Equivalence of (ii) and (iii) is trivial and so we omit the proof.

$(i) \Rightarrow (ii)$: Suppose that $a\mathcal{H}^+b$ in S. Then $a\rho_{\mathcal{V}}\mathcal{H}^+b\rho_{\mathcal{V}}$ in $S/\mathcal{H}^+$. Since each $\mathcal{H}^+$-class contains at most one additive idempotent, $a\rho_{\mathcal{V}} = b\rho_{\mathcal{V}}$. Hence $\mathcal{H}^+ \subseteq \rho_{\mathcal{V}}$ and it follows that $\mu \subseteq \mathcal{H}^+ \subseteq \rho_{\mathcal{V}}$. Now $S/\mathcal{H}^+ \in \mathcal{V}$ implies that $\rho_{\mathcal{V}} \subseteq \mathcal{H}^+$. Since $\mathcal{H}^+$ is an additive idempotent separating congruence, $\mathcal{H}^+ \subseteq \mu$. Thus $\mu = \mathcal{H}^+ = \rho_{\mathcal{V}}$.

$(ii) \Rightarrow (i)$: Suppose that the condition (ii) holds. Then $\mathcal{H}^+ = \rho_{\mathcal{V}}$ implies that $S/\mathcal{H}^+ \in \mathcal{V}$. Let H be an $\mathcal{H}^+$-class in S. Since $\mathcal{H}^+$ is an idempotent semiring congruence on S, H is an additive regular subsemiring of S and, by Lallement's Lemma, contains an additive idempotent. Hence $(H, +)$ is a group. Now for every $x, y \in S$, $x + y^o + x^o + y = x^o + y + x + y^o$ implies that $(H, +)$ is an abelian group. Thus H is a subring of S and so S is a completely regular semiring. □

Now we have the following immediate consequence. Though it is a particular case of the above lemma, but useful.

Corollary 2.2. *Let S be any additive regular semiring. Then the following statements are equivalent.*

(*i*) *S is a completely regular semiring (b-lattice of rings, distributive lattice of rings);*

(*ii*) *$\mu = \mathcal{H}^+ = \beta\ (\ \delta,\ \eta\ )$ and $x + y^o + x^o + y = x^o + y + x + y^o$ for every $x, y \in S$;*

(*iii*) *$\mu = \beta\ (\ \delta,\ \eta\ )$ and $x + y^o + x^o + y = x^o + y + x + y^o$ for every $x, y \in S$.*

3. The relation τ on C(S)

The relation τ on $C(S)$ has many interesting properties when S is a completely regular semiring. Before coming into the main features let us first prove some lemmas.

The proof of the following result is similar to Lemma 2.1 [15], still for the sake of completeness we would like to include a proof.

Lemma 3.1. *Let S be an additive regular semiring and α be an additive idempotent separating congruence on S. Then for every $\gamma \in C(S)$, $(\alpha \vee \gamma, \gamma) \in \tau$.*

Proof. Let $\gamma \in C(S)$. Consider the relation $h = \{(a,b) \in S \times S : (a\gamma, b\gamma) \in \mathcal{H}^+\}$ on S. Then h is an equivalence relation on S and $\mathcal{H}^+ \subseteq h$ and $\gamma \subseteq h$. Let $(e,f) \in h \cap (E^+(S) \times E^+(S))$. Then $(e\gamma)\ \mathcal{H}^+\ (f\gamma)$. Since $\mathcal{H}^+$ is an additive idempotent separating congruence, $e\gamma = f\gamma$ and hence $(e,f) \in \gamma \cap (E^+(S) \times E^+(S))$. Therefore $h \cap (E^+(S) \times E^+(S)) \subseteq \gamma \cap (E^+(S) \times E^+(S))$. Since α separates additive idempotents, $\gamma \subseteq \alpha \vee \gamma \subseteq \mathcal{H}^+ \vee \gamma \subseteq h$ and consequently $\gamma \cap (E^+(S) \times E^+(S)) = h \cap (E^+(S) \times E^+(S)) = (\alpha \vee \gamma) \cap (E^+(S) \times E^+(S))$. Therefore $(\alpha \vee \gamma, \gamma) \in \tau$. □

Since $\mathcal{H}^+$ is an additive idempotent separating congruence on a completely regular semiring, we have, in particular:

Corollary 3.2. *Let S be a completely regular semiring. Then for every $\alpha \in C(S)$, $(\alpha \vee \mathcal{H}^+, \alpha) \in \tau$.*

We omit the proof of the following result, since it is similar to the proof of Theorem 2.2 [15].

Lemma 3.3. *If S is an additive regular semiring, then the relation τ is a complete lattice congruence on $C(S)$.*

Let $\mathcal{V}$ be a variety of idempotent semirings. Then a congruence σ on an additive regular semiring S is a $\mathcal{V}$-congruence if and only if σ contains $\rho_{\mathcal{V}}$, the minimum $\mathcal{V}$-congruence on S. Therefore we have:

Theorem 3.4. *Let S be an additive regular semiring and $\mathcal{V}$ be a variety of idempotent semirings. Then each τ-class in $C(S)$ contains at most one $\mathcal{V}$-congruence on S. In addition, if S is a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$, then each τ-class contains exactly one $\mathcal{V}$-congruence.*

Proof. Let α, γ be two $\mathcal{V}$-congruences on S such that $(\alpha, \gamma) \in \tau$. Then $\rho_{\mathcal{V}} \subseteq \alpha$ and $\rho_{\mathcal{V}} \subseteq \gamma$. Let $x\alpha y$. Since $S/\rho_{\mathcal{V}}$ is an idempotent semiring it follows, by Lallement's Lemma, that there exist $e, f \in E^+(S)$ such that $e\rho_{\mathcal{V}}x$ and $f\rho_{\mathcal{V}}y$. Then $e\rho_{\mathcal{V}}x\alpha y\rho_{\mathcal{V}}f$ which implies that $(e,f) \in \alpha$. Since $(\alpha, \gamma) \in \tau$ it follows that $(e,f) \in \gamma$, and so $\rho_{\mathcal{V}} \subseteq \gamma$ implies that $x\gamma y$. Therefore $\alpha \subseteq \gamma$. Similarly we have $\gamma \subseteq \alpha$, and finally $\alpha = \gamma$.

Now suppose that S is a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then for every $\alpha \in C(S)$, $\rho_{\mathcal{V}} \subseteq \mathcal{H}^+ \subseteq \alpha \vee \mathcal{H}^+$ implies that $\alpha \vee \mathcal{H}^+$ is a $\mathcal{V}$-congruence. Also it follows from Lemma 3.1 that $\alpha \vee \mathcal{H}^+$ is in the τ-class of α. □

In particular, we have:

Corollary 3.5. *Let S be an additive regular semiring. Then each τ-class in $C(S)$ contains at most one idempotent semiring (b-lattice, distributive lattice) congruence. If moreover, S is a completely regular semiring (b-lattice of rings, distributive lattice of rings), then each τ-class contains exactly one idempotent semiring (b-lattice, distributive lattice) congruence.*

Following result shows that for every congruence α on S, the join $\alpha \vee \mathcal{H}^+$ in $C(S)$ gives important information about α.

Theorem 3.6. *Let S be a completely regular semiring and $\alpha, \gamma \in C(S)$. Then $(\alpha, \gamma) \in \tau$ if and only if $\alpha \vee \mathcal{H}^+ = \gamma \vee \mathcal{H}^+$.*

Proof. First suppose that $(\alpha, \gamma) \in \tau$. Then, by Corollary 3.2, $(\alpha \vee \mathcal{H}^+, \alpha) \in \tau$ and $(\gamma \vee \mathcal{H}^+, \gamma) \in \tau$ and it follows that $(\alpha \vee \mathcal{H}^+, \gamma \vee \mathcal{H}^+) \in \tau$. Also both $\alpha \vee \mathcal{H}^+$ and $\gamma \vee \mathcal{H}^+$ are idempotent semiring congruences and so, by Theorem 3.4, it follows that $\alpha \vee \mathcal{H}^+ = \gamma \vee \mathcal{H}^+$.

Conversely suppose that $\alpha \vee \mathcal{H}^+ = \gamma \vee \mathcal{H}^+$. Then $(\alpha \vee \mathcal{H}^+, \alpha) \in \tau$ and $(\gamma \vee \mathcal{H}^+, \gamma) \in \tau$ implies that $(\alpha, \gamma) \in \tau$. $\square$

Following result can be proved similarly to Theorem 3.4 (ii) [14]. So we omit the proof.

Lemma 3.7. *Let S be an additive regular semiring. Then each τ-class in $C(S)$ is a complete modular sublattice of $C(S)$ with the greatest and least elements.*

The following theorem gives a necessary and sufficient condition for the greatest element of each τ-class in $C(S)$ to be a $\mathcal{V}$-congruence on S, where $\mathcal{V}$ is a variety of idempotent semirings.

Theorem 3.8. *Let S be a completely regular semiring and $\mathcal{V}$ be a variety of idempotent semirings. Then the greatest element of each τ-class in $C(S)$ is a $\mathcal{V}$-congruence if and only if $S/\mathcal{H}^+ \in \mathcal{V}$.*

Proof. First suppose that S is a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then $\mathcal{H}^+ = \rho_{\mathcal{V}}$, by Theorem 2.1. Let $\alpha \in C(S)$ and γ be the greatest element of the τ-class of α. Now, by Corollary 3.2, $(\alpha \vee \mathcal{H}^+, \alpha) \in \tau$ and so $\mathcal{H}^+ \subseteq \alpha \vee \mathcal{H}^+ \subseteq \gamma$. Thus γ is a $\mathcal{V}$-congruence.

Conversely, suppose that the greatest element of each τ-class is a $\mathcal{V}$-congruence. Since μ is the greatest element of the τ-class of Δ_S, μ is a $\mathcal{V}$-congruence. Also, on every additive regular semiring, $\mu \subseteq \mathcal{H}^+ \subseteq \rho_{\mathcal{V}}$. Therefore $\mu = \mathcal{H}^+ = \rho_{\mathcal{V}}$ and it follows, by Theorem 2.1, that $S/\mathcal{H}^+ \in \mathcal{V}$. $\square$

Now we have the following important corollary.

Corollary 3.9. *Let S be a completely regular semiring. Then the greatest element of each τ-class is an idempotent semiring congruence.*

Moreover, the greatest element of each τ-class is a b-lattice (distributive lattice) congruence if and only if S is a b-lattice (distributive lattice) of rings.

4. Embedding of C(S) in a product lattice

Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then for every $\alpha \in C(S)$, $\mathcal{H}^+ \subseteq \alpha \vee \mathcal{H}^+$ implies that

$\alpha \vee \mathcal{H}^+ \in \mathcal{V}(S)$ and $\alpha \wedge \mathcal{H}^+ \subseteq \mathcal{H}^+$ implies that $\alpha \wedge \mathcal{H}^+$ is an additive idempotent separating congruence. Thus we have a mapping $\phi : C(S) \to \mathcal{V}(S) \times M(S)$ defined by: for every $\alpha \in C(S)$,

$$\phi(\alpha) = (\alpha \vee \mathcal{H}^+, \alpha \wedge \mathcal{H}^+).$$

Lemma 4.1. *Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then ϕ is one-to-one.*

Proof. Suppose that $\alpha, \gamma \in C(S)$ are such that $\phi(\alpha) = \phi(\gamma)$. Then $\alpha \vee \mathcal{H}^+ = \gamma \vee \mathcal{H}^+$ and $\alpha \wedge \mathcal{H}^+ = \gamma \wedge \mathcal{H}^+$. It follows, by Theorem 3.6, $(\alpha, \gamma) \in \tau$. Let $(x, y) \in \alpha$ and $e \in E^+(S) \cap H_x^+$, $f \in E^+(S) \cap H_y^+$. Then $e = x^o \ \alpha \ y^o = f$ implies that $(e, f) \in \alpha$, and so $(e, f) \in \gamma$. Now $x = (x + e)\gamma(x + f)$ and $y = (f + y)\gamma(e + y)$ together with $e\alpha f$ imply that $(e + y)\alpha(f + y) = y\alpha x = (x + e)\alpha(x + f)$, and so $(e + y)\alpha(x + f)$. Also $e \ \mathcal{H}^+ \ x$ and $y \ \mathcal{H}^+ \ f$ imply that $(e + y)\mathcal{H}^+(x + f)$. Thus $(e + y)\mathcal{H}^+ \wedge \alpha(x + f)$ and so $(e + y)\gamma \wedge \mathcal{H}^+(x + f)$. Hence $x\gamma(x + f)\gamma(e + y)\gamma y$ and it follows that $\alpha \subseteq \gamma$. Similarly $\gamma \subseteq \alpha$. Thus $\alpha = \gamma$. □

Theorem 4.2. *Let S be a completely regular semiring and $\alpha \in C(S)$. Then $\alpha = \bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)$, where $\bar{\alpha}$ is the smallest element of τ-class of α.*

Proof. Let $\alpha \in C(S)$. Then $\alpha \wedge \mathcal{H}^+$ is an additive idempotent separating congruence, which implies that $\alpha \wedge \mathcal{H}^+ \tau \mathcal{H}^+$ and so, by Corollary 3.2, $\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+) \ \tau \ \bar{\alpha} \vee \mathcal{H}^+ \ \tau \ \bar{\alpha} \ \tau \ \alpha$. Therefore, by Theorem 3.6, $\alpha \vee \mathcal{H}^+ = [\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)] \vee \mathcal{H}^+$. Now $\bar{\alpha}, \alpha \wedge \mathcal{H}^+ \subseteq \alpha$ implies that $[\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)] \wedge \mathcal{H}^+ \subseteq \alpha \wedge \mathcal{H}^+$. Also, $\alpha \wedge \mathcal{H}^+ \subseteq \bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+), \mathcal{H}^+$ and hence $\alpha \wedge \mathcal{H}^+ \subseteq [\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)] \wedge \mathcal{H}^+$. Thus we have $\alpha \wedge \mathcal{H}^+ = [\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)] \wedge \mathcal{H}^+$. Therefore $\phi(\alpha) = \phi(\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+))$ and so $\alpha = \bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)$. □

Theorem 4.3. *Let S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then ϕ is $\wedge$-preserving.*

Proof. We have $(\alpha \vee \mathcal{H}^+) \wedge (\gamma \vee \mathcal{H}^+) \ \tau \ (\alpha \wedge \gamma)$ and $(\alpha \wedge \gamma) \vee \mathcal{H}^+ \ \tau \ (\alpha \wedge \gamma)$. Then it follows that $(\alpha \wedge \gamma) \vee \mathcal{H}^+ \ \tau \ (\alpha \vee \mathcal{H}^+) \wedge (\gamma \vee \mathcal{H}^+)$. Since both $(\alpha \wedge \gamma) \vee \mathcal{H}^+$ and $(\alpha \vee \mathcal{H}^+) \wedge (\gamma \vee \mathcal{H}^+)$ are $\mathcal{V}$-congruences it follows, by Theorem 3.4, that $(\alpha \wedge \gamma) \vee \mathcal{H}^+ = (\alpha \vee \mathcal{H}^+) \wedge (\gamma \vee \mathcal{H}^+)$. Also we have $(\alpha \wedge \gamma) \wedge \mathcal{H}^+ = (\alpha \wedge \mathcal{H}^+) \wedge (\gamma \wedge \mathcal{H}^+)$. Therefore ϕ is $\wedge$-preserving. □

Corollary 4.4. *Let S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then $\mathcal{V}(S)$ is lattice isomorphic with $C(S)/\tau$.*

Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Denote the restriction of ϕ to $D^+(S)$ by $\widetilde{\phi}$. Thus the mapping $\widetilde{\phi} : D^+(S) \to \mathcal{V}(S) \times M(S)$ is given by: for every $\alpha \in D^+(S)$,

$$\widetilde{\phi}(\alpha) = (\alpha \vee \mathcal{H}^+, \alpha \wedge \mathcal{H}^+).$$

Theorem 4.5. *Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then $\widetilde{\phi}$ is $\vee$-preserving.*

Proof. Let $\alpha, \gamma \in D^+(S)$. Then $\alpha \wedge \mathcal{H}^+$, $\gamma \wedge \mathcal{H}^+ \subseteq (\alpha \vee \gamma) \wedge \mathcal{H}^+$ implies that $(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+) \subseteq (\alpha \vee \gamma) \wedge \mathcal{H}^+$. Suppose that $(x, y) \in (\alpha \vee \gamma) \wedge \mathcal{H}^+$. Then $(x, y) \in \alpha \vee \gamma$ implies that there exists a positive integer n and $x_i, y_i \in S$, $i = 1, 2, \ldots, n$ such that

$$x\alpha x_1 \gamma y_1 \alpha x_2 \gamma y_2 \alpha \cdots \alpha x_n \gamma y_n = y.$$

Since $\alpha, \gamma \subseteq \mathcal{D}^+$, it follows that x_i, $y_i \in D_x^+ = D_y^+$. Also $x\mathcal{H}^+ y$ implies that $H_x^+ = H_y^+$. Suppose that e is the identity element in H_x^+. Then

$$x = (e+x+e)\alpha(e+x_1+e)\tau(e+y_1+e)\alpha \cdots \alpha(e+x_n+e)\gamma(e+y_n+e) = (e+y+e) = y.$$

Since $D_x^+ = D_y^+$ is a completely simple semiring, $e + x_i + e$, $e + y_i + e \in e + D_x^+ + e = H_x^+$ for each i. Therefore we have $(x, y) \in (\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)$ and so $(\alpha \vee \gamma) \wedge \mathcal{H}^+ \subseteq (\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)$. Hence $(\alpha \vee \gamma) \wedge \mathcal{H}^+ = (\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)$. Also we have $(\alpha \vee \gamma) \vee \mathcal{H}^+ = (\alpha \vee \mathcal{H}^+) \vee (\gamma \vee \mathcal{H}^+)$. This completes the proof. $\square$

Let L be a lattice, and ζ a lattice congruence on L. We say that L is ζ- *modular* if for every $a, b, c \in L$, the conditions $a \geq b$, $(a, b) \in \zeta$, $a \wedge c = b \wedge c$ and $a \vee c = b \vee c$, imply that $a = b$. A semiring S is said to be τ-modular if the lattice $C(S)$ of all congruences on S is τ-modular. Thus:

Definition 4.6. *A semiring S is called τ-modular if for every $\rho, \sigma, \xi \in C(S)$, the conditions $\sigma \subseteq \rho$, $\sigma\tau\rho$, $\rho \wedge \xi = \sigma \wedge \xi$ and $\rho \vee \xi = \sigma \vee \xi$ imply that $\rho = \sigma$.*

Lemma 4.7. *Let S be a τ-modular completely regular semiring. Then for every $\alpha, \gamma \in C(S)$, $\alpha \vee [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)] = \alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]$.*

Proof. Let $\alpha, \gamma \in C(S)$. Then $(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+) \subseteq (\alpha \vee \gamma) \wedge \mathcal{H}^+$ implies that $\alpha \vee [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)] \subseteq \alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]$. Also $\alpha \vee [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)] = [\alpha \vee (\alpha \wedge \mathcal{H}^+)] \vee (\gamma \wedge \mathcal{H}^+) = \alpha \vee (\gamma \wedge \mathcal{H}^+)$. Now $(\gamma \wedge \mathcal{H}^+)\tau(\alpha \vee \gamma) \wedge \mathcal{H}^+$ implies that $[\alpha \vee (\gamma \wedge \mathcal{H}^+)]\ \tau\ \alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]$. Thus by τ-modularity, it suffices to show $\gamma \vee [\alpha \vee (\gamma \wedge \mathcal{H}^+)] = \gamma \vee [\alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]]$ and $\gamma \wedge [\alpha \vee (\gamma \wedge \mathcal{H}^+)] = \gamma \wedge [\alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]]$. Now $\gamma \vee \alpha \subseteq \gamma \vee [\alpha \vee (\gamma \wedge \mathcal{H}^+)] \subseteq \gamma \vee [\alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]] \subseteq \gamma \vee [\alpha \vee (\alpha \vee \gamma)] = \gamma \vee (\alpha \vee \gamma) = \gamma \vee \alpha$, implies the first equality. For the other equality, we have $\gamma \wedge [\alpha \vee (\gamma \wedge \mathcal{H}^+)] \subseteq \gamma \wedge [\alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+)]] \subseteq \gamma \wedge [\alpha \vee \mathcal{H}^+]$. Hence it suffices to show that $\gamma \wedge [\alpha \vee \mathcal{H}^+] \subseteq \gamma \wedge [\alpha \vee (\gamma \wedge \mathcal{H}^+)]$. For this, it is sufficient to show that $\gamma \wedge (\alpha \vee \mathcal{H}^+) \subseteq \alpha \vee (\gamma \wedge \mathcal{H}^+)$. Suppose that $(x, y) \in \gamma \wedge (\alpha \vee \mathcal{H}^+)$. Then $(x^o, y^o) \in \alpha \vee \mathcal{H}^+$. Since $(\alpha, \alpha \vee \mathcal{H}^+) \in \tau$, we have $(x^o, y^o) \in \alpha$. Thus $x^o = (x^o + x^o)\ \alpha\ (x^o + y^o)\ \mathcal{H}^+\ (x + y)\ \mathcal{H}^+\ (x + y)^o$, so that $(x^o, (x + y)^o) \in \alpha \vee \mathcal{H}^+$. Then $(\alpha, \alpha \vee \mathcal{H}^+) \in \tau$ implies that $(x^o, (x + y)^o) \in \alpha$, and so $(y^o, (x + y)^o) \in \alpha$. Since the $\mathcal{D}^+$-class $D_{(x+y)}$ is completely simple, we have $((x+y)^o + x + (x+y)^o)\ \mathcal{H}^+\ ((x + y)^o + y + (x + y)^o)$. Thus $x = (x^o + x + x^o)\ \alpha\ ((x + y)^o + x + (x + y)^o)\ \gamma\ \wedge \mathcal{H}^+ ((x + y)^o + y + (x + y)^o)\ \alpha\ (y^o + y + y^o) = y$, and so $(x, y) \in \alpha \vee (\gamma \wedge \mathcal{H}^+)$. This completes the proof. $\square$

Theorem 4.8. *Let $\mathcal{V}$ be a variety of idempotent semirings and S be a τ-modular completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then ϕ is $\vee$-preserving.*

Proof. Let $\alpha, \gamma \in C(S)$. Then $(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+) \subseteq (\alpha \vee \gamma) \wedge \mathcal{H}^+$. Since both $(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)$, $(\alpha \vee \gamma) \wedge \mathcal{H}^+$ are contained in $\mathcal{H}^+$, $(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+) \ \tau \ (\alpha \vee \gamma) \wedge \mathcal{H}^+$. Therefore, by τ-modularity, it suffices to show that $\alpha \vee [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)] = \alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]$ and $\alpha \wedge [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)] = \alpha \wedge [(\alpha \vee \gamma) \wedge \mathcal{H}^+]$. First equality holds by Lemma 4.7. Since $\alpha \wedge \mathcal{H}^+ \subseteq (\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)$, we have $\alpha \wedge \mathcal{H}^+ = \alpha \wedge (\alpha \wedge \mathcal{H}^+) \subseteq \alpha \wedge [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)) \subseteq \alpha \wedge [(\alpha \vee \gamma) \wedge \mathcal{H}^+] = [\alpha \wedge (\alpha \vee \gamma)] \wedge \mathcal{H}^+] = \alpha \wedge \mathcal{H}^+$. Therefore $(\rho \vee \gamma) \wedge \mathcal{H}^+ = (\rho \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)$. Also we have $(\rho \vee \gamma) \vee \mathcal{H}^+ = (\rho \vee \mathcal{H}^+) \vee (\gamma \vee \mathcal{H}^+)$. This completes the proof. □

Theorem 4.9. *Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring. If $\phi : C(S) \longrightarrow \mathcal{V}(S) \times M(S)$ is an embedding, then S is τ-modular.*

Proof. Let σ, ρ be two congruences on S such that $\sigma \subseteq \rho$, $\sigma \tau \rho$ and ξ is a congruence such that $\sigma \vee \xi = \rho \vee \xi$ and $\sigma \wedge \xi = \rho \wedge \xi$. Clearly $\sigma \wedge \mathcal{H}^+ \subseteq \rho \wedge \mathcal{H}^+$, and since ϕ is an embedding, we have $(\sigma \wedge \mathcal{H}^+) \vee (\xi \wedge \mathcal{H}^+) = (\sigma \vee \xi) \wedge \mathcal{H}^+ = (\rho \vee \xi) \wedge \mathcal{H}^+ = (\rho \wedge \mathcal{H}^+) \vee (\xi \wedge \mathcal{H}^+)$. Also, $(\sigma \wedge \mathcal{H}^+) \wedge (\xi \wedge \mathcal{H}^+) = (\sigma \wedge \xi) \wedge \mathcal{H}^+ = (\rho \wedge \xi) \wedge \mathcal{H}^+ = (\rho \wedge \mathcal{H}^+) \wedge (\xi \wedge \mathcal{H}^+)$. Since, by Lemma 3.7, the τ-class of $\mathcal{H}^+$ is a modular sublattice of $C(S)$, we have $\sigma \wedge \mathcal{H}^+ = \rho \wedge \mathcal{H}^+$. Also $\sigma \ \tau \ \rho$ implies that $\sigma \vee \mathcal{H}^+ = \rho \vee \mathcal{H}^+$, by Theorem 3.6. Since ϕ is one-to-one, $\sigma = \rho$. Thus S is τ-modular. □

Now combining Theorem 4.1, 4.3, 4.8 and 4.9, we get the following result.

Theorem 4.10. *Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then the function $\phi : C(S) \to \mathcal{V}(S) \times M(S)$ defined by $\phi(\alpha) = (\alpha \vee \mathcal{H}^+, \alpha \wedge \mathcal{H}^+)$ is an embedding if and only if S is τ-modular.*

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