On nuclei and conuclei of $S$-quantales

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Abstract. $S$-quantales have been proved to be injectives in the category of $S$-posets with $S$-submultiplicative order-preserving mappings as morphisms. In this work, algebraic investigations on $S$-quantales are presented. A representation theorem of an $S$-quantale according to nuclei is obtained, quotients of an $S$-quantale with respect to nuclei and congruences are completely studied. Simultaneously, the relationship between $S$-subquantales and conuclei of an $S$-quantale is established.

1. Preliminary

Various quantale-like structures (quantales, locales, quantale modules, quantale algebras, unital quantales etc.) have been studied for decades and they have useful applications in algebra, logic and computer science ([3], [6], [11], [12]). In [11], algebraic properties and applications of quantales are well studied. The idea was then extended to groupoid quantales [7], involutive quantales [9], [6], sup-lattices [10], quantale modules [4], [14], [13], and quantale algebras [15], [8], etc. Recently, Zhang and Laan in [16] introduced a new kind of quantale-like structure, named $S$-quantales. It has been shown that $S$-quantales play an important role in the theory of injectivity on the category of $S$-posets with $S$-submultiplicative order-preserving mappings as morphisms. In fact, injectives in this category are exactly $S$-quantales. The purpose of this paper is to make a contribution on algebraic investigations of $S$-quantales. Let us first recall some basic definitions.

In this work, $S$ is always a pomonoid, i.e., a monoid $S$ equipped with a partial order $\leq$ such that $ss' \leq tt'$ whenever $s \leq t$, $s' \leq t'$ in $S$. A poset $(A, \leq)$ together with a mapping $A \times S \to A$ (under which a pair $(a, s)$ maps to an element of $A$ denoted by $as$) is called a right $S$-poset, denoted by $A_S$, if for any $a, b \in A$, $s, t \in S$,

1. $a(st) = (as)t$,
2. $a1 = a$,
3. $a \leq b$, $s \leq t$ imply that $as \leq bt$.

A left $S$-poset can be defined similarly. In this paper we only consider right $S$-posets, therefore we will omit the word “right”.

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Let \( A_S \) and \( B_S \) be \( S \)-posets. A mapping \( f : A_S \rightarrow B_S \) is said to be \( S \)-submultiplicative if \( f(a)s \leq f(as) \) for any \( a \in A_S, \ s \in S \). We call \( f \) an \( S \)-poset homomorphism if it preserves both \( S \)-actions and orders.

An \( S \)-poset \( A_S \) is said to be an \( S \)-quantale \([16]\) if

1. the poset \( A \) is a complete lattice;
2. \( \bigvee M = \bigvee \{ms \mid m \in M \} \) for each subset \( M \) of \( A \) and each \( s \in S \).

An \( S \)-quantale homomorphism is a mapping between \( S \)-quantales which preserves both \( S \)-actions and arbitrary joins. An \( S \)-subquantale of an \( S \)-quantale \( A_S \) is indeed the relative subposet of \( A_S \) which closed under \( S \)-actions and arbitrary joins.

We begin with properties of \( S \)-quantale homomorphisms and mappings between \( S \)-quantales with right adjoints. Then a representation theorem of quotients for \( S \)-quantales by nuclei is presented. The important topic of relations between the lattices of nuclei and congruences of an \( S \)-quantale is fully investigated. Dually, the connection on \( S \)-subquantales and counuclei is studied.

2. Mappings and homomorphisms

Let \( f : P \rightarrow Q \) be a join-preserving mapping of posets. By the adjoint functor theorem ([1]), \( f \) has a unique right adjoint \( f_* : Q \rightarrow P \), fulfilling

\[
f(x) \leq y \iff x \leq f_*(y),
\]

for any \( x \in P, \ y \in Q \), and hence

\[
f(f_*(y)) \leq y, \ x \leq f_*(f(x)).
\]

Given an \( S \)-quantale \( Q_S \), and any \( s \in S \), the mapping \( s_- : Q_S \rightarrow Q_S \) defined by \( s(a) = as \) for each \( a \in Q_S \), preserves all joins, and thus has a unique right adjoint, denoted by \( s_* \), satisfying

\[
s(a) \leq b \iff a \leq s_*(b),
\]

and

\[
s(s_*(a)) \leq a, \ a \leq s_*(s(a)),
\]

for each \( a,b \in Q_S \). It holds evidently that \( s_*(a)s \leq a, \ \forall a \in Q_S \).

**Proposition 2.1.** Let \( Q_S \) be an \( S \)-quantale. Then for any \( b \in Q_S, \ s,t \in S \), the following statements hold.

1. \( s_*(t_*(b)) = (st)_*(b) \),
2. \( s_*(b)s = b \iff (\exists a \in Q_S) as = b \),
3. \( s_*(bs) = b \iff (\exists a \in Q_S) b = s_*(a) \).
Proof. We note that for any \( x, b \in Q_S \), \( s, t \in S \),
\[
x \leq s_*(t_*(b)) \iff xs \leq t_*(b) \iff xst \leq b \iff x \leq (st)_*(b),
\]
by (3), so we obtain 1. 2 and 3 can be proved similarly. \qed

**Proposition 2.2.** Let \( f : P_S \to Q_S \) be an \( S \)-quantale homomorphism. Then
\[
f_*(s_*(a)) = s_*(f_*(a))
\]
for any \( a \in Q_S \), \( s \in S \).

**Proof.** By (1) and \( f \) preserving \( S \)-actions, we have
\[
f_*(s_*(a)) \leq s_*(f_*(a)) \iff f_*(s_*(a))s \leq f_*(a) \iff f(f_*(s_*(a)))s \leq a
\]
\[
\iff f(f_*(s_*(a)))s \leq a \iff f(f_*(s_*(a)))) \leq s_*(a),
\]
for each \( a \in Q_S \). But the final inequality natural follows by (2), we soon get that
\( f_*(s_*(a)) \leq s_*(f_*(a)) \). One may dually gain that \( s_*(f_*(a)) \leq f_*(s_*(a)) \). \qed

Recall that for a poset \( P \), a monotone mapping \( j \) on \( P \) is said to be a closure operator if it is both increasing and idempotent.

**Definition 2.3.** Let \( Q_S \) be an \( S \)-quantale, \( j \) a closure operator on \( Q_S \). We call \( j \) a nucleus if it is \( S \)-submultiplicative, i.e.,
\[
j(a)s \leq j(as)
\]
for each \( a \in Q_S \), \( s \in S \).

**Lemma 2.4.** Let \( Q_S \) be an \( S \)-quantale, \( j \) a nucleus on \( Q_S \). Then
\[
j(s_*(a)) \leq s_*(j(a))
\]
for all \( a \in Q_S \), \( s \in S \).

**Proof.** Keep in mind that \( s_*(a)s \leq a \), \( \forall a \in Q_S \), \( s \in S \), we immediately get that
\( j(s_*(a))s \leq j(s_*(a)s) \leq j(a) \), and thus \( j(s_*(a)) \leq s_*(j(a)) \) by (3). \qed

**Lemma 2.5.** Let \( f : P_S \to Q_S \) be an \( S \)-quantale homomorphism. Then \( f_* : Q_S \to P_S \) is \( S \)-submultiplicative.

**Proof.** By (2), \( f_*(f_*(a)) \leq a \), \( \forall a \in Q_S \), it follows that \( f(f_*(a)s) = f_*(a)s \leq as \), and hence \( f_*(a)s \leq f_*(as) \) by (1). \qed

**Lemma 2.6.** Let \( f : P_S \to Q_S \) be an \( S \)-quantale homomorphism. Then \( f_*f \) is a nucleus on \( P_S \).

**Proof.** If \( a \leq b \) for \( a, b \in P_S \), then \( f(a) \leq f(b) \), and thus \( f_*f(a) \leq f_*f(b) \) by the fact that \( f_* \) preserves arbitrary meets.

Directly applying (2), we hence obtain that \( a \leq f_*f(a) \) and
\[
f_*f(a) \leq f_*(f_*f(a)) = f_*(f_*(f_*(a))) \leq f_*(f_*(a)),
\]
for any $a \in P_S$. So $f_* f$ is a closure operator.
In addition, Lemma 2.5 provides that

$$(f_* f)(a)s = (f_*(f(a))s \leq f_*(f(a)s) = (f_* f)(as),$$

for any $a \in Q_S$, $s \in S$. Consequently, $f_* f$ is a nucleus as desired. \hfill \Box

### 3. Nuclei and a representation theorem

For an $S$-quantale $Q_S$, we write $\text{Nuc}(Q_S)$ for the set of all nuclei on $Q_S$. $\text{Nuc}(Q_S)$ will therefore become a complete lattice if it is equipped with the pointwise order. The following properties of nuclei can be easily gained.

**Lemma 3.1.** (cf. [16]) Let $Q_S$ be an $S$-quantale, $j$ a nucleus on $Q_S$. Then for any $a \in Q_S$, $s \in S$, $j(\ldots) = j(j(\ldots))$.

**Lemma 3.2.** Let $Q_S$ be an $S$-quantale, $j$ a nucleus on $Q_S$. Then

$$j\left(\bigvee_{i \in I} j(a_i)\right) = j\left(\bigvee_{i \in I} a_i\right), \quad \forall a_i \in Q_S, \ i \in I.$$

**Proof.** Follows from the property of $j$ being a closure operator. \hfill \Box

**Lemma 3.3.** Let $Q_S$ be an $S$-quantale, $j$, $\tilde{j}$ $\in \text{Nuc}(Q_S)$. Then the following statements hold.

1. $j \leq \tilde{j} \iff \tilde{j}j = j;$
2. $j \leq \tilde{j} \iff \forall x, y \in Q_S, \ j(x) = j(y) \Rightarrow \tilde{j}(x) = \tilde{j}(y).$

Given a nucleus $j$ on an $S$-quantale $Q_S$. Write

$$Q_j = \{a \in Q_S \mid j(a) = a\}.$$

Then $Q_j$ becomes an $S$-quantale with the $S$-action defined by

$$a \circ s = j(as), \ a \in A, \ s \in S,$$

and the order inherited from $Q_S$, where the joins are given by

$$\bigvee D = j(\bigvee D)$$

for any $D \subseteq Q_j$ (cf. [16]).

**Proposition 3.4.** Let $Q_S$ be an $S$-quantale, $P_S \subseteq Q_S$ an $S$-subquantale. Then $P_S = Q_j$ for some nucleus $j$ iff $P_S$ is closed under meets and $s_*(a) \in P_S$ whenever $a \in P_S$.

**Proof.** Suppose that $P_S = Q_j$ for some nucleus $j$ on $Q_S$. It is routine to check that $\bigwedge A \in P_S$ for any $A \subseteq P_S$. Note that for any $a \in P_S$, $j(s_*(a)) \leq s_*(j(a)) = s_*(a)$ by Lemma 2.4, one gets that $s_*(a) \in P_S$, as well.
On the contrary, define a mapping \( j \) on \( Q_S \) by
\[
j(x) = \bigwedge \{ a \in P_S \mid x \leq a \}, \forall x \in Q_S.
\]

Straightforward verification shows that \( j \) is a closure operator.

For any \( x \in Q_S, s \in S, a \in P_S \), since \( xs \leq a \) if \( x \leq s_*(a) \) by (3), and \( s_*(a) \in P_S \) by the assumption, it follows that
\[
j(xs) \leq a \Rightarrow xs \leq j(xs) \leq a \Rightarrow j(x) \leq s_*(a) \Rightarrow j(x)s \leq a,
\]
and results in \( j(xs) \leq j(xs) \). Therefore, \( j \) is a nucleus on \( Q_S \).

By the definition of \( j \) and the fact that \( P_S \) being closed under meets, we finally achieve that \( P_S = Q_j \).

Let \( Q_S \) be an \( S \)-quantale, \( \mathcal{P}(Q) \) the power set of \( Q \). Define an \( S \)-action on \( \mathcal{P}(Q) \) by
\[
I \cdot s = \{ as \mid a \in I, s \in S \}, \forall I \subseteq Q.
\]
Then \( (\mathcal{P}(Q)_S, \cdot, \subseteq) \) becomes an \( S \)-quantale. The following theorem provides a representation of an \( S \)-quantale according to quotients w.r.t. nuclei.

**Theorem 3.5.** (Representation Theorem) Let \( Q_S \) be an \( S \)-quantale. Then there exists a nucleus \( j \) on \( \mathcal{P}(Q)_S \) such that \( Q_S \cong \mathcal{P}(Q)_j \).

**Proof.** Define a mapping \( j \) on \( \mathcal{P}(Q)_S \) by
\[
j(I) = (\bigvee I) \downarrow, \forall I \in \mathcal{P}(Q)_S.
\]
Clearly, \( j \) is a closure operator. Suppose that \( I \subseteq Q_S \) and \( x \in j(I) \). Then \( xs \leq (\bigvee I)s = \bigvee(Is) \) for all \( s \in S \), giving that \( xs \in j(Is) \). Thus \( j(I) \cdot s \subseteq j(Is) \).

We note that for any \( I \subseteq Q_S, j(I) = I \) iff \( I = a \downarrow \) for some \( a \in Q_S \). Therefore,
\[
\mathcal{P}(Q)_j = \{ I \in \mathcal{P}(Q)_S \mid I = j(I) \} = \{ I \subseteq Q_S \mid I = a \downarrow \text{ for some } a \in Q_S \}.
\]

Now define a mapping \( \sigma : Q_S \rightarrow \mathcal{P}(Q)_j \) by
\[
\sigma(a) = a \downarrow, \forall a \in Q_S.
\]
Then \( \sigma \) is certainly bijective. We remain to show that \( \sigma \) is a homomorphism. By virtue of
\[
\sigma\left( \bigvee_{i \in I} a_i \right) = \left( \bigvee_{i \in I} a_i \right) \downarrow = \left( \bigvee_{i \in I} (\bigcup a_i \downarrow) \right) \downarrow = j\left( \bigcup_{i \in I} a_i \downarrow \right) = j\left( \bigvee_{i \in I} \sigma(a_i) \right),
\]
for any \( a_i \in Q_S, i \in I \), and
\[
\sigma(a) \circ s = j(\sigma(a) \cdot s) = j(a \downarrow \cdot s) = \left( \bigvee (a \downarrow \cdot s) \right) \downarrow = \left( \bigvee \{ xs \mid x \leq a \} \right) \downarrow = (as) \downarrow = \sigma(as),
\]
for each \( a \in Q_S, s \in S \), we finally achieve that \( \sigma \) is an isomorphism between \( S \)-quantales \( Q_S \) and \( \mathcal{P}(Q)_j \).

\qed
4. Quotients of $S$-quantales

Let $Q_S$ be an $S$-quantale. A congruence $\rho$ on $Q_S$ is an equivalence relation on $Q_S$ which is compatible both with $S$-actions and joins, and has the further property that $Q/\rho$ equipped with a partial order becomes an $S$-quantale, and the canonical mapping $\pi : Q_S \to (Q/\rho)_S$ is an $S$-quantale homomorphism. Similar to the case of $S$-poets ([2]), a simple way for $Q/\rho$ being an $S$-quantale is that $Q/\rho$ accompanies an order “$\sqsubseteq$” defined by a $\rho$-chain, that is,

$$[a]_\rho \sqsubseteq [b]_\rho \iff a \leq \rho b, \ \forall a, b \in Q_S,$$

where $a \leq \rho b$ is given by a sequence

$$a \leq a_1 \rho a_1' \leq a_2 \rho a_2' \leq \cdots \leq a_n \rho a_n' \leq b,$$

for $a_i, a_i' \in Q_S$, $i = 1, 2, \ldots, n$. We see at once that in the $S$-quantale $(Q/\rho, \sqsubseteq)$,

$$\bigvee_{i \in I} [a_i]_\rho = \big[ \bigvee_{i \in I} a_i \big]_\rho, \ \forall a_i \in Q_S.$$

Let us denote by $\mathrm{Con}(Q_S)$ the set of all congruences on $Q_S$. Then $\mathrm{Con}(Q_S)$ is a complete lattice with the inclusion as order.

This section is devoted to presenting the intrinsic relationship between the posets $\mathrm{Nuc}(Q_S)$ and $\mathrm{Con}(Q_S)$, respectively. We begin with the following results.

**Lemma 4.1.** Let $Q_S$ be an $S$-quantale, $\rho \in \mathrm{Con}(Q_S)$, $\pi : Q_S \to (Q/\rho)_S$ be the canonical mapping. Then $\pi = \pi \pi \pi$.

**Proof.** By Lemma 2.6, $\pi \pi \pi$ is a nucleus on $Q_S$. So for any $a \in Q_S$, one has that $a \leq \pi \pi \pi(a)$, and hence $\pi(a) \leq \pi \pi \pi(a)$. However, (1) indicates that $\pi \pi \pi(a) \leq \pi(a)$. Consequently, we get that $\pi(a) = \pi \pi \pi(a)$.

Let us write $\pi \pi \pi$ in Lemma 4.1 as $j_\rho$. As usual, $\pi$ is a homomorphism on $Q_S$ such that $\rho = \ker \pi$.

**Lemma 4.2.** Let $Q_S$ be an $S$-quantale, $\rho \in \mathrm{Con}(Q_S)$, $\pi : Q_S \to (Q/\rho)_S$ be the canonical mapping. Then $\ker j_\rho = \ker \pi$.

**Proof.** Follows by Lemma 4.1. \hfill $\Box$

**Lemma 4.3.** Let $Q_S$ be an $S$-quantale, $j$ a nucleus on $Q_S$. Then $\ker j$ is a congruence on $Q_S$.

**Proof.** From Lemma 3.1, we have $j(as) = j(j(a)s)$, $\forall a \in Q_S, s \in S$. Thus for any $(a, b) \in \ker j, s \in S$,

$$j(as) = j(j(a)s) = j(j(b)s) = j(bs),$$

that is, $(as, bs) \in \ker j$. Moreover, derived from Lemma 3.2, we obtain that

$$j \left( \bigvee_{i \in I} a_i \right) = j \left( \bigvee_{i \in I} j(a_i) \right) = j \left( \bigvee_{i \in I} j(b_i) \right) = j \left( \bigvee_{i \in I} b_i \right),$$

for any $(a_i, b_i) \in \ker j, i \in I$. Therefore, $(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i) \in \ker j$ as needed. \hfill $\Box$
Now we are ready to characterize the concrete relationship between nuclei and congruences of an \( S \)-quantale.

**Theorem 4.4.** Let \( Q_S \) be an \( S \)-quantale. Then there exists an isomorphism \( \varphi : \text{Nuc}(Q_S) \rightarrow \text{Con}(Q_S) \) as posets. Moreover, for each \( j \in \text{Nuc}(Q_S) \), \( Q_j \equiv (Q/\varphi(j))_S \) as \( S \)-quantales.

**Proof.** Define a mapping \( \varphi : \text{Nuc}(Q_S) \rightarrow \text{Con}(Q_S) \) by
\[
\varphi(j) = \ker j,
\]
for each \( j \in \text{Nuc}(Q_S) \). Then by Lemma 4.3, \( \ker j \) is a congruence on \( Q_S \). From Lemma 3.3(2), we obtain that \( \varphi \) is an order embedding.

Suppose that \( \rho \in \text{Con}(Q_S) \), and \( \pi : Q_S \rightarrow (Q/\rho)_S \) is the canonical mapping. Then by Lemma 4.2, we have
\[
\varphi(\rho) = \ker \rho = \ker \pi = \rho.
\]
We hence conclude that \( \text{Nuc}(Q_S) \) is isomorphic to \( \text{Con}(Q_S) \) as posets.

For each \( j \in \text{Nuc}(Q_S) \), define \( f : (Q/\ker j)_S \rightarrow Q_j \) and \( g : Q_j \rightarrow (Q/\ker j)_S \) as
\[
f([a]_{\ker j}) = j(a),
\]
for each \( [a]_{\ker j} \in (Q/\ker j)_S \), and
\[
g(a) = [a]_{\ker j},
\]
for any \( a \in Q_j \). We need to show that \( f \) and \( g \) are invertible \( S \)-quantale homomorphisms.

Obviously, \( f \) is well-defined. For any \( a \in Q_S, s \in S \), since \( j(j(a)s) = j(as) \) by Lemma 3.1, we obtain that
\[
f([a]_{\ker j}s) = f([as]_{\ker j}) = j(as) = j(j(a)s) = j(a) \circ s = f([a]_{\ker j}) \circ s.
\]
Moreover, Lemma 3.2 yields that
\[
f \left( \bigvee_{i \in I} [a_i]_{\ker j} \right) = f \left( \bigvee_{i \in I} a_i \right) = j \left( \bigvee_{i \in I} a_i \right) = \bigvee_{i \in I} j(a_i) = \bigvee_{i \in I} f(a_i)_{\ker j},
\]
for each \( [a_i]_{\ker j} \in (Q/\ker j)_S, i \in I \). Therefore, \( f \) is an \( S \)-quantale homomorphism.

It is clear that \( g \) is an \( S \)-poset homomorphism. Furthermore, the equalities
\[
g \left( \bigvee_{i \in I} a_i \right) = g \left( j \left( \bigvee_{i \in I} a_i \right) \right) = \left[ \bigvee_{i \in I} a_i \right]_{\ker j} = \bigvee_{i \in I} [a_i]_{\ker j} = \bigvee_{i \in I} g(a_i),
\]
for any \( a_i \in Q_j, i \in I \) indicate that \( g \) is an \( S \)-quantale homomorphism. We then achieve our aim by the final step, that is, for all \( a \in Q_j \),
\[
f(g(a)) = f([a]_{\ker j}) = j(a) = a,
\]
and
\[
g(f([a]_{\ker j})) = g(j(a)) = [j(a)]_{\ker j} = [a]_{\ker j},
\]
for any \( a \in Q_S \). \( \square \)
5. Conuclei and \( S \)-subquantales

In this section, we introduce the notion of conuclei on an \( S \)-quantale \( Q_S \), and discuss the relationship between conuclei and \( S \)-subquantales of \( Q_S \).

**Definition 5.1.** Let \( Q_S \) be an \( S \)-quantale. We call a co closure operator \( g \) on \( Q_S \) a conucleus if it is \( S \)-submultiplicative.

Dually to Theorem 3.5, which represented quotients of an \( S \)-quantale by nuclei, the following theorem establishes the relation between conuclei and \( S \)-subquantales of an \( S \)-quantale.

**Theorem 5.2.** Let \( Q_S \) be an \( S \)-quantale, \( g \) a conucleus on \( Q_S \). Then

\[
Q_g = \{ a \in Q_S \mid g(a) = a \}
\]

is an \( S \)-subquantale of \( Q_S \). Moreover, for any \( S \)-subquantale \( P_S \) of \( Q_S \), there is a conucleus \( g \) on \( Q_S \), such that \( P_S = Q_g \).

**Proof.** Firstly, we have

\[
\bigvee A = \bigvee \{ g(a) \mid a \in A \} \leq g(\bigvee \{ a \mid a \in A \}) = g(\bigvee A),
\]

for any \( A \subseteq Q_S \), and

\[
as = g(a)s \leq g(as) \leq as,
\]

for each \( a \in Q_S \), \( s \in S \). It turns out that \( Q_g \) is an \( S \)-subquantale of \( Q_S \).

Next, suppose that \( P_S \) is an \( S \)-subquantale of \( Q_S \). Define a mapping \( g \) on \( Q_S \) as

\[
g(b) = \bigvee \{ a \in P_S \mid a \leq b \}, \ \forall b \in Q_S.
\]

Straightforward proving shows that \( g \) is order-preserving and \( g(b) \leq b, \ \forall b \in Q_S \).

Recall that \( P_S \) is join closed, \( g(b) \in P_S \), and hence

\[
g(b) = \bigvee \{ a \in P_S \mid a \leq g(b) \} = g(g(b)).
\]

So \( g \) is a co closure operator. Together with the inequalities

\[
g(b)s = \bigvee \{ a \in P_S \mid a \leq b \} \cdot s = \bigvee \{ as \in P_S \mid a \leq b \}
\]

\[
\leq \bigvee \{ a \in P_S \mid a \leq bs \} = g(bs),
\]

for any \( b \in Q_S \), \( s \in S \), we consequently obtain that \( g \) is a conucleus on \( Q_S \).

By the definition of \( g \), we immediately get that \( b \leq g(b), \ \forall b \in P_S \). So \( P_S \subseteq Q_g \). Another inclusion is clear. Therefore, \( P_S = Q_g \) as required.

Given an \( S \)-quantale \( Q_S \), write \( \text{CoNuc}(Q_S) \) as the poset of all conuclei on \( Q_S \) equipped with pointwise order, and \( \text{Sub}(Q_S) \) the poset of all \( S \)-subquantales of \( Q_S \) with inclusion as order, respectively. Theorem 5.3 describes the potential connection between the posets \( \text{Sub}(Q_S) \) and \( \text{CoNuc}(Q_S) \).
Theorem 5.3. Let $Q_S$ be a fixed $S$-quantale. Then there is an isomorphism $k : \text{Sub}(Q_S) \to \text{CoNuc}(Q_S)$ as posets, such that for any $M_S \in \text{Sub}(Q_S)$ we have $M_S = Q_{k(M_S)}$.

Proof. Define mappings $h : \text{CoNuc}(Q_S) \to \text{Sub}(Q_S)$ and $k : \text{Sub}(Q_S) \to \text{CoNuc}(Q_S)$ as

$$h(g) = Q_g, \; \forall g \in \text{CoNuc}(Q_S),$$

and

$$k(M_S) = g_{M_S}, \; \forall M_S \in \text{Sub}(Q_S),$$

respectively, where $g_{M_S}$ is given by

$$g_{M_S}(a) = \bigvee \{ m \in M_S \mid m \leq a \} = \bigvee \{ M_S \cap a \downarrow \}, \; \forall a \in Q_S.$$

It is routine to check that $g_{M_S}$ is a co-closure operator. In addition, for any $s \in S$, $a \in Q_S$, the inequalities

$$g_{M_S}(a)s = \left( \bigvee \{ m \in M_S \mid m \leq a \} \right)s = \bigvee \{ ms \in M_S \mid m \leq a \} \leq \bigvee \{ m \in M_S \mid m \leq as \} = g_{M_S}(as)$$

show that $g_{M_S}$ is $S$-submultiplicative, and hence a conucleus. $k$ being order-preserving is clear.

By Theorem 5.2, $h$ is well-defined. Assume that $m, n \in \text{CoNuc}(Q_S)$ with $m \leq n$. Then $a = m(a) \leq n(a) \leq a$, for any $a \in Q_m$, indicate that $a \in Q_n$. Thus $h$ is order-preserving.

We next show that $hk = \text{id}_{\text{Sub}(Q_S)}$, i.e., $Q_{g_{M_S}} = M_S, \forall M_S \in \text{Sub}(Q_S)$. This follows by the fact that

$$g_{M_S}(x) = \bigvee (M_S \cap x \downarrow) = x,$$

for any $x \in M_S$, and conversely, $Q_{g_{M_S}} \subseteq M_S$ by the reason that $M_S$ is closed under joins.

It remains to prove that $kh = \text{id}_{\text{CoNuc}(Q_S)}$, i.e., $g_{Q_f} = f, \forall f \in \text{CoNuc}(Q_S)$. Suppose that $a \in Q_S$. Then

$$f(a) \leq a \leq \bigvee (Q_f \cap a \downarrow) = g_{Q_f}(a).$$

Conversely, for any $x \in Q_f \cap a \downarrow$, $x = f(x) \leq f(a)$ give rise to that $f(a)$ is an upper bound of $Q_f \cap a \downarrow$. Therefore, we achieve that $g_{Q_f}(a) = f(a)$ and finally, $\text{Sub}(Q_S) \cong \text{CoNuc}(Q_S)$ as needed.

References


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