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Unit and unitary Cayley graphs for the ring of Gaussian integers modulo n

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Abstract. Let $\mathbb{Z}_n[i]$ be the ring of Gaussian integers modulo n and $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ be the unit graph and the unitary Cayley graph of $\mathbb{Z}_n[i]$, respectively. In this paper, we study $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$. Among many results, it is shown that for each positive integer n, the graphs $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are *Hamiltonian*. We also find a necessary and sufficient condition for the unit and unitary Cayley graphs of $\mathbb{Z}_n[i]$ to be Eulerian.

1. Introduction

Finding the relationship between the algebraic structure of rings using properties of graphs associated to them has become an interesting topic in the last years. There are many papers on assigning a graph to a ring, see [1], [3], [4], [5], [7], [6], [8], [10], [11], [12], [17], [19], and [20].

Let R be a commutative ring with non-zero identity. We denote by U(R), J(R) and Z(R) the group of units of R, the Jacobson radical of R and the set of zero divisors of R, respectively. The *unitary Cayley graph* of R, denoted by G_R, is the graph whose vertex set is R, and in which $\{a, b\}$ is an edge if and only if $a - b \in U(R)$. The unit graph $G(R)$ of R is the *simple graph* whose vertices are elements of R and two distinct vertices a and b are adjacent if and only if a + b in U(R) . There are many papers where these two concepts have been discussed. See for examples [4], [8], [19], [20], [22] and [23].

The following facts are well known, see for examples Silverman (2006), [2] and [16]. The set of all complex numbers $a + ib$, where a and b are integers, form an Euclidean domain with the usual complex number operations and Euclidian norm $N(a+ib) = a^2+b^2$. This domain will be denoted by $\mathbb{Z}[i]$ and will be called the ring of Gaussian integers. Let n be a natural number and let (n) be the principal ideal generated by n in $\mathbb{Z}[i]$, and let $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{n-1}\}$ be the ring of integers modulo n. Then the factor ring $\frac{\mathbb{Z}[i]}{(n)}$ is isomorphic to $\mathbb{Z}_n[i]$, which implies that $\mathbb{Z}_n[i]$ is a principal ideal ring. The ring $\mathbb{Z}_n[i]$ is called the ring of *Gaussian integers modulo n*. Let p be a prime integer. Then $p \equiv 1 (\text{mod } 4)$ if and only if there are integers a , b such that $p = a^2 + b^2$ if and only if there exists an integer c such that

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$c^2 \equiv -1 (\mathrm{mod}\ p)$. Moreover, if n is a natural number, then there exist integers a and b, relatively prime to p such that $p^n = a^2 + b^2$, and there exists an integer z such that $z^2 \equiv -1 (\mathrm{mod}\ p^n)$. It was shown that $\overline{a} + i\overline{b}$ is a unit in $\mathbb{Z}_n[i]$ if and only if $\overline{a}^2 + \overline{b}^2$ is a unit in $\mathbb{Z}_n$. If $n = \prod_{j=1}^{s} a_j^{k_j}$ is the prime power decomposition of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings $\mathbb{Z}_{a_j^{k_j}}[i]$. Also if $m = t^k$ for some prime t and positive integer k, then $\mathbb{Z}_m[i]$ is local ring if and only if $t = 2$ or $t \equiv 3 (\mathrm{mod}\ 4)$.

In this article, some properties of the graphs $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are studied. The diameter, the girth, chromatic number, clique number and independence number, in terms of n, are found. Moreover, we prove that for each $n > 1$, the graphs $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are *Hamiltonian*. We also find a necessary and sufficient condition for the unit and unitary Cayley graphs of $\mathbb{Z}_n[i]$ to be Eulerian.

A *local ring* is a ring with exactly one maximal ideal. A local ring with finitely many maximal ideals is called $semi-local$ ring. For classical theorem and notations in commutative algebra, the interested reader is referred to [9].

Throughout this paper all graphs are simple (with no loop and multiple edges). For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. The set of vertices adjacent to a vertex v in the graph G is denoted by $N(v)$. The degree $deg(v)$ of a vertex v in the graph G is the number of edges of G incident with v. The graph G is called $k-regular$ if all vertices of G have degree k, where k is a fixed positive integer. A *walk* (of length k) in a graph G between two vertices a, b is an alternating sequence $a = v_0, e_1, v_1, e_2, \ldots, e_k, v_k = b$ of vertices and edges in G, denoted by

$$a = v_0 \longrightarrow v_1 \longrightarrow \ldots \longrightarrow v_k = b,$$

such that $e_i = \{v_{i-1}, v_i\}$ for all $1 \leqslant i \leqslant k$. If the vertices in a walk are all distinct, it defines a *path* in G. A *trail* between two vertices a, b is a walk between a and b without repeated vertices. A *cycle* of a graph is a path such that the start and end vertices are the same. A *Hamiltonian* path (cycle) in G is a path (cycle) in G that visits every vertex exactly once. A graph is called *Hamitonian* if it contains a Hamiltonian cycle. Also a graph G is called *connected* if for any vertices a and b of a graph G there is a path between a and b. A connected graph G is called *Eulerian* if there exists a closed trail containing every edge of G. For distinct vertices a and b of a graph G, let $d(a, b)$ be the length of a shortest path from a to b; if no such paths exists, we set $d(a, b) = \infty$. The *diameter* of G is defined as

$$diam(G) = sup\{d(a, b); a, b \in V(G)\}.$$

The *girth* of G, denoted by $gr(G)$ is the length of a shortest cycle in G, ($gr(G) = \infty$ if G contain no cycle). For a positive integer r, a graph is called $r - partite$ if the vertex set admits a partition into r classes such that vertices in the same partition class are not adjacent. A $r-$partite graph is called *complete* if every two vertices in different parts are adjacent. The complete $2 - partite$ graph (also called the *complete bipartite* graph) with exactly two partitions of size n and m, is denoted by $K_{n,m}$. A graph G is called a *complete graph* if every two distinct vertices in G are adjacent. A complete graph with n vertices is denoted by K_n. A *clique* of a graph is a complete subgraph. A maximum clique is a clique of the largest possible size in a given graph. the *clique number*, $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G. An independent set in a graph is a set of pairwise non-adjacent vertices. The *independence number*, $\alpha(G)$ of a graph G is the size of a largest independet set of G. A subset M of the edge set of G, is called a *matching* in G if no two of the edges in M are adjacent. In other words, if for any two edges e and f in M, both the end vertices of e are different from the end vertices of f. A $perfect\ matching$ of a graph G is a matching of G containing $\frac{n}{2}$ edges, the largest possible, meaning perfect matchings are only possible on graphs with an even number of vertices. A perfect matching sometimes called a *complete matching* or $1-factor$. A *coloring* of a graph is a labeling of the vertices with colors such that no two adjacent vertices have the same color. The smallest number of colors need to color the vertices of a graph G is called its *chromatic number*, and denoted by $\chi(G)$. Let G_1 and G_2 be two vertex-disjoint graphs. The *tensor product* or *Kronecker product* of G_1 and G_2 is denoted by $G_1 \otimes G_2$. That is, $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$; two distinct vertices (a,b) and (c,d) are adjacent if and only if a is adjacent to c in G_1 and b adjacent to d in G_2. We refer the reader to [13] and [15] for general references on graph theory.

2. The unit and unitary Cayley graphs for $\mathbb{Z}_{t^n}[i]$

In this section we find the diameter and girth of the unit and unitary Cayley graphs of $\mathbb{Z}_{t^n}[i]$ where t is a prime integer. Three cases are considered: When $t = 2$, $t \equiv 3 (\mathrm{mod}\ 4)$ or $t \equiv 1 (\mathrm{mod}\ 4)$. In this article p and q will denote prime integers such that $p \equiv 1 (\mathrm{mod}\ 4)$ and $q \equiv 3 (\mathrm{mod}\ 4)$.

2.1. The unit and unitary Cayley graphs for $\mathbb{Z}_{2^n}[i]$

Proposition 2.1. [4, Proposition 2.2]

(*a*) *Let R be a ring. Then G_R is a regular graph of degree $|U(R)|$.*

(*b*) *Let S be a local ring with mamximal ideal m. Then G_S is a complete mutipartite graph whose partite sets are the cosets of m in S. In paticular, G_S is a compelete graph if and only if S is a field.*

Lemma 2.2. *For each positive integer n, $G_{\mathbb{Z}_{2^n}[i]}$ is a complete bipartite graph $K_{2^{2n-1},2^{2n-1}}$.*

Proof. For each positive integer n, $\mathbb{Z}_{2^n}[i]$ is a local ring with its only maximal ideal $m = (\bar{1} + \bar{1}i)$ and the number of units in $\mathbb{Z}_{2^n[i]}$ is 2^{2n-1}, see [2] and [14]. Since $|\frac{\mathbb{Z}_{2^n}[i]}{(\bar{1}+\bar{1}i)}| = 2$, by Proposition 2.1, we conclude that $G_{\mathbb{Z}_{2^n}[i]}$ is a complete bipartite graph $K_{2^{2n-1},2^{2n-1}}$. $\square$

Lemma 2.3. [20, Lemma 4.1] *Let R be a finite ring. For $j \in R$, the following statements are equivalent:*

(*a*) $j \in J_R$

(*b*) $j + u \in U(R)$ *for any* $u \in U(R)$.

Theorem 2.4. [19, Theorem 2.6] *Let R be a finite ring. Then the following statements hold.*

(*a*) *If* (R, m) *is a local ring of even order, then* $G(R) \cong G_R$.

(*b*) *If R is a ring of odd order, then* $G(R) \ncong G_R$

Proposition 2.5. [19, Corollary 2.3] *Let R be a finite ring. Then $2 \in U(R)$ if and only if $|R|$ is odd.*

Theorem 2.6. *Let R be a finite ring and $R \cong R_1 \times R_2 \times \cdots \times R_n$. Then the following statements are equivalent:*

(*i*) $2 \in J(R)$.

(*ii*) $G_R = G(R)$.

(*iii*) *For every i with $1 \leqslant i \leqslant n$, $|R_i|$ is even.*

Proof. Let $2 \in J(R)$ and a, b be two distict elements of R. Since

$$(a - b) + (a + b) = 2a.$$

By Lemma 2.3,

$$a - b \in U(R) \quad \text{if and only if} \quad a + b \in U(R).$$

This means that $G_R = G(R)$.

Now supoose that $G_R = G(R)$ and $R \cong R_1 \times R_2 \times \cdots \times R_n$. If $n = 1$ then by Proposition 2.5, we deduce that $|R|$ is even. Now assume that $n > 1$. Since $G_R = G(R)$, we have for every i with $1 \leqslant i \leqslant n$, $G_{R_i} = G(R_i)$. Hence by the first case, for every i with $1 \leqslant i \leqslant n$, $|R_i|$ is even.

Finally, if for every i with $1 \leqslant i \leqslant n$, $|R_i|$ be even. Then by Proposition 2.5, we have $2 \notin U(R_i)$; $1 \leqslant i \leqslant n$. This implies that $2 \in J(R_i)$, and therefore $2 \in J(R)$. This completes the proof. □

Corollary 2.7. *For each positive integer n,* $G_{\mathbb{Z}_{2^n}[i]} = G(\mathbb{Z}_{2^n}[i])$.

Proof. Since $|\mathbb{Z}_{2^n}[i]|$ is even, by Proposition 2.5, we have $2 \notin U(\mathbb{Z}_{2^n}[i])$. Therefore $2 \in J(\mathbb{Z}_{2^n}[i])$. By using Theorem 2.6 we conclude that $G_{\mathbb{Z}_{2^n}[i]} = G(\mathbb{Z}_{2^n}[i])$. □

Corollary 2.8. *Let n be a positive integer. Then the following statements hold:*

(*i*) $diam(G_{\mathbb{Z}_{2^n}[i]}) = diam(G(\mathbb{Z}_{2^n}[i])) = 2$

(ii) $gr(G_{\mathbb{Z}_{2^n}[i]}) = gr(G(\mathbb{Z}_{2^n}[i])) = 4$.

Proof. For each positive integer n, $G_{\mathbb{Z}_{2^n}} = G(\mathbb{Z}_{2^n}[i])$ is a complete bipartite graph with $|\mathbb{Z}_{2^n}[i]| \geq 4$, Hence

$$diam(G_{\mathbb{Z}_{2^n}[i]}) = diam(G(\mathbb{Z}_{2^n}[i])) = 2$$

and

$$gr(G_{\mathbb{Z}_{2^n}[i]}) = gr(G(\mathbb{Z}_{2^n}[i])) = 4. \qquad \square$$

2.2. The unit and unitary Cayley graphs for $\mathbb{Z}_{q^n}[i], q \equiv 3(\mathrm{mod} 4)$

Theorem 2.9. *Let n be a positive integer. Then the following statements hold:*

(i) $G_{\mathbb{Z}_{q^n}[i]}$ *is a complete q^2- partite graph.*

(ii) $G_{\mathbb{Z}_{q^n}[i]} \ncong G(\mathbb{Z}_{q^n}[i])$

Proof. If $q \equiv 3(\mathrm{mod}\ 4)$, then $\mathbb{Z}_q[i]$ is a field with q^2 elements see [2]. By Proposition 2.1, $G_{\mathbb{Z}_q[i]}$ is a complete graph with q^2 vertices. If $n > 1$, then $\mathbb{Z}_{q^n[i]} \cong \frac{\mathbb{Z}[i]}{(q^n)}$ is a local ring with maximal ideal $m = (\overline{q})$ see [2]. Also, the number of units in $\mathbb{Z}_{q^n}[i]$ is $q^{2n} - q^{2n-2}$, see [14]. Clearly, $|\frac{\mathbb{Z}_{q^n}[i]}{m}| = q^2$. Hence by proposition 2.1, $G_{\mathbb{Z}_{q^n}[i]}$ is a complete q^2- partite graph.

Since $|\mathbb{Z}_{q^n}[i]|$ is odd, by Theorem 2.4, $G_{\mathbb{Z}_{q^n}[i]} \ncong G(\mathbb{Z}_{q^n}[i])$ $\square$

Corollary 2.10. *For each positive integer n, the following statements hold:*

(i) $diam(G_{\mathbb{Z}_{q^n[i]}} = \begin{cases} 1 & for \quad n = 1 \\ 2 & for \quad n > 1 \end{cases}$.

(ii) $diam(G(\mathbb{Z}_{q^n[i]})) = 2$.

(iii) $gr(G_{\mathbb{Z}_{q^n[i]}}) = gr(G(\mathbb{Z}_{q^n}[i])) = 3$.

Proof. Let $n = 1$, then $G(G_{\mathbb{Z}_q[i]})$ is a complete graph with q^2 vertices. This implies that $diam(G_{\mathbb{Z}_q[i]}) = 1$ and $gr(G_{\mathbb{Z}_q[i]}) = 3$. Also in this case $G(\mathbb{Z}_q[i])$ is a complete $\frac{q^2+1}{2}-$ partite graph. Thus

$$diam(G(\mathbb{Z}_q[i])) = 2 \text{ and } gr(G(\mathbb{Z}_q[i])) = 3.$$

Now suppose that $n > 1$. In this case, $G_{\mathbb{Z}_{q^n}[i]}$ is a complete q^2- partite graph. Therefore,

$$diam(G_{\mathbb{Z}_{q^n}[i]}) = 2 \text{ and } gr(G_{\mathbb{Z}_{q^n}[i]}) = 3.$$

Since, $G(\dfrac{\mathbb{Z}_{q^n}[i]}{(q)})$ is a complete $\frac{q^2+1}{2}-$ partite graph, we obtain that

$$diam(G(\mathbb{Z}_{q^n[i]})) = 2 \text{ and } gr(G(\mathbb{Z}_{q^n[i]})) = 3. \qquad \square$$

2.3. The unit and unitary Cayley graphs for $\mathbb{Z}_{p^n}[i], p \equiv 1(\mathrm{mod} 4)$

Theorem 2.11. *Let n be a positive integer. Then the following statements hold:*

(*i*) $diam(G_{\mathbb{Z}_{p^n}[i]}) = diam(G(\mathbb{Z}_{p^n}[i])) = 2.$

(*ii*) $gr(G_{\mathbb{Z}_{p^n}[i]}) = gr(G(\mathbb{Z}_{p^n}[i])) = 3.$

Proof. Let p be a prime integer that is congruent to 1 modulo 4. Then there exist integer numbers a, b such that

$$p = a^2 + b^2 = (a + ib)(a - ib)$$

and

$$\mathbb{Z}_p[i] \cong \frac{\mathbb{Z}[i]}{(p)} \cong (\frac{\mathbb{Z}[i]}{(a+ib)}) \times (\frac{\mathbb{Z}[i]}{(a-ib)}).$$

Also the ideals $(a + ib)$ and $(a - ib)$ are the only maximal ideals in $\mathbb{Z}_p[i]$ see [2]. The number of units in $\mathbb{Z}_p[i]$ is $(p - 1)^2$, see [14]. By [19, Theorem 3.5], we have

$$diam(G_{\mathbb{Z}_{p^n}[i]}) = diam(G(\mathbb{Z}_{p^n}[i])) = 2.$$

On the other hand, in view of the proof of [8, Proposition 5.10] and [4, Theorem3.2], we obtain

$$gr(G_{\mathbb{Z}_{p^n}[i]}) = gr(G(\mathbb{Z}_{p^n}[i])) = 3.$$

To investigate the more general case, let $p \equiv 1(\mathrm{mod}\ 4)$, $n > 1$. By an argoment similar to that above, we conclude that

$$\mathbb{Z}_{p^n}[i] \cong \frac{\mathbb{Z}[i]}{(p^n)} \cong (\frac{\mathbb{Z}[i]}{((a+ib)^n)}) \times (\frac{\mathbb{Z}[i]}{((a-ib)^n)}) \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}.$$

The number of units in $\mathbb{Z}_{p^n}[i]$ is $(p^n - p^{n-1})^2$, see [14]. Note that, $\mathbb{Z}_{p^n}$ is a local ring with only maximal ideal, $m = (p)$, and hence $|\frac{\mathbb{Z}_{p^n}}{m}| = p$. Hence by [19, Theorem 3.5], we have that

$$diam(G_{\mathbb{Z}_{p^n}[i]}) = diam(G(\mathbb{Z}_{p^n}[i])) = 2.$$

On the other hand, in view of the proof of [8, Proposition 5.10] and [4, Theorem3.2], we obtain $gr(G_{\mathbb{Z}_{p^n}[i]}) = gr(G(\mathbb{Z}_{p^n}[i])) = 3.$ □

3. The unit and unitary Cayley graphs for $\mathbb{Z}_n[i]$

In this section, the integers q_j and p_s are used implicitly to denote primes congruent to 3 modulo 4 and primes congruent to 1 modulo 4 respectively. The general case is now investigated.

3.1. Diameter and girth for the graphs $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$

Now we find the diameter and girth of the unit and unitary Cayley graphs of $G(\mathbb{Z}_n[i])$ where $n > 1$ is an integer.

Remark 3.1. If R is a finite commutative ring, then $R \cong R_1 \times R_2 \times \cdots \times R_t$, where each R_i is a finite commutative local ring with maximal ideal m_i by [9, Theorem 8.7]. This decomposition is unique up to permutation of factors. Since $(u_1, \ldots, u_t)$ is a unit of R if and only if each u_i is a unit in R_i, we see immediately that

$$G_R \cong G_{R_1} \otimes G_{R_2} \cdots \otimes G_{R_t} \text{ and } G(R) \cong G(R_1) \otimes G(R_2) \cdots \otimes G(R_t)$$

We denote by K_i the (finite) residue field $\dfrac{R_i}{m_i}$ and $f_i = |K_i|$. We also assume (after appropriate permutation of factors) that $f_1 \leqslant f_2 \leqslant \ldots \leqslant f_t$.

Remark 3.2. If $n = 2^k \times \prod_{j=1}^m q_j^{\alpha_j} \times \prod_{s=1}^l p_s^{\beta_s}$ is the prime power decomposition of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^m \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^l \mathbb{Z}_{p_s^{\beta_s}}[i].$$

Also the number of units in $\mathbb{Z}_n[i]$ is

$$2^{2k-1} \times \prod_{j=1}^m (q_j^{2\alpha_j} - q_j^{2\alpha_j - 2}) \times \prod_{s=1}^l (p_s^{\beta_s} - p_s^{\beta_s - 1})^2 \text{ see [2] and [14].}$$

Theorem 3.3. *Let $n > 1$ be an integer with at least two distinct prime factors. Then* $diam(G_{\mathbb{Z}_n[i]}) = diam(G(\mathbb{Z}_n[i])) = \begin{cases} 2 & \text{for } 2 \nmid n, \\ 3 & \text{for } 2 \mid n. \end{cases}$

Proof. Let $n = 2^k \times \prod_{j=1}^m q_j^{\alpha_j} \times \prod_{s=1}^l p_s^{\beta_s}$. By Remark 3.2,

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^m \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^l \mathbb{Z}_{p_s^{\beta_s}}[i].$$

This shows that, $\mathbb{Z}_n[i]$ is isomorphic to a direct product of finite local rings, R_i such that for every i, $|\dfrac{R_i}{m_i}| = 2$ or q_j^2 or p_s. Since $n > 1$ is an integer with at least two distinct prime factors, we have $J(\mathbb{Z}_n[i]) \neq \{0\}$.

By [4, Theorem 3.1], we conclude that

$$diam(G_{\mathbb{Z}_n[i]}) = \begin{cases} 2 & \text{for } 2 \nmid n, \\ 3 & \text{for } 2 \mid n. \end{cases}$$

On the other hand, by [8, Theorem 5.7] we have

$$diam(G(\mathbb{Z}_n[i])) = \begin{cases} 2 & \text{for } 2 \nmid n, \\ 3 & \text{for } 2 \mid n. \end{cases} \qquad \square$$

Theorem 3.4. *Let $n > 1$ be an integer with at least two distinct prime factors. Then*

$$gr(G_{\mathbb{Z}_n[i]}) = \begin{cases} 4 & for \quad 2 \nmid n \\ 3 & for \quad 2 \mid n \end{cases}$$

and

$$gr(G(\mathbb{Z}_n[i])) \in \{3, 4\}.$$

Proof. By an argoment similar to that above, we conclude that

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^{m} \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i].$$

Thus, by [4, Theorem 3.2], we obtain

$$gr(G_{\mathbb{Z}_n[i]}) = \begin{cases} 4 & \text{for} \quad 2 \nmid n, \\ 3 & \text{for} \quad 2 \mid n. \end{cases}$$

On the other hand, $J(\mathbb{Z}_n[i]) \neq \{0\}$. Thus, in view of the proof of [8, Theorem 5.10], we have $gr(G(\mathbb{Z}_n[i])) \in \{3, 4\}$. □

3.2. When are $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$ Hamiltonian or Eulerian?

In the following, we prove that for each integer $n > 1$, the graphs $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are *Hamiltonian*

Theorem 3.5. *For each integer $n > 1$, the graphs $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are Hamitonian.*

Proof. Let $n > 1$ be an integer. By Corollary 2.10, Corollary 2.8, Theorem 2.11 and Theorem 3.3, the graphs ,$G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are connected. Thus by [23, Theorem 2.1], $G(\mathbb{Z}_n[i])$ is Hamiltonian graph. On the other hand, by [21, Lemma 4], we conclude that $G_{\mathbb{Z}_n[i]}$ is Hamiltonian graph. □

Now, we are going to find a necessary and sufficient condition for $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ to be Eulerian. we recall the following well-known propossition.

Proposition 3.6. *A connected graph G is Eulerian if and only if the degree of each vertex of G is even.*

Theorem 3.7. *Let $n > 1$ be an integer. Then the following statements hold:*

(*i*) *The graph $G(\mathbb{Z}_n[i])$ is Eulerian if and only if $2 \mid n$.*

(*ii*) *The graph $G_{\mathbb{Z}_n[i]}$ is Eulerian if and only if $2 \mid n$.*

Proof. First Suppose that $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are Eulerian. Since these graphs are connected, by Propossition 3.6 we deduce that the degree of each vertex of $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are even. On the other hand

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^{m} \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i]$$

and so

$$|U(\mathbb{Z}_n[i])| = 2^{2k-1} \times \prod_{j=1}^{m}(q_j^{2\alpha_j} - q_j^{2\alpha_j-2}) \times \prod_{s=1}^{l}(p_s^{\beta_s} - p_s^{\beta_s-1})^2.$$

Since $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$ are $|U(\mathbb{Z}_n[i])|-$regular graph by Proposition 2.1, and [8, Proposition 2.4], we deduce that $\mathbb{Z}_n[i]$ has a direct factor of the form $\mathbb{Z}_{2^k}[i]$, and so $2 \mid n$. Conversely, suppose that $2 \mid n$. Thus $|\mathbb{Z}_n[i)|$ is even. Hence by Proposition 2.5, $2 \notin U(\mathbb{Z}_n[i])$. On the other hand, $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$ are connected and $|U(\mathbb{Z}_n[i])|-$regular graphs by Proposition 2.1 and [8, Proposition 2.4]. This means that

$$|U(\mathbb{Z}_n[i])| = 2^{2k-1} \times \prod_{j=1}^{m}(q_j^{2\alpha_j} - q_j^{2\alpha_j-2}) \times \prod_{s=1}^{l}(p_s^{\beta_s} - p_s^{\beta_s-1})^2$$

is even and so the degree of each vertex of $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$ are even, and therefore these graphs are Eulirian. □

3.3. Some graph invariants of $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$

In the following, we study chromatic, clique and independence numbers of the Graphs $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$.

Theorem 3.8. *Let $n > 1$ be an integer and $n = 2^k \times \prod_{j=1}^{m} q_j^{\alpha_j} \times \prod_{s=1}^{l} p_s^{\beta_s}$.*

(*i*) *If $2 \mid n$, then $\chi(G_{\mathbb{Z}_n[i]}) = \omega(G_{\mathbb{Z}_n[i]}) = 2$ and $\alpha(G_{\mathbb{Z}_n[i]}) = \dfrac{n^2}{2}$.*

(*ii*) *If $2 \nmid n$, then*

$$\chi(G_{\mathbb{Z}_n[i]}) = \omega(G_{\mathbb{Z}_n[i]}) = \min\{p_s, q_j^2 \mid 1 \leqslant s \leqslant l,\ 1 \leqslant j \leqslant m,\ p_s \mid n, q_j \mid n\}$$

and

$$\alpha(G_{\mathbb{Z}_n[i]}) = \frac{n^2}{\min\{p_s, q_j^2 \mid 1 \leqslant s \leqslant l,\ 1 \leqslant j \leqslant m,\ p_s \mid n,\ q_j \mid n\}}.$$

Proof. Let $2 \mid n$, and k, be the biggest positive integer such that $2^k \mid n$. Since

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^{m} \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i],$$

thus $\mathbb{Z}_n[i]$ has a direct factor of the form $\mathbb{Z}_{2^k}[i]$. Since $|\dfrac{\mathbb{Z}_{2^k}[i]}{m}| = 2$, by [4, Proposition 6.1], we conclude that $\chi(G_{\mathbb{Z}_n[i]}) = \omega(G_{\mathbb{Z}_n[i]}) = 2$ and $\alpha(G_{\mathbb{Z}_n[i]}) = \dfrac{n^2}{2}$.
Now suppose that $2 \nmid n$. This yields that $\mathbb{Z}_n[i]$ is isomorphic to a direct product of finite local rings, R_i such that for every i, $|\dfrac{R_i}{m_i}| = q_j^2$ *or* p_s. Thus by [4, Proposition 6.1], we have

$$\chi(G_{\mathbb{Z}_n[i]}) = \omega(G_{\mathbb{Z}_n[i]}) = \min\{p_s, q_j^2 \mid 1 \leqslant s \leqslant l, 1 \leqslant j \leqslant m, p_s \mid n, q_j \mid n\}$$

and

$$\alpha(G_{\mathbb{Z}_n[i]}) = \frac{n^2}{\min\{p_s, q_j^2 \mid 1 \leqslant s \leqslant l, 1 \leqslant j \leqslant m, p_s \mid n, q_j \mid n\}}. \qquad \square$$

Proposition 3.9. [13, Corollary 16.6] *Every nonempty regular bipartite graph has a perfect matching*

Lemma 3.10. [18, Lemma 2.3] *If G is a bipartite graph with a perfect matching and H is a Hamiltonian graph, then $\alpha(G \otimes H) = \dfrac{|V(G)| \times |V(H)|}{2}$.*

Theorem 3.11. *Let $n > 1$ be an integer and $n = 2^k \times \prod_{j=1}^m q_j^{\alpha_j} \times \prod_{s=1}^l p_s^{\beta_s}$.*

(i) If $2 \mid n$, then $\chi(G(\mathbb{Z}_n[i])) = \omega(G(\mathbb{Z}_n[i])) = 2$ and $\alpha(G(\mathbb{Z}_n[i])) = \dfrac{n^2}{2}$

(ii) If $2 \nmid n$, then

$$\chi(G(\mathbb{Z}_n[i])) = \omega(G(\mathbb{Z}_n[i])) = \frac{1}{2^{m+l}} \times \prod_{j=1}^{m} (q_j^{2\alpha_j} - q_j^{2\alpha_j - 2}) \times \prod_{s=1}^{l} (p_s^{\beta_s} - p_s^{\beta_s - 1})^2 + m + 2l$$

and $\alpha(G(\mathbb{Z}_n[i])) \leqslant \dfrac{n^2}{2}$.

Proof. Let $n = 2^k \times \prod_{j=1}^m q_j^{\alpha_j} \times \prod_{s=1}^l p_s^{\beta_s}$. Then

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^m \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^l \mathbb{Z}_{p_s^{\beta_s}}[i].$$

Assume that $2 \mid n$. Then by Proposition 2.5, $2 \notin U(\mathbb{Z}_n[i])$. Hence, in view of the proof of [22, Theorem 2.2], we have

$$\chi(G(\mathbb{Z}_n[i])) = \omega(G(\mathbb{Z}_n[i])) = 2.$$

Since $2 \mid n$, $\mathbb{Z}_n[i]$ has a direct factor of the form $\mathbb{Z}_{2^k}[i]$. Moreover, $G(\mathbb{Z}_{2^k}[i])$ is a nonempty regular graph. Thus, by Proposition 3.9 $G(\mathbb{Z}_{2^k}[i])$ has a perfect matching. On the otherhand, by Theorem 3.5, $G(\prod_{j=1}^m \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^l \mathbb{Z}_{p_s^{\beta_s}}[i])$ is Hamiltonian graph. Therefore, by Lemma 3.10,

$$\alpha(G(\mathbb{Z}_n[i])) = \frac{n^2}{2}.$$

Now suppose that $2 \nmid n$. Thus $2 \in U(\mathbb{Z}_n[i])$. By an argoment similar to that above, we conclude that

$$\chi(G(\mathbb{Z}_n[i])) = \omega(G(\mathbb{Z}_n[i])) = \frac{1}{2^{m+l}} \times \prod_{j=1}^{m} (q_j^{2\alpha_j} - q_j^{2\alpha_j - 2}) \times \prod_{s=1}^{l} (p_s^{\beta_s} - p_s^{\beta_s - 1})^2 + m + 2l.$$

Let $n = \prod_{j=1}^m \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^l \mathbb{Z}_{p_s^{\beta_s}}[i]$. Then

$$\mathbb{Z}_n[i] \cong \prod_{j=1}^m \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^l \mathbb{Z}_{p_s^{\beta_s}}[i]$$

and so we have

$$\mathbb{Z}_2[i] \times \mathbb{Z}_n[i] \cong \mathbb{Z}_2[i] \times \prod_{j=1}^{m} \mathbb{Z}_{qj^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i].$$

Thus,

$$\alpha(G(\mathbb{Z}_2[i] \times \mathbb{Z}_n[i])) \cong \alpha(G(\mathbb{Z}_2[i] \times \prod_{j=1}^{m} \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i])).$$

Now by part (i), we coclude that

$$\alpha(G(\mathbb{Z}_2[i] \times \mathbb{Z}_n[i])) = 2n^2.$$

On the other hand,

$$\alpha(G(\mathbb{Z}_2[i] \times \mathbb{Z}_n[i])) \geq \alpha(G(\mathbb{Z}_2[i])) \times |\mathbb{Z}_n[i]| = 4 \times \alpha(G(\mathbb{Z}_n[i])).$$

This implies that $\alpha(G(\mathbb{Z}_n[i])) \leqslant \frac{n^2}{2}$. □

References

[1] **G. Aalipour, S. Akbari**, *On the Cayley graph of a commutative ring with respect to its zero-divisors*, Comm. Algebra **44** (2016), 1443 – 1459.

[2] **E.A. Abu Osba, S. Al-Addasi, N. Abu Jaradeh**, *Zero divisor graph for the ring of Gaussian integers modulo n*, Comm. Algebra **36** (2008), 3865 – 3877.

[3] **M. Afkhami, K. Khashyarmanesh, R. Zohreh**, *Some results on the annihilator graph of a commutative ring*, Czechoslovak Math. J. **67(142)** (2017), 151 – 169.

[4] **R. Akhtar, T. Jackson-Henderson, R. Karpman, M. Boggess, I. Jiménez, A. Kinzel, D. Pritikin**, *On the unitary Cayley graph of a finite ring*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 117, 13 pp.

[5] **D.F. Anderson, A. Badawi**, *The total graph of a commutative ring*, J. Algebra **320** (2008), 2706 – 2719.

[6] **D.F. Anderson, A. Badawi**, *The generalized total graph of a commutative ring*, J. Algebra Appl. **12** (2013), no.5, paper 1250212.

[7] **D.F. Anderson, P.S. Livingston**, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434 – 447.

[8] **N. Ashrafi, H.R. Maimani, M.R. Pournaki, S. Yassemi**, *Unit graphs associated with rings*, Comm. Algebra **38** (2010), 2851 – 2871.

[9] **M.F. Atiyah, I.G. Macdonald**, *Introduction to commutative algebra*, Addison-Wesley Publishing Co. (1969).

[10] **A. Badawi**, *On the annihilator graph of a commutative ring*, Comm. Algebra **42** (2014), 108 – 121.

[11] **A. Badawi**, *On the dot product graph of a commutative ring*, Comm. Algebra **43** (2015), 43 – 50.

[12] **Z. Barati, K. Khashyarmanesh, F. Mohammadi, Kh. Nafar**, *On the associated graphs to a commutative ring*, J. Algebra Appl. **11** (2012), no.2, paper 1250037.

[13] **J.A. Bondy, U.S.R. Murty** *Graph theory with applications*, American Elsevier Publishing Co. (1976).

[14] **J.T. Cross**, *The Euler φ-function in the Gaussian integers*, Amer. Math. Monthly **90** (1983), 518 – 528.

[15] **R. Diestel**, *Graph theory*, Springer-Verlag (2001).

[16] **G. Dresden, W.M. Dymáček**, *Finding factors of factor rings over the Gaussian integers*, Amer. Math. Monthly **112** (2005), 602 – 611.

[17] **R.P. Grimaldi**, *Graphs from rings*, Congr. Numer. **71** (1990), 95 – 103.

[18] **P. Jha, K. Pranava, S. Klavžar**, *Independence in direct-product graphs*, Ars Combin. **50** (1998), 53 – 63.

[19] **K. Khashyarmanesh, M.R. Khorsandi**, *A generalization of the unit and unitary Cayley graphs of a commutative ring*, Acta Math. Hungar. **137** (2012), no. 4, 242 – 253.

[20] **D. Kiani, M.M.A. Aghaei**, *On the unitary Cayley graph of a ring*, Electron. J. Combin. **19** (2012), no. 2, Paper 10, 10 pp.

[21] **C. Lanski, A. Maróti**, *Ring elements as sums of units*, Cent. Eur. J. Math. **7** (2009), 395 – 399.

[22] **H.R. Maimani, M.R. Pournaki, S. Yassemi**, *Weakly perfect graphs arising from rings* Glasg. Math. J. **52** (2010), 417 – 425.

[23] **H.R. Maimani, M.R. Pournaki, S. Yassemi**, *Necessary and sufficient conditions for unit graphs to be Hamiltonian*, Pacific J. Math. **249** (2011), 419 – 429.

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