Quasigroups and Related Systems **25** (2017), 59 − 71

Identities in right Hom-alternative superalgebras

A. Nourou Issa

Abstract. Some fundamental identities characterizing right Hom-alternative superalgebras are found. These identities are the $\mathbb{Z}_2$-graded Hom-versions of well-known identities in right alternative algebras.

1. Introduction

A *right alternative algebra* is an algebra satisfying the *right alternative identity*:

$$(xy)y = x(yy).$$

If, moreover, it satisfies the *left alternative identity* $(xx)y = x(xy)$, then it is called an *alternative algebra*. Alternative algebras were studied ([28]) in connection with some problems related to projective planes (see also [17]). The 8-dimensional Cayley algebra is an example of an alternative algebra that is not associative. For fundamentals on alternative algebras, one may refer to [5], [19], [27].

As a generalization of alternative algebras, right alternative algebras were first studied in [1], where an example of a five-dimensional right alternative algebra that is not left alternative is constructed. For further studies on right alternative algebras one may refer, e.g., to [11], [21], [22] (see also [27] and references therein).

A $\mathbb{Z}_2$-graded generalization of Lie theory is considered in [4] and [16] with the introduction of the $\mathbb{Z}_2$-graded version of Lie algebras (now called *Lie superalgebras*). Next, the $\mathbb{Z}_2$-gradation of algebras is extended to other types of algebras in [10], [20] and [26].

Another generalization of usual algebras is the one of Hom-type generalization of algebras with the introduction of Hom-Lie algebras in [8] (see also [12], [13]). The defining identity of a Hom-Lie algebra is a twisted version of the usual Jacobi identity by a linear map, and the corresponding twisted associative algebra, called *Hom-associative algebra*, is introduced in [15]. Since then, various Hom-type algebras were defined and studied (see, e.g., [15], [14], [2], [23], [24], [9], [7], [3]). Observe that, in general, the twisting map in a Hom-algebra is neither injective nor surjective (see, e.g., [6] for a study on this topic). A $\mathbb{Z}_2$-graded generalization of Hom-Lie algebras is defined in [2].

In [25] the Hom-versions of some well-known identities in right alternative algebras ([11], [21], [22]) are found. The purpose of this short paper is to discuss

2010 Mathematics Subject Classification: 17A30, 17A70, 17D15.
Keywords: right alternative superalgebra, Hom-algebra, right Hom-alternative superalgebra.

the $\mathbb{Z}_2$-graded versions of the identities found in [25]. Other identities are also proposed. These Hom-super identities could be useful as a working tool in further investigations related to Hom-alternative superalgebras.

In Section 2 we recall some useful notions on Hom-superalgebras and prove some general identities that hold in any Hom-superalgebra. In Section 3 we define the $\mathbb{Z}_2$-graded Hom-version of the function $g(w, x, y, z)$ (that is first defined in [11] for right alternative algebras, and its Hom-version is defined in [25]) and we prove that it is identically zero. Next, using essentially the identity $g(w, x, y, z) = 0$ along with the Hom-Teichmüller identity, we prove some fundamental identities characterizing right Hom-alternative superalgebras. As a consequence, we obtained the $\mathbb{Z}_2$-graded Hom-version of the right Bol identity.

All vector spaces and algebras are considered over a ground field of characteristic not 2.

2. Definitions and some general results

Let $\mathbb{Z}_2 = \{0, 1\}$ be the field of integers modulo 2. A vector space A is said to be $\mathbb{Z}_2$*-graded* if $A = A_0 \oplus A_1$ (then A is also called a *superspace*).

Definition 2.1. A triple $(A, \cdot, \alpha)$ is called a (*binary*) *Hom-superalgebra* (i.e., a $\mathbb{Z}_2$*-graded binary Hom-algebra*), if A is a superspace, "$\cdot$" a binary operation on A such that $A_i \cdot A_j \subseteq A_{i+j}$, $i, j \in \mathbb{Z}_2$, and α a linear self-map of A such that $\alpha(A_i) \subseteq A_i$ (and then α is said to be *even*). The subspaces A_0 and A_1 are called respectively the *even* and *odd* parts of the Hom-superalgebra A; so are also called the elements from A_0 and A_1 respectively.

All elements in A are assumed to be *homogeneous*, i.e., either even or odd. For a given homogeneous element $x \in A_i$ $(i = 0, 1)$, by $\overline{x} = i$ we denote its parity. Since α is even, $\overline{\alpha(x)} = \overline{x}$ (we shall use this fact in the sequel without any further comment). In order to reduce the number of braces, we use juxtaposition whenever applicable and so, e.g., $xy \cdot z$ means $(x \cdot y) \cdot z$. Moreover, for simplicity and where there is no danger of confusion, we write xy in place of $x \cdot y$.

In a Hom-superalgebra $(A, \cdot, \alpha)$, the *supercommutator* and the *super Jordan product* of any two elements $x, y \in A$ are defined respectively as

$$[x, y] := xy - (-1)^{\overline{x}\,\overline{y}} yx \quad \text{and} \quad x \circ y := xy + (-1)^{\overline{x}\,\overline{y}} yx.$$

For any $x, y, z \in A$, the *Hom-associator* (x, y, z) is defined as

$$(x, y, z) := xy \cdot \alpha(z) - \alpha(x) \cdot yz.$$

Definition 2.2. ([2]). A Hom-superalgebra $(A, \cdot, \alpha)$ is called a *Hom-Lie superalgebra* if it is *super anticommutative* and satisfies the *super Hom-Jacobi identity*, i.e.,

$xy = -(-1)^{\overline{xy}} yx, \quad$ and

$$xy \cdot \alpha(z) + (-1)^{\overline{x}(\overline{y}+\overline{z})} yz \cdot \alpha(x) + (-1)^{\overline{z}(\overline{x}+\overline{y})} zx \cdot \alpha(y) = 0$$

for all $x, y, z \in A$. A Hom-superalgebra $(A, \cdot, \alpha)$ is said to be *Hom-Lie admissible*, if $(A, [,], \alpha)$ is a Hom-Lie superalgebra.

Definition 2.3. A Hom-superalgebra A is said to be *right Hom-alternative* if

$$(x, y, z) = -(-1)^{\overline{y}\,\overline{z}}(x, z, y) \quad (\textit{right superalternativity}) \tag{2.1}$$

for all $x, y, z \in A$. If the *left superalternativity* $(x, y, z) = -(-1)^{\overline{x}\,\overline{y}}(y, x, z)$ holds in A, then A is said to be *Hom-alternative* i.e., $-(-1)^{\overline{x}\,\overline{y}}(y, x, z) = (x, y, z) = -(-1)^{\overline{y}\,\overline{z}}(x, z, y)$ (*superalternativity*).

If A has zero odd part, then (2.1) reads as $(x, y, z) = -(x, z, y)$ which is the linearized form of the right Hom-alternativity $xy \cdot \alpha(y) = \alpha(x) \cdot yy$.

The following trilinear function is introduced in [2]:

$$S(x, y, z) := (x, y, z) + (-1)^{\overline{x}(\overline{y}+\overline{z})}(y, z, x) + (-1)^{\overline{z}(\overline{x}+\overline{y})}(z, x, y)$$

(this definition differs from the one in [2] by the factor $(-1)^{\overline{x}\,\overline{z}}$).

Consider in a Hom-superalgebra A the following multilinear function

$$\begin{aligned} f(w, x, y, z) := {} & (wx, \alpha(y), \alpha(z)) - (\alpha(w), xy, \alpha(z)) + (\alpha(w), \alpha(x), yz) \\ & -\alpha^2(w)(x, y, z) - (w, x, y)\alpha^2(z). \end{aligned}$$

The following identities hold in any Hom-superalgebra.

Proposition 2.4. *Let $(A, \cdot, \alpha)$ be a Hom-superalgebra. Then*

- $$f(w, x, y, z) = 0, \tag{2.2}$$
- $$\begin{aligned} & [xy, \alpha(z)] - \alpha(x)[y, z] - (-1)^{\overline{y}\,\overline{z}}[x, z]\alpha(y) = \\ & \qquad (x, y, z) - (-1)^{\overline{y}\,\overline{z}}(x, z, y) + (-1)^{\overline{z}(\overline{x}+\overline{y})}(z, x, y), \end{aligned} \tag{2.3}$$
- $$\begin{aligned} & [xy, \alpha(z)] - [x, y]\alpha(z) + (-1)^{\overline{y}\,\overline{z}}[xz, \alpha(y)] - (-1)^{\overline{y}\,\overline{z}}[x, z]\alpha(y) = \\ & \qquad (-1)^{\overline{x}\,\overline{y}}(y, x, z) + (-1)^{\overline{z}(\overline{x}+\overline{y})}(z, x, y), \end{aligned} \tag{2.4}$$
- $$[xy, \alpha(z)] + (-1)^{\overline{x}(\overline{y}+\overline{z})}[yz, \alpha(x)] + (-1)^{\overline{z}(\overline{x}+\overline{y})}[zx, \alpha(y)] = S(x, y, z), \tag{2.5}$$
- $$\begin{aligned} & (x \circ y) \circ \alpha(z) - (-1)^{\overline{y}\,\overline{z}}(x \circ z) \circ \alpha(y) = \\ & S(x, y, z) - (-1)^{\overline{x}\,\overline{y}} S(y, x, z) - 2(-1)^{\overline{z}(\overline{x}+\overline{y})}(z, x, y) + [\alpha(x), [y, z]] \end{aligned} \tag{2.6}$$

for all w, x, y, z in A.

Proof. The identity (2.2) follows by direct expansion of associators in $f(w, x, y, z)$. Next we have

$$[xy, \alpha(z)] - \alpha(x)[y, z] - (-1)^{\overline{y}\,\overline{z}}[x, z]\alpha(y) = xy \cdot \alpha(z) - (-1)^{\overline{z}(\overline{x}+\overline{y})}\alpha(z) \cdot xy$$

$$-\alpha(x)(yz-(-1)^{\overline{y}\,\overline{z}}zy)-(-1)^{\overline{y}\,\overline{z}}(xz-(-1)^{\overline{y}\,\overline{z}}zx)\alpha(y)=\{xy\cdot\alpha(z)-\alpha(x)\cdot yz\}$$
$$-(-1)^{\overline{y}\,\overline{z}}\{xz\cdot\alpha(y)-\alpha(x)\cdot zy\}+(-1)^{\overline{z}(\overline{x}+\overline{y})}\{zx\cdot\alpha(y)-\alpha(z)\cdot xy\}$$
$$=(x,y,z)-(-1)^{\overline{y}\,\overline{z}}(x,z,y)+(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)$$

and so we get (2.3). As for (2.4), we compute

$$[xy,\alpha(z)]-[x,y]\alpha(z)+(-1)^{\overline{y}\,\overline{z}}[xz,\alpha(y)]-(-1)^{\overline{y}\,\overline{z}}[x,z]\alpha(y)=xy\cdot\alpha(z)$$
$$-(-1)^{\overline{z}(\overline{x}+\overline{y})}\alpha(z)\cdot xy-(xy-(-1)^{\overline{x}\overline{y}}yx)\alpha(z)+(-1)^{\overline{y}\overline{z}}(xz\cdot\alpha(y)-(-1)^{\overline{y}(\overline{x}+\overline{z})}\alpha(y)\cdot xz)$$
$$-(-1)^{\overline{y}\,\overline{z}}(xz-(-1)^{\overline{x}\,\overline{z}}zx)\alpha(y)=(-1)^{\overline{x}\,\overline{y}}(yx\cdot\alpha(z)-\alpha(y)\cdot xz)$$
$$+(-1)^{\overline{z}(\overline{x}+\overline{y})}(zx\cdot\alpha(y)-\alpha(z)\cdot xy)=(-1)^{\overline{x}\,\overline{y}}(y,x,z)+(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y),$$

which gives (2.4).

The identity (2.5) follows by expansion of associators in the right-hand side and next rearrangement of terms.

Starting from the right-hand side of (2.6), we have

$$S(x,y,z)-(-1)^{\overline{x}\,\overline{y}}S(y,x,z)-2(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)+[\alpha(x),[y,z]]$$
$$=(x,y,z)-(-1)^{\overline{x}\,\overline{y}+\overline{x}\overline{z}+\overline{y}\,\overline{z}}(z,y,x)+(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)$$
$$-(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)+(-1)^{\overline{x}\,\overline{y}}(y,x,z)-(-1)^{\overline{y}\,\overline{z}}(x,z,y)+[\alpha(x),[y,z]]$$
$$=(x\circ y)\cdot\alpha(z)+(-1)^{\overline{z}(\overline{x}+\overline{y})}\alpha(z)\cdot(x\circ y)-(-1)^{\overline{y}\,\overline{z}}(x\circ z)\cdot\alpha(y)-(-1)^{\overline{x}\,\overline{y}}\alpha(y)\cdot(x\circ z)$$

(developing associators and commutators and next rearranging terms)

$$=(x\circ y)\circ\alpha(z)-(-1)^{\overline{y}\,\overline{z}}(x\circ z)\circ\alpha(y)$$

and so we get (2.6). □

The identity (2.2) is usually called the *Hom-Teichmüller identity* ([24], [25]). Note that, up to $(-1)^{\overline{y}\,\overline{z}}$, the identity (2.4) is symmetric with respect to y and z.

Upon the additional requirement of right superalternativity or alternativity on $(A,\cdot,\alpha)$, the following corollaries hold.

Corollary 2.5. *If $(A,\cdot,\alpha)$ is a right Hom-alternative superalgebra, then*

- $(x\circ y)\circ\alpha(z)-(-1)^{\overline{y}\,\overline{z}}(x\circ z)\circ\alpha(y)=2(x,y,z)+[\alpha(x),[y,z]],$ (2.7)
- $[x,y]\alpha(z)-\alpha(x)[y,z]-(-1)^{\overline{y}\,\overline{z}}[xz,\alpha(y)]=2(x,y,z)-(-1)^{\overline{x}\,\overline{y}}(y,x,z),$ (2.8)
- $S(x,y,z)+(-1)^{\overline{y}\,\overline{z}}S(x,z,y)=0,$ (2.9)
- $[x\circ y,\alpha(z)]+(-1)^{\overline{x}(\overline{y}+\overline{z})}[y\circ z,\alpha(x)]+(-1)^{\overline{z}(\overline{x}+\overline{y})}[z\circ x,\alpha(y)]=0,$ (2.10)
- $[[x,y],\alpha(z)]+(-1)^{\overline{x}(\overline{y}+\overline{z})}[[y,z],\alpha(x)]+(-1)^{\overline{z}(\overline{x}+\overline{y})}[[z,x],\alpha(y)]=2S(x,y,z)$ (2.11)

for all x,y,z in A. In particular, $(A,\cdot,\alpha)$ is Hom-Lie admissible if and only if $S(x,y,z)=0$.

Proof. The application of the right superalternativity (2.1) to the right-hand side of (2.6) gives (2.7). Subtracting memberwise (2.4) from (2.3) and next using

(2.1), we get (2.8). The identity (2.9) follows by direct expansion of $S(x,y,z)$ and $S(x,z,y)$ in terms of associators and the use of (2.1). In order to prove (2.10), one starts from (2.9) by replacing $S(x,y,z)$ and $S(x,z,y)$ with their respective expressions from (2.5). Next, rearranging terms with the definition of the super Jordan product in mind, one gets (2.10).

In (2.3) let permute x and y and next multiply by $(-1)^{\overline{x}\,\overline{y}}$ to get

$$\begin{aligned}&[(-1)^{\overline{x}\,\overline{y}}yx,\alpha(z)]-(-1)^{\overline{x}\,\overline{y}}\alpha(y)[x,z]-(-1)^{\overline{x}(\overline{y}+\overline{z})}[y,z]\alpha(x)\\&=(-1)^{\overline{x}\,\overline{y}}(y,x,z)-(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)+(-1)^{\overline{x}\,\overline{y}+\overline{z}(\overline{x}+\overline{y})}(z,y,x).\end{aligned}\tag{2.12}$$

Now, subtracting memberwise (2.12) from (2.3), we get

$$\begin{aligned}&[xy,\alpha(z)]-\alpha(x)[y,z]-(-1)^{\overline{y}\,\overline{z}}[x,z]\alpha(y)-[(-1)^{\overline{x}\,\overline{y}}yx,\alpha(z)]-(-1)^{\overline{x}\,\overline{y}}\alpha(y)[x,z]\\&-(-1)^{\overline{x}\,\overline{y}}[y,z]\alpha(x)=(x,y,z)-(-1)^{\overline{y}\,\overline{z}}(x,z,y)+(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)\\&-(-1)^{\overline{x}\,\overline{y}}(y,x,z)+(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)-(-1)^{\overline{x}\,\overline{y}+\overline{z}(\overline{x}+\overline{y})}(z,y,x)\end{aligned}$$

i.e.,

$$\begin{aligned}&\{[xy,\alpha(z)]-[(-1)^{\overline{x}\,\overline{y}}yx,\alpha(z)]\}+\{(-1)^{\overline{x}(\overline{y}+\overline{z})}[y,z]\alpha(x)-\alpha(x)[y,z]\}\\&+\{-(-1)^{\overline{y}\,\overline{z}}[x,z]\alpha(y)+(-1)^{\overline{x}\,\overline{y}}\alpha(y)[x,z]\}\\&=\{(x,y,z)-(-1)^{\overline{y}\,\overline{z}}(x,z,y)\}+\{-(-1)^{\overline{x}\,\overline{y}}(y,x,z)+(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)\}\\&+\{(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)-(-1)^{\overline{x}\,\overline{y}+\overline{z}(\overline{x}+\overline{y})}(z,y,x)\}\end{aligned}$$

and so, by the definition of the supercommutator and the right superalternativity (2.1), we come to (2.11).

The last assertion is obvious. □

Corollary 2.6. *If* $(A,\cdot,\alpha)$ *is a right Hom-alternative superalgebra, then*

$$[x\circ y,\alpha(z)]=(-1)^{\overline{y}\,\overline{z}}[x,z]\circ\alpha(y)+\alpha(x)\circ[y,z]+2(x,y,z)+2(-1)^{\overline{x}\,\overline{y}}(y,x,z)\tag{2.13}$$

for all x,y,z *in* A. *Moreover, if* $(A,\cdot,\alpha)$ *is Hom-alternative, then*

$$[x\circ y,\alpha(z)]=(-1)^{\overline{y}\,\overline{z}}[x,z]\circ\alpha(y)+\alpha(x)\circ[y,z].\tag{2.14}$$

Proof. Adding (2.3) and (2.12) and next rearranging terms, we obtain

$$\begin{aligned}&[x\circ y,\alpha(z)]-\alpha(x)\circ[y,z]-(-1)^{\overline{y}\,\overline{z}}[x,z]\circ\alpha(y)\\&=\{(x,y,z)-(-1)^{\overline{y}\,\overline{z}}(x,z,y)\}+\{(-1)^{\overline{x}\,\overline{y}}(y,x,z)-(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)\}\\&+\{(-1)^{\overline{x}\,\overline{y}+\overline{z}(\overline{x}+\overline{y})}(z,y,x)+(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)\}\\&=2(x,y,z)+2(-1)^{\overline{x}\,\overline{y}}(y,x,z)\text{ (by the right superalternativity)}\end{aligned}$$

which proves (2.13).

The identity (2.14) follows from (2.13) by the left superalternativity. □

Remark. *If* $(A,\cdot,\alpha)$ *has zero odd part, then the identities* (2.2) – (2.14) *reduce to their ungraded counterparts in Hom-algebras.*

3. Main results

Throughout this section, unless stated otherwise, $(A, \cdot, \alpha)$ denotes a right Hom-alternative superalgebra and we will prove some fundamental identities characterizing right Hom-alternative superalgebras.

First, we define on $(A, \cdot, \alpha)$ the following multilinear function

$$g(x,w,y,z) := (-1)^{\overline{w}(\overline{y}+\overline{z})}(\alpha(x),\alpha(w),yz) + (-1)^{\overline{w}\,\overline{z}}(\alpha(x),\alpha(y),wz)$$
$$-(-1)^{\overline{w}\,\overline{z}+\overline{w}\,\overline{y}+\overline{y}\,\overline{z}}(x,w,z)\alpha^2(y) - (x,y,z)\alpha^2(w).$$

One observes that if A has zero odd part and $\alpha = Id$, then the function $g(x,w,y,z)$ is precisely the one defined in [11]. As a tool in the proof of part of the results below, we show that $g(x,w,y,z)$ is identically zero.

Lemma 3.1. *For all w,x,y,z in A, the following identity holds:*

$$g(x,w,y,z) = 0. \tag{3.1}$$

Proof. By (2.2) and right superalternativity (2.1), we have

$$0 = (-1)^{\overline{w}(\overline{y}+\overline{z})}f(x,w,y,z) - (-1)^{\overline{y}\,\overline{z}}f(x,z,y,w) + (-1)^{\overline{w}\,\overline{z}+\overline{w}\,\overline{y}+\overline{y}\,\overline{z}}f(x,w,z,y)$$
$$+(-1)^{\overline{w}\,\overline{z}}f(x,y,w,z) - (-1)^{\overline{y}(\overline{w}+\overline{z})}f(x,z,w,y) + f(x,y,z,w)$$
$$= (-1)^{\overline{w}(\overline{y}+\overline{z})}\{(xw,\alpha(y),\alpha(z)) - (\alpha(x),wy,\alpha(z)) + (\alpha(x),\alpha(w),yz)$$
$$-\alpha^2(x)(w,y,z) - (x,w,y)\alpha^2(z)\}$$
$$-(-1)^{\overline{y}\,\overline{z}}\{(xz,\alpha(y),\alpha(w)) - (\alpha(x),zy,\alpha(w)) + (\alpha(x),\alpha(z),yw)$$
$$-\alpha^2(x)(z,y,w) - (x,z,y)\alpha^2(w)\}$$
$$+(-1)^{\overline{w}\,\overline{z}+\overline{w}\,\overline{y}+\overline{y}\,\overline{z}}\{(xw,\alpha(z),\alpha(y)) - (\alpha(x),wz,\alpha(y)) + (\alpha(x),\alpha(w),zy)$$
$$-\alpha^2(x)(w,z,y) - (x,w,z)\alpha^2(y)\}$$
$$+(-1)^{\overline{w}\,\overline{z}}\{(xy,\alpha(w),\alpha(z)) - (\alpha(x),yw,\alpha(z)) + (\alpha(x),\alpha(y),wz)$$
$$-\alpha^2(x)(y,w,z) - (x,y,w)\alpha^2(z)\}$$
$$-(-1)^{\overline{y}(\overline{w}+\overline{z})}\{(xz,\alpha(w),\alpha(y)) - (\alpha(x),zw,\alpha(y)) + (\alpha(x),\alpha(z),wy)$$
$$-\alpha^2(x)(z,w,y) - (x,z,w)\alpha^2(y)\}$$
$$+\{(xy,\alpha(z),\alpha(w)) - (\alpha(x),yz,\alpha(w)) + (\alpha(x),\alpha(y),zw)$$
$$-\alpha^2(x)(y,z,w) - (x,y,z)\alpha^2(w)\}$$
$$= 2[(-1)^{\overline{w}(\overline{y}+\overline{z})}(\alpha(x),\alpha(w),yz) + (-1)^{\overline{w}\,\overline{z}}(\alpha(x),\alpha(y),wz)$$
$$-(-1)^{\overline{w}\,\overline{z}+\overline{w}\,\overline{y}+\overline{y}\,\overline{z}}(x,w,z)\alpha^2(y) - (x,y,z)\alpha^2(w)] \text{ (after rearranging terms)}$$
$$= 2g(x,w,y,z)$$

and so we get (3.1). □

We can now prove the following

Theorem 3.2. *In* $(A, \cdot, \alpha)$*, the identities*

$$(wx, \alpha(y), \alpha(z)) + (\alpha(w), \alpha(x), [y, z]) = (-1)^{\overline{x}(\overline{y}+\overline{z})}(w, y, z)\alpha^2(x) + \alpha^2(w)(x, y, z), \tag{3.2}$$

$$(\alpha(x), \alpha(z), y \circ w) = (\alpha(x), z \circ y, \alpha(w)) + (-1)^{\overline{w}\,\overline{y}}(\alpha(x), z \circ w, \alpha(y)) \tag{3.3}$$

hold for all w, x, y, z *in* A.

Proof. We have

$$\begin{aligned}
0 &= f(w, x, y, z) - g(w, z, x, y) \text{ (by (2.2) and (3.1))}\\
&= (wx, \alpha(y), \alpha(z)) - (\alpha(w), xy, \alpha(z)) + (\alpha(w), \alpha(x), yz) - \alpha^2(w)(x, y, z)\\
&\quad -(w, x, y)\alpha^2(z) - (-1)^{\overline{z}(\overline{x}+\overline{y})}(\alpha(w), \alpha(z), xy) - (-1)^{\overline{y}\,\overline{z}}(\alpha(w), \alpha(x), zy)\\
&\quad +(-1)^{\overline{x}(\overline{y}+\overline{z})+\overline{y}\,\overline{z}}(w, z, y)\alpha^2(x) + (w, x, y)\alpha^2(z)\\
&= (wx, \alpha(y), \alpha(z)) + (\alpha(w), \alpha(x), [y, z])\\
&\quad -(-1)^{\overline{x}(\overline{y}+\overline{z})}(w, y, z)\alpha^2(x) - \alpha^2(w)(x, y, z), \quad \text{(by right superalternativity)}
\end{aligned}$$

which yields (3.2). As for (3.3), we proceed as follows.

$$\begin{aligned}
0 &= (-1)^{\overline{w}\,\overline{y}} f(x, z, w, y) + f(x, z, y, w) \text{ (by (2.2))}\\
&= \{(-1)^{\overline{w}\,\overline{y}}(xz, \alpha(w), \alpha(y)) - (-1)^{\overline{w}\,\overline{y}}(\alpha(x), zw, \alpha(y)) + (-1)^{\overline{w}\,\overline{y}}(\alpha(x), \alpha(z), wy)\\
&\quad -(-1)^{\overline{w}\,\overline{y}}\alpha^2(x)(z, w, y) - (-1)^{\overline{w}\,\overline{y}}(x, z, w)\alpha^2(y)\}\\
&\quad +\{(xz, \alpha(y), \alpha(w)) - (\alpha(x), zy, \alpha(w)) + (\alpha(x), \alpha(z), yw)\\
&\quad -\alpha^2(x)(z, y, w) - (x, z, y)\alpha^2(w)\}\\
&= -(-1)^{\overline{w}\,\overline{y}}(\alpha(x), zw, \alpha(y)) - (\alpha(x), zy, \alpha(w)) + (\alpha(x), \alpha(z), yw)\\
&\quad +(-1)^{\overline{w}\,\overline{y}}(\alpha(x), \alpha(z), wy) + (-1)^{\overline{w}(\overline{y}+\overline{z})}(x, w, z)\alpha^2(y) + (-1)^{\overline{y}\,\overline{z}}(x, y, z)\alpha^2(w)\\
&\quad +[(-1)^{\overline{w}\overline{y}}(xz, \alpha(w), \alpha(y)) + (xz, \alpha(y), \alpha(w)) - (-1)^{\overline{w}\overline{y}}\alpha^2(x)(z,w,y) - \alpha^2(x)(z,y,w)]\\
&= (\alpha(x), zw, \alpha(y)) - (\alpha(x), zy, \alpha(w)) + (\alpha(x), \alpha(z), yw) + (-1)^{\overline{w}\,\overline{y}}(\alpha(x), \alpha(z), wy)\\
&\quad +(-1)^{\overline{w}(\overline{y}+\overline{z})}(x, w, z)\alpha^2(y) + (-1)^{\overline{y}\,\overline{z}}(x, y, z)\alpha^2(w)\\
&\quad \text{(since, by right superalternativity, the expression in bracket above is zero)}\\
&= -(-1)^{\overline{w}\,\overline{y}}(\alpha(x), zw, \alpha(y)) - (\alpha(x), zy, \alpha(w)) + (\alpha(x), \alpha(z), yw)\\
&\quad +(-1)^{\overline{w}\,\overline{y}}(\alpha(x), \alpha(z), wy) + (-1)^{\overline{w}(\overline{y}+\overline{z})+\overline{y}\,\overline{z}}(\alpha(x), \alpha(w), yz)\\
&\quad +(-1)^{\overline{z}(\overline{w}+\overline{y})}(\alpha(x), \alpha(y), wz) \quad \text{(by (3.1))}\\
&= -(-1)^{\overline{w}\,\overline{y}}(\alpha(x), zw, \alpha(y)) - (\alpha(x), zy, \alpha(w)) + (\alpha(x), \alpha(z), y \circ w)\\
&\quad -(-1)^{\overline{y}\,\overline{z}}(\alpha(x), yz, \alpha(w)) - (-1)^{\overline{w}(\overline{y}+\overline{z})}(\alpha(x), wz, \alpha(y)),
\end{aligned}$$

which leads to (3.3). □

In order to prove the identity (3.5) below, we first prove that the following identity holds in $(A, \cdot, \alpha)$.

Lemma 3.3. *The identity*

$$(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\bar{y}}[(\alpha(x),yz,\alpha(w))+(-1)^{\overline{w}\,\overline{z}}(\alpha(x),yw,\alpha(z))+(x,y,z)\alpha^2(w)$$
$$+(-1)^{\overline{w}\,\overline{z}}(x,y,w)\alpha^2(z)]\cdot\alpha^3(t)+[(\alpha(x),tz,\alpha(w))+(-1)^{\overline{w}\,\overline{z}}(\alpha(x),tw,\alpha(z))$$
$$+(x,t,z)\alpha^2(w)+(-1)^{\overline{w}\,\overline{z}}(x,t,w)\alpha^2(z)]\cdot\alpha^3(y)$$
$$=(-1)^{\bar{y}(\bar{t}+\overline{w}+\overline{z})}(\alpha^2(x),\alpha^2(y),\alpha(t)\cdot(z\circ w))+(-1)^{\bar{y}(\overline{w}+\overline{z})}(\alpha^2(x),\alpha^2(t),\alpha(y)\cdot(z\circ w)) \quad (3.4)$$

holds for all t,w,x,y,z *in* A.

Proof. Starting from the left-hand side of (3.4), we have

$$(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\bar{y}}[(\alpha(x),yz,\alpha(w))+(-1)^{\overline{w}\,\overline{z}}(\alpha(x),yw,\alpha(z))+(x,y,z)\alpha^2(w)$$
$$+(-1)^{\overline{w}\,\overline{z}}(x,y,w)\alpha^2(z)]\cdot\alpha^3(t)+[(\alpha(x),tz,\alpha(w))+(-1)^{\overline{w}\,\overline{z}}(\alpha(x),tw,\alpha(z))$$
$$+(x,t,z)\alpha^2(w)+(-1)^{\overline{w}\,\overline{z}}(x,t,w)\alpha^2(z)]\cdot\alpha^3(y)$$
$$=(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\bar{y}}(\alpha(x),yz,\alpha(w))\alpha^3(t)$$
$$+(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\bar{y}+\overline{w}\,\overline{z}}(\alpha(x),yw,\alpha(z))\alpha^3(t)$$
$$+(\alpha(x),tz,\alpha(w))\alpha^3(y)+(-1)^{\overline{w}\,\overline{z}}(\alpha(x),tw,\alpha(z))\alpha^3(y)$$
$$-(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\bar{y}+\overline{w}(\bar{y}+\overline{z})}[(-1)^{\bar{y}(\overline{w}+\overline{z})+\overline{w}\,\overline{z}}(x,z,y)\alpha^2(w)+(x,w,y)\alpha^2(z)]\cdot\alpha^3(t)$$
$$-(-1)^{\overline{w}(\bar{t}+\overline{z})}[(-1)^{\bar{t}(\overline{w}+\overline{z})+\overline{w}\,\overline{z}}(x,z,t)\alpha^2(w)+(x,w,t)\alpha^2(z)]\cdot\alpha^3(y)$$
(by right superalternativity)
$$=(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\bar{y}}(\alpha(x),yz,\alpha(w))\alpha^3(t)$$
$$+(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\,\bar{y}+\overline{w}\,\overline{z}}(\alpha(x),yw,\alpha(z))\alpha^3(t)+(\alpha(x),tz,\alpha(w))\alpha^3(y)$$
$$+(-1)^{\overline{w}\,\overline{z}}(\alpha(x),tw,\alpha(z))\alpha^3(y)$$
$$-(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\bar{y}}[(-1)^{\overline{z}(\overline{w}+\bar{y})+\overline{w}(\bar{y}+\overline{z})}(\alpha(x),\alpha(z),wy)$$
$$+(-1)^{\bar{y}\,\overline{z}+\overline{w}(\bar{y}+\overline{z})}(\alpha(x),\alpha(w),zy)]\cdot\alpha^3(t)-[(-1)^{\overline{z}(\bar{t}+\overline{w})+\overline{w}(\bar{t}+\overline{z})}(\alpha(x),\alpha(z),wt)$$
$$+(-1)^{\bar{t}\overline{z}+\overline{w}(\bar{t}+\overline{z})}(\alpha(x),\alpha(w),zt)]\cdot\alpha^3(y) \quad \text{(by (3.1))}$$
$$=(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\,\bar{y}}(\alpha(x),yz,\alpha(w))\alpha^3(t)$$
$$+(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\,\bar{y}+\overline{w}\,\overline{z}}(\alpha(x),yw,\alpha(z))\alpha^3(t)+(\alpha(x),tz,\alpha(w))\alpha^3(y)$$
$$+(-1)^{\overline{w}\,\overline{z}}(\alpha(x),tw,\alpha(z))\alpha^3(y)+(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\bar{y}}[(-1)^{\overline{w}(\bar{y}+\overline{z})}(\alpha(x),wy,\alpha(z))$$
$$+(-1)^{\bar{y}\,\overline{z}}(\alpha(x),zy,\alpha(w))]\cdot\alpha^3(t)+[(-1)^{\overline{w}(\bar{t}+\overline{z})}(\alpha(x),wt,\alpha(z))$$
$$+(-1)^{\bar{t}\overline{z}}(\alpha(x),zt,\alpha(w))]\cdot\alpha^3(y) \quad \text{(by right superalternativity)}$$
$$=(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\bar{y}}[(\alpha(x),yz,\alpha(w))+(-1)^{\overline{w}\,\overline{z}}(\alpha(x),yw,\alpha(z))$$
$$+(-1)^{\overline{w}(\bar{y}+\overline{z})}(\alpha(x),wy,\alpha(z))+(-1)^{\bar{y}\,\overline{z}}(\alpha(x),zy,\alpha(w))]\cdot\alpha^3(t)$$
$$+[(\alpha(x),tz,\alpha(w))+(-1)^{\overline{w}\,\overline{z}}(\alpha(x),tw,\alpha(z))+(-1)^{\overline{w}(\bar{t}+\overline{z})}(\alpha(x),wt,\alpha(z))$$
$$+(-1)^{\bar{t}\overline{z}}(\alpha(x),zt,\alpha(w))]\cdot\alpha^3(y)=(-1)^{(\bar{t}+\bar{y})(\overline{w}+\overline{z})+\bar{t}\bar{y}}[(-1)^{\overline{w}\,\overline{z}}(\alpha(x),\alpha(y),wz)$$
$$+(\alpha(x),\alpha(y),zw)]\cdot\alpha^3(t)+[(-1)^{\overline{w}\,\overline{z}}(\alpha(x),\alpha(t),wz)+(\alpha(x),\alpha(t),zw)]\cdot\alpha^3(y)$$
(applying (3.3) to each of the expressions in brackets above)

$= \{(-1)^{(\bar{t}+\bar{y})(\overline{w}+\bar{z})+\bar{t}\bar{y}}(\alpha(x), \alpha(y), zw)\alpha^3(t) + (\alpha(x), \alpha(t), zw)\alpha^3(y)\}$
$+\{(-1)^{\overline{w}\,\bar{z}+(\bar{t}+\bar{y})(\overline{w}+\bar{z})+\bar{t}\bar{y}}(\alpha(x), \alpha(y), wz)\alpha^3(t) + (-1)^{\overline{w}\,\bar{z}}(\alpha(x), \alpha(t), wz)\alpha^3(y)\}$
$= (-1)^{\bar{y}(\bar{t}+\overline{w}+\bar{z})}(\alpha^2(x), \alpha^2(y), \alpha(t) \cdot zw) + (-1)^{\bar{y}(\overline{w}+\bar{z})}(\alpha^2(x), \alpha^2(t), \alpha(y) \cdot zw)$
$+(-1)^{\overline{w}\bar{z}}[(-1)^{\bar{y}(\bar{t}+\overline{w}+\bar{z})}(\alpha^2(x),\alpha^2(y),\alpha(t)\cdot wz)+(-1)^{\bar{y}(\overline{w}+\bar{z})}(\alpha^2(x),\alpha^2(t),\alpha(y)\cdot wz)]$
(applying (3.1) to each of the expressions in $\{\cdots\}$ above) and so we get (3.4). □

We are now in position to prove the following

Theorem 3.4. *The identity*

$$(-1)^{(\bar{t}+\bar{y})(\overline{w}+\bar{z})+\bar{t}\bar{y}}[(x, y, z)\alpha^2(w) + (-1)^{\overline{w}\,\bar{z}}(x, y, w)\alpha^2(z)] \cdot \alpha^3(t)$$
$$+[(x, t, z)\alpha^2(w) + (-1)^{\overline{w}\,\bar{z}}(x, t, w)\alpha^2(z)] \cdot \alpha^3(y)$$
$$-(-1)^{\bar{t}(\bar{y}+\bar{z})+\bar{y}(\overline{w}+\bar{z})}\alpha((x, y, z))\alpha^2(tw) - (-1)^{(\bar{t}+\bar{z})(\overline{w}+\bar{y})+\overline{w}\,\bar{y}}\alpha((x, y, w))\alpha^2(tz)$$
$$-(-1)^{\overline{w}\,\bar{y}}\alpha((x, t, z))\alpha^2(yw) - (-1)^{\bar{z}(\overline{w}+\bar{y})}\alpha((x, t, w))\alpha^2(yz) = 0 \qquad (3.5)$$

holds for all t, w, x, y, z in A.

Proof. Relying essentially on (3.1) and (3.4), we compute

$0 = g(\alpha(x), \alpha(y), tz, \alpha(w)) + (-1)^{(\bar{t}+\bar{y})(\overline{w}+\bar{z})+\bar{t}\bar{y}}g(\alpha(x), \alpha(t), yz, \alpha(w))$
$+(-1)^{\overline{w}\,\bar{z}}g(\alpha(x), \alpha(y), tw, \alpha(z)) + (-1)^{(\bar{t}+\bar{z})(\overline{w}+\bar{y})+\bar{t}\bar{z}+\overline{w}\,\bar{y}}g(\alpha(x), \alpha(t), yw, \alpha(z))$
$= (-1)^{\bar{y}(\bar{t}+\overline{w}+\bar{z})}[(\alpha^2(x), \alpha^2(y), tz \cdot \alpha(w)) + (-1)^{\overline{w}\,\bar{z}}(\alpha^2(x), \alpha^2(y), tw \cdot \alpha(z))]$
$+(-1)^{\bar{y}(\overline{w}+\bar{z})}[(\alpha^2(x), \alpha^2(t), yz \cdot \alpha(w)) + (-1)^{\overline{w}\,\bar{z}}(\alpha^2(x), \alpha^2(t), yw \cdot \alpha(z))]$
$+(-1)^{\bar{y}(\overline{w}+\bar{z})+\bar{t}(\bar{y}+\bar{z})}[(\alpha^2(x), \alpha(yz), \alpha(tw))+(-1)^{(\bar{y}+\bar{z})(\bar{t}+\overline{w})}(\alpha^2(x), \alpha(tw), \alpha(yz))]$
$-(-1)^{\bar{t}(\bar{y}+\bar{z})+\bar{y}(\overline{w}+\bar{z})}(\alpha(x), \alpha(y), \alpha(z))\alpha^2(tw)$
$-(-1)^{(\bar{t}+\bar{z})(\overline{w}+\bar{y})+\overline{w}\,\bar{y}}(\alpha(x), \alpha(y), \alpha(w))\alpha^2(tz)$
$-(-1)^{\bar{y}\,\overline{w}}(\alpha(x), \alpha(t), \alpha(z))\alpha^2(yw) - (-1)^{\bar{z}(\overline{w}+\bar{y})}(\alpha(x), \alpha(t), \alpha(w))\alpha^2(yz)$
$-(-1)^{(\bar{t}+\bar{y})(\overline{w}+\bar{z})+\bar{t}\bar{y}}(\alpha(x), yz, \alpha(w))\alpha^3(t)$
$-(-1)^{(\bar{t}+\bar{y})(\overline{w}+\bar{z})+\bar{t}\bar{y}+\overline{w}\,\bar{z}}(\alpha(x), yw, \alpha(z))\alpha^3(t)$
$-(\alpha(x), tz, \alpha(w))\alpha^3(y)-(-1)^{\overline{w}\,\bar{z}}(\alpha(x), tw, \alpha(z))\alpha^3(y)$ (after rearranging terms)
$= (-1)^{\bar{y}(\bar{t}+\overline{w}+\bar{z})}(\alpha^2(x), \alpha^2(y), \alpha(t) \cdot zw + (-1)^{\overline{w}\,\bar{z}}\alpha(t) \cdot wz)$
$+(-1)^{\bar{y}(\overline{w}+\bar{z})}(\alpha^2(x), \alpha^2(t), \alpha(y) \cdot zw + (-1)^{\overline{w}\,\bar{z}}\alpha(y) \cdot wz)$
$+[(-1)^{\bar{y}(\overline{w}+\bar{z})+\bar{t}(\bar{y}+\bar{z})}(\alpha^2(x), \alpha(yz), \alpha(tw))+(-1)^{(\bar{y}+\bar{z})(\bar{t}+\overline{w})}(\alpha^2(x), \alpha(tw), \alpha(yz))]$
$-(-1)^{\bar{t}(\bar{y}+\bar{z})+\bar{y}(\overline{w}+\bar{z})}(\alpha(x), \alpha(y), \alpha(z))\alpha^2(tw)$
$-(-1)^{(\bar{t}+\bar{z})(\overline{w}+\bar{y})+\overline{w}\,\bar{y}}(\alpha(x), \alpha(y), \alpha(w))\alpha^2(tz)$
$-(-1)^{\bar{y}\,\overline{w}}(\alpha(x), \alpha(t), \alpha(z))\alpha^2(yw) - (-1)^{\bar{z}(\overline{w}+\bar{y})}(\alpha(x), \alpha(t), \alpha(w))\alpha^2(yz)$
$-(-1)^{(\bar{t}+\bar{y})(\overline{w}+\bar{z})+\bar{t}\bar{y}}(\alpha(x), yz, \alpha(w))\alpha^3(t)$
$-(-1)^{(\bar{t}+\bar{y})(\overline{w}+\bar{z})+\bar{t}\bar{y}+\overline{w}\,\bar{z}}(\alpha(x), yw, \alpha(z))\alpha^3(t)$

$-(\alpha(x), tz, \alpha(w))\alpha^3(y) - (-1)^{\overline{w}\,\overline{z}}(\alpha(x), tw, \alpha(z))\alpha^3(y)$
(by right superalternativity)
$=(-1)^{\overline{y}(\overline{t}+\overline{w}+\overline{z})}(\alpha^2(x),\alpha^2(y),\alpha(t)\cdot(z\circ w))+(-1)^{\overline{y}(\overline{w}+\overline{z})}(\alpha^2(x),\alpha^2(t),\alpha(y)\cdot(z\circ w))$
$+(-1)^{\overline{y}(\overline{w}+\overline{z})+\overline{t}(\overline{y}+\overline{z})}[(\alpha^2(x), \alpha(y)\alpha(z), \alpha(t)\alpha(w))$
$+(-1)^{(\overline{t}+\overline{w})(\overline{y}+\overline{z})}(\alpha^2(x), \alpha(t)\alpha(w), \alpha(y)\alpha(z))]$
$-(-1)^{\overline{t}(\overline{y}+\overline{z})+\overline{y}(\overline{w}+\overline{z})}\alpha((x, y, z))\alpha^2(tw) - (-1)^{(\overline{t}+\overline{z})(\overline{w}+\overline{y})+\overline{w}\,\overline{y}}\alpha((x, y, w))\alpha^2(tz)$
$-(-1)^{\overline{y}\,\overline{w}}\alpha((x, t, z))\alpha^2(yw) - (-1)^{\overline{z}(\overline{w}+\overline{y})}\alpha((x, t, w))\alpha^2(yz)$
$-(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\overline{y}}(\alpha(x), yz, \alpha(w))\alpha^3(t)$
$-(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\overline{y}+\overline{w}\,\overline{z}}(\alpha(x), yw, \alpha(z))\alpha^3(t) - (\alpha(x), tz, \alpha(w))\alpha^3(y)$
$-(-1)^{\overline{w}\,\overline{z}}(\alpha(x), tw, \alpha(z))\alpha^3(y)$
(by linearity of the associator and multiplicativity)
$=\{(-1)^{\overline{y}(\overline{t}+\overline{w}+\overline{z})}(\alpha^2(x),\alpha^2(y),\alpha(t)\cdot(z\circ w))+(-1)^{\overline{y}(\overline{w}+\overline{z})}(\alpha^2(x),\alpha^2(t),\alpha(y)\cdot(z\circ w))$
$-(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\overline{y}}(\alpha(x), yz, \alpha(w))\alpha^3(t)$
$-(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\overline{y}+\overline{w}\,\overline{z}}(\alpha(x), yw, \alpha(z))\alpha^3(t)$
$-(\alpha(x), tz, \alpha(w))\alpha^3(y) - (-1)^{\overline{w}\,\overline{z}}(\alpha(x), tw, \alpha(z))\alpha^3(y)\}$
$-(-1)^{\overline{t}(\overline{y}+\overline{z})+\overline{y}(\overline{w}+\overline{z})}\alpha((x, y, z))\alpha^2(tw) - (-1)^{(\overline{t}+\overline{z})(\overline{w}+\overline{y})+\overline{w}\,\overline{y}}\alpha((x, y, w))\alpha^2(tz)$
$-(-1)^{\overline{y}\,\overline{w}}\alpha((x, t, z))\alpha^2(yw) - (-1)^{\overline{z}(\overline{w}+\overline{y})}\alpha((x, t, w))\alpha^2(yz)$
$= \{(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\overline{y}}(x, y, z)\alpha^2(w)\alpha^3(t)$
$+(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\overline{y}+\overline{w}\,\overline{z}}(x, y, w)\alpha^2(z) \cdot \alpha^3(t)$
$+(x, t, z)\alpha^2(w) \cdot \alpha^3(y) + (-1)^{\overline{w}\,\overline{z}}(x, t, w)\alpha^2(z) \cdot \alpha^3(y)\}$
$-(-1)^{\overline{t}(\overline{y}+\overline{z})+\overline{y}(\overline{w}+\overline{z})}\alpha((x, y, z))\alpha^2(tw) - (-1)^{(\overline{t}+\overline{z})(\overline{w}+\overline{y})+\overline{w}\,\overline{y}}\alpha((x, y, w))\alpha^2(tz)$
$-(-1)^{\overline{y}\,\overline{w}}\alpha((x, t, z))\alpha^2(yw) - (-1)^{\overline{z}(\overline{w}+\overline{y})}\alpha((x, t, w))\alpha^2(yz)$
(applying (3.4) to the expression in $\{\cdots\}$ above), which is (3.5). □

Remark. *It is easily seen that the identities* (3.1) – (3.5) *are the* $\mathbb{Z}_2$*-graded generalization of identities*

$$(\alpha(x), \alpha(w), yz) + (\alpha(x), \alpha(y), wz) - (x, w, z)\alpha^2(y) - (x, y, z)\alpha^2(w) = 0, \tag{3.6}$$
$$(wx, \alpha(y), \alpha(z)) + (\alpha(w), \alpha(x), [y, z]) = \alpha^2(w)(x, y, z) + (w, y, z)\alpha^2(x), \tag{3.7}$$
$$(\alpha(x), y^2, \alpha(z)) = (\alpha(x), \alpha(y), yz + zy), \tag{3.8}$$
$$(\alpha^2(x), \alpha^2(y), \alpha(y) \cdot z^2) = (\alpha(x), yz, \alpha(z))\alpha^3(y) + (x, y, z)\alpha^2(z) \cdot \alpha^3(y), \tag{3.9}$$
$$(x, y, z)\alpha^2(y) \cdot \alpha^3(z) = (x, y, z)\alpha^2(zy) \tag{3.10}$$

respectively, all of which could be found in [25].

As it could be seen below, some $\mathbb{Z}_2$-graded Moufang-type identities hold in right Hom-alternative superalgebras.

Theorem 3.5. *In* $(A, \cdot, \alpha)$ *the identity*

$$(xy \cdot \alpha(z))\alpha^2(w) + (-1)^{\overline{y}\,\overline{z}+\overline{w}\,\overline{y}+\overline{w}\,\overline{z}}(xw \cdot \alpha(z))\alpha^2(y)$$
$$= \alpha^2(x)(yz \cdot \alpha(w)) + (-1)^{\overline{y}\,\overline{z}+\overline{w}\,\overline{y}+\overline{w}\,\overline{z}}\alpha^2(x)(wz \cdot \alpha(y)) \quad (3.11)$$

holds for all w, x, y, z *in* A.

Proof. In $(A, \cdot, \alpha)$, we have $(-1)^{\overline{w}(\overline{y}+\overline{z})}(\alpha(x), \alpha(w), yz) + (-1)^{\overline{w}\,\overline{z}}(\alpha(x), \alpha(y), wz)$ $= (-1)^{\overline{y}\,\overline{z}+\overline{w}\,\overline{y}+\overline{w}\,\overline{z}}(x, w, z)\alpha^2(y) + (x, y, z)\alpha^2(w)$ (by (3.1))
i.e., by the right superalternativity,

$$-(\alpha(x), yz, \alpha(w)) - (-1)^{\overline{y}\,\overline{z}+\overline{w}\,\overline{y}+\overline{w}\,\overline{z}}(\alpha(x), wz, \alpha(y))$$
$$= (-1)^{\overline{y}\,\overline{z}+\overline{w}\,\overline{y}+\overline{w}\,\overline{z}}(x, w, z)\alpha^2(y) + (x, y, z)\alpha^2(w). \quad (3.12)$$

Therefore one gets (3.11) if expand associators in (3.12). □

Remark. *The ungraded version of* (3.11) *is proved in* [25]. *For* $\alpha = Id$ *in* (3.11), *one gets*

$$(xy \cdot z)w + (-1)^{\overline{y}\,\overline{z}+\overline{w}\,\overline{y}+\overline{w}\,\overline{z}}(xw \cdot z)y = x(yz \cdot w) + (-1)^{\overline{y}\,\overline{z}+\overline{w}\,\overline{y}+\overline{w}\,\overline{z}}x(wz \cdot y)$$

and if, moreover, A *has zero odd part and* $y = w$, *one gets the right Bol identity* $(xy \cdot z)y = x(yz \cdot y)$ *formerly called the "right Moufang identity"* (*see, e.g.,* [17] *and* [27]). *Consistent with this observation,* (3.11) *may be called the "right super Hom-Bol identity".*

Remark. *In case when* $(A, \cdot, \alpha)$ *is Hom-alternative, then* (3.12) *yields*

$$(-1)^{\overline{x}(\overline{y}+\overline{z})}(yz, \alpha(x), \alpha(w)) + (-1)^{\overline{x}(\overline{w}+\overline{z})+\overline{w}(\overline{y}+\overline{z})+\overline{y}\,\overline{z}}(wz, \alpha(x), \alpha(y))$$
$$= (-1)^{\overline{z}(\overline{x}+\overline{y})}(z, x, y)\alpha^2(w) + (-1)^{\overline{z}(\overline{x}+\overline{y})+\overline{w}\,\overline{y}}(z, x, w)\alpha^2(y). \quad (3.13)$$

If, moreover, A *has zero odd part and* $\alpha = Id$, *then* (3.13) *reads as*

$$(yz, x, w) + (wz, x, y) = (z, x, y)w + (z, x, w)y$$

which is the linearized form of the middle Moufang identity ([27])

$$(yz, x, y) - (z, x, y)y = 0.$$

In this sense, the identity (3.12) *is* (*in part*) *close to the middle Moufang identity.*

In [18] (identity (9)) the following identity is proved to hold in right alternative algebras:

$$(x, z, y \circ w) = 2(x, z, w)y - 2(x, y, z)w + (x, [z, y], w) + (x, [z, w], y).$$

Its $\mathbb{Z}_2$-graded Hom-version is given by

Theorem 3.6. *In* $(A, \cdot, \alpha)$ *the identity*

$$(\alpha(x), \alpha(z), y \circ w) = 2(-1)^{\overline{w}\,\overline{y}}(x, z, w)\alpha^2(y) - 2(-1)^{\overline{y}\,\overline{z}}(x, y, z)\alpha^2(w)$$
$$+(\alpha(x), [z, y], \alpha(w)) + (-1)^{\overline{w}\,\overline{y}}(\alpha(x), [z, w], \alpha(y)) \quad (3.14)$$

holds for all w, x, y, z *in* A.

Proof. We have

$$(\alpha(x), \alpha(z), y \circ w) = (\alpha(x), z \circ y, \alpha(w)) + (-1)^{\overline{w}\,\overline{y}}(\alpha(x), z \circ w, \alpha(y)) \text{ (see (3.3))}$$
$$= (\alpha(x), zy, \alpha(w)) + (-1)^{\overline{y}\,\overline{z}}(\alpha(x), yz, \alpha(w)) + (-1)^{\overline{w}\,\overline{y}}(\alpha(x), zw, \alpha(y))$$
$$+(-1)^{\overline{w}(\overline{y}+\overline{z})}(\alpha(x), wz, \alpha(y))$$
$$= (\alpha(x), [z, y], \alpha(w)) + 2(-1)^{\overline{y}\,\overline{z}}(\alpha(x), yz, \alpha(w)) + (-1)^{\overline{w}\,\overline{y}}(\alpha(x), [z, w], \alpha(y))$$
$$+2(-1)^{\overline{w}(\overline{y}+\overline{z})}(\alpha(x), wz, \alpha(y))$$
$$= (\alpha(x), [z, y], \alpha(w)) + (-1)^{\overline{w}\,\overline{y}}(\alpha(x), [z, w], \alpha(y))$$
$$+2\{(-1)^{\overline{y}\,\overline{z}}(\alpha(x), yz, \alpha(w)) + (-1)^{\overline{w}(\overline{y}+\overline{z})}(\alpha(x), wz, \alpha(y))\}$$
$$= (\alpha(x), [z, y], \alpha(w)) + (-1)^{\overline{w}\,\overline{y}}(\alpha(x), [z, w], \alpha(y))$$
$$-2(-1)^{\overline{y}\,\overline{z}}\{(-1)^{\overline{y}\,\overline{z}+\overline{w}\,\overline{y}+\overline{w}\,\overline{z}}(x, w, z)\alpha^2(y) + (x, y, z)\alpha^2(w)\} \quad \text{(by (3.1))}$$
$$= (\alpha(x), [z, y], \alpha(w)) + (-1)^{\overline{w}\,\overline{y}}(\alpha(x), [z, w], \alpha(y))$$
$$+2(-1)^{\overline{w}\,\overline{y}}(x, z, w)\alpha^2(y) - 2(-1)^{\overline{y}\,\overline{z}}(x, y, z)\alpha^2(w)$$

(by the right superalternativity) and so we get (3.14). □

References

[1] **A.A. Albert**, *On right alternative algebras*, Ann. Math. **50** (1949), 318 – 328.

[2] **F. Ammar and A. Makhlouf**, *Hom-Lie superalgebras and Hom-Lie admissible superalgebras*, J. Algebra **324** (2010), 1513 – 1528.

[3] **S. Attan and A.N. Issa**, *Hom-Bol algebras*, Quasigroups and Related Systems **21** (2013), 131 – 146.

[4] **F.A. Berezin and G.I. Kats**, *Lie groups with commuting and anticommuting parameters*, (Russian), Mat. Sb. **82(124)** (1970), 343 – 359.

[5] **R.H. Bruck and E. Kleinfeld**, *The structure of alternative division rings*, Proc. Amer. Math. Soc. **2** (1951), 878 – 890.

[6] **Y. Frégier, A. Gohr and S.D. Silvestrov**, *Unital algebras of Hom-associative type and surjective or injective twistings*, J. Gen. Lie Theory Appl. **3** (2009), no. 4, 285 – 295.

[7] **D. Gaparayi and A.N. Issa**, *A twisted generalization of Lie-Yamaguti algebras*, Int. J. Algebra **6** (2012), no. 7, 339 – 352.

[8] **J.T. Hartwig, D. Larsson and S.D. Silvestrov**, *Deformations of Lie algebras using σ-derivations*, J. Algebra **295** (2006), 314 – 361.

[9] **A.N. Issa**, *Hom-Akivis algebras*, Comment. Math. Univ. Carolin. **52** (2011), no. 4, 485 – 500.

[10] **V.G. Kac**, *Classification of simple* $\mathbb{Z}$*-graded Lie superalgebras and simple Jordan superalgebras*, Commun. Algebra **5** (1977), 1375 – 1400.

[11] **E. Kleinfeld**, *Right alternative rings*, Proc. Amer. Math. Soc. **4** (1953), 939 − 944.

[12] **D. Larsson and S. Silvestrov**, *Quasi-Hom-Lie algebras, central extensions and 2-cycle-like identities*, J. Algebra **288** (2005), 321 − 344.

[13] **D. Larsson and S. Silvestrov**, *Quasi-Lie algebras*, Contemp. Math. **391** (2005), 241 − 248.

[14] **A. Makhlouf**, *Hom-alternative algebras and Hom-Jordan algebras*, Int. Electron. J. Alg. **8** (2010), 177 − 190.

[15] **A. Makhlouf and S.D. Silvestrov**, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2008), no. 2, 51 − 64.

[16] **W. Milnor and I.C. Moore**, *On the structure of Hopf algebras*, Ann. Math. **81** (1965), 211 − 264.

[17] **R. Moufang**, *Zur Struktur von Alternative Korpern*, Math. Ann. **110** (1935), 416 − 430.

[18] **S.V. Pchelintsev**, *Free* $(-1,1)$*-algebra with two generators*, (Russian), Algebra i Logika **13** (1974), 425 − 449.

[19] **R.D. Schafer**, *An introduction to nonassociative algebras*, Dover Pub., New York, 1995.

[20] **I.P. Shestakov**, *Prime Mal'tsev superalgebras*, (Russian), Mat. Sb. **182** (1991), 1357 − 1366.

[21] **L.A. Skornyakov**, *Right alternative division rings*, (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **15** (1952), 177 − 184.

[22] **A. Thedy**, *Right alternative rings*, J. Algebra **37** (1975), 1 − 43.

[23] **D. Yau**, *Hom-Novikov algebras*, J. Phys. A **44** (2011), 085202.

[24] **D. Yau**, *Hom-Maltsev, Hom-alternative, and Hom-Jordan algebras*, Int. Electron. J. Algebra **11** (2012), 177 − 217.

[25] **D. Yau**, *Right Hom-alternative algebras*, ArXiv:1010.3407v1 [math. RA], 17 Oct 2010.

[26] **E.I. Zel'manov and I.P. Shestakov**, *Prime alternative superalgebras and nilpotency of the radical of a free alternative algebra*, (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), 676 − 693.

[27] **K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov and A.I. Shirshov**, *Rings that are nearly associative*, Academic Press, New York, 1982.

[28] **M. Zorn**, *Theorie der alternativen Ringe*, Abh. Math. Sem. Univ. Hamburg **8** (1930), 123 − 147.

Received December 11, 2016

Département de Mathématiques, Université d'Abomey-Calavi, 01 BP 4521, Cotonou 01, Bénin
E-mail: woraniss@yahoo.fr