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Nilpotency of *gb*-triple systems

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Abstract. *gb*-triple systems are among the generalizations of Leibniz algebras (which includes Lie algebras) to ternary algebras. In this paper, we extend several results established on nilpotent Lie algebras to *gb*-triple systems. In particular we prove an analogue of Engel's theorem for *gb*-triple systems and establish some properties of nilpotent *gb*-triple systems in connection with their Frattini ideal. Also, we show the invariance of the nilradical under derivations.

1. Introduction

In recent years, Lie algebras have been generalized to several algebraic structures endowed with a multilinear operation. In particular, 3-Lie algebras [12] and Lie triple systems [8, 14] are generalizations of Lie algebras to ternary algebras. Another ternary algebra in this picture is Leibniz 3-algebras [10] which generalizes Leibniz algebras introduced by J. L. Loday [17] as a non commutative version of Lie algebras. A considerable amount of research (see [2, 3, 9, 11, 16]) has been devoted in extending classical theorems of Lie algebras to these generalizations. This paper is a continuation of investigations on *gb*-triple systems; a new algebraic structure recently introduced in [6] as another generalization of Leibniz algebras to ternary operations, and further investigated in [7].

Our purpose in this work is the study of nilpotency on *gb*-triple systems. In Section 3 we introduce the Frattini subalgebra and ideal of *gb*-triple systems and extend their classical properties known on Lie algebras to *gb*-triple systems. In Section 4, we prove that a *gb*-triple system $\mathfrak{g}$ for which the Frattini ideal $\phi(\mathfrak{g})$ is a 3-sided ideal is nilpotent if and only if the quotient *gb*-triple system $\mathfrak{g}/\phi(\mathfrak{g})$ is nilpotent. We also prove an analogue of Engel's theorem for *gb*-triple systems, thanks to the fact that the bracket operator generates the Lie algebra of inner derivations as in the case of all algebras mentioned above. In Section 5, we show that the nilradical 2-sided (right) ideal of a *gb*-triple system is invariant under derivations.

For the remainder of this paper, we assume that $\mathfrak{K}$ is a field of characteristic different to 2, all tensor products are taken over $\mathfrak{K}$ and all algebras are finite dimensional.

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2. *gb*-triple systems

In this section, we recall preliminaries about *gb*-triple systems and define the quotient *gb*-triple system.

Definition 2.1. (cf. [6]) A *gb-triple system* is a $\mathfrak{K}$ -vector space $\mathfrak{g}$ equipped with a trilinear operation $[-,-,-]_{\mathfrak{g}} : \mathfrak{g}^{\times 3} \longrightarrow \mathfrak{g}$ satisfying the identity

$$[x,y,[a,b,c]_{\mathfrak{g}}]_{\mathfrak{g}} = [a,[x,y,b]_{\mathfrak{g}},c]_{\mathfrak{g}} - [[a,x,c]_{\mathfrak{g}},y,b]_{\mathfrak{g}} - [x,[a,y,c]_{\mathfrak{g}},b]_{\mathfrak{g}}. \tag{2.1}$$

Definition 2.2. (cf. [6]) Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}})$ be a *gb*-triple system. A subspace S of $\mathfrak{g}$ is a *subalgebra* of $\mathfrak{g}$ if $(S,[-,-,-]_{\mathfrak{g}})$ be a *gb*-triple system.

Example 2.3. See Example 2 and Example 8 in [6].

Definition 2.4. (cf. [6]) A subalgebra $\mathfrak{I}$ of a *gb*-triple system $\mathfrak{g}$ is called *ideal* (resp. *left ideal*, *right ideal*) of $\mathfrak{g}$ if it satisfies the condition $\big[\mathfrak{g},\mathfrak{I},\mathfrak{g}\big]_{\mathfrak{g}} \subseteq \mathfrak{I}$ (resp. $\big[\mathfrak{g},\mathfrak{g},\mathfrak{I}\big]_{\mathfrak{g}} \subseteq \mathfrak{I}$, resp. $\big[\mathfrak{I},\mathfrak{g},\mathfrak{g}\big]_{\mathfrak{g}} \subseteq \mathfrak{I}$). If $\mathfrak{I}$ satisfies these three conditions, then $\mathfrak{I}$ is called a 3-*sided ideal.*

Remark 2.5. (cf. [6]) If S is a subalgebra of a *gb*-triple system $\mathfrak{g}$, then the *left normalizer*

$$\mathfrak{N}^l_{\mathfrak{g}}(S) := \big\{x \in \mathfrak{g} \ : \ \big[x,S,\mathfrak{g}\big]_{\mathfrak{g}} \subseteq S\big\}$$

and the *right normalizer*

$$\mathfrak{N}^r_{\mathfrak{g}}(S) := \big\{x \in \mathfrak{g} \ : \ \big[\mathfrak{g},S,x\big]_{\mathfrak{g}} \subseteq S\big\}$$

of S in $\mathfrak{g}$ are also subalgebras of $\mathfrak{g}$. Note that this statement is not true for Leibniz algebras since the right normalizer of a subalgebra of a (left) Leibniz algebra need not be a subalgebra (see [5, Example 1.7]).

Moreover, S is an ideal of $\mathfrak{g}$ if and only if $\mathfrak{N}^l_{\mathfrak{g}}(S) = \mathfrak{g} = \mathfrak{N}^r_{\mathfrak{g}}(S)$.

Definition 2.6. (cf. [6]) Given a *gb*-triple system $\mathfrak{g}$, the *center* $Z(\mathfrak{g})$ and the *derived algebra* of $\mathfrak{g}$ are defined respectively by

$$Z(\mathfrak{g}) = \Big\{x \in \mathfrak{g} \ : \ \big[\mathfrak{g},x,\mathfrak{g}\big]_{\mathfrak{g}} = 0\Big\}$$

and

$$\big[\mathfrak{g},\mathfrak{g},\mathfrak{g}\big] = \big\{\big[a_1,a_2,a_3\big]_{\mathfrak{g}},\ a_1,a_2,a_3 \in \mathfrak{g}\big\}.$$

$\mathfrak{g}$ is said to be *perfect* if $\big[\mathfrak{g},\mathfrak{g},\mathfrak{g}\big]_{\mathfrak{g}} = \mathfrak{g}$, and *abelian* if $\big[\mathfrak{g},\mathfrak{g},\mathfrak{g}\big]_{\mathfrak{g}} = 0$.

Definition 2.7. The *full center* $Z_f(\mathfrak{g})$ of a *gb*-triple system $\mathfrak{g}$ is defined by

$$Z_f(\mathfrak{g}) = Z(\mathfrak{g}) \cap \Big\{x \in \mathfrak{g} \ : \ \big[\mathfrak{g},\mathfrak{g},x\big]_{\mathfrak{g}} = 0 \text{ and } \big[x,\mathfrak{g},\mathfrak{g}\big]_{\mathfrak{g}} = 0\Big\}.$$

The following result is straightforward.

Proposition 2.8. *A gb-triple system* $\mathfrak{g}$ *is abelian if and only if* $Z_f(\mathfrak{g}) = \mathfrak{g}$.

Note that $Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$ while $Z_f(\mathfrak{g})$ and $\left[\mathfrak{g},\mathfrak{g},\mathfrak{g}\right]_{\mathfrak{g}}$ are 3-sided ideals of $\mathfrak{g}$. Let $\mathfrak{I}$ be a 3-sided ideal of a *gb*-triple system $\mathfrak{g}$. Then the quotient space $\mathfrak{g}/\mathfrak{I}$ has a natural *gb*-triple system structure given by the bracket

$$[x+\mathfrak{I}, y+\mathfrak{I}, z+\mathfrak{I}]_{\mathfrak{g}/\mathfrak{I}} = [x,y,z]+\mathfrak{I}. \tag{2.2}$$

Notice that if $x+\mathfrak{I} = x'+\mathfrak{I}$, $y+\mathfrak{I} = y'+\mathfrak{I}$ and $z+\mathfrak{I} = z'+\mathfrak{I}$, then

$$\begin{aligned}[x,y,z]_{\mathfrak{g}} &= [x'+(x-x'), y'+(y-y'), z'+(z-z')]_{\mathfrak{g}}\\ &= [x',y',z']_{\mathfrak{g}} + [x'+(x-x'), y-y', z'+(z-z')]_{\mathfrak{g}}\\ &\quad + [(x-x'), y', z'+(z-z')]_{\mathfrak{g}} + [x',y',(z-z')]_{\mathfrak{g}}\end{aligned}$$

and thus $[x,y,z]_{\mathfrak{g}}+\mathfrak{I} = [x',y',z']_{\mathfrak{g}}+\mathfrak{I}$ since $x-x'\in\mathfrak{I}$, $y-y'\in\mathfrak{I}$ and $z-z'\in\mathfrak{I}$ as $\mathfrak{I}$ is a 3-sided ideal. That the bracket (2.2) satisfies the identity (2.1) follows by definition.

Definition 2.9. $\mathfrak{g}/\mathfrak{I}$ endowed with the bracket (2.2) is called *quotient gb-triple system* of $\mathfrak{g}$ by $\mathfrak{I}$.

Recall that if V is a vector space endowed with a trilinear operation $\sigma : V\times V\times V \longrightarrow V$, then a map $d : V \longrightarrow V$ is called a *derivation with respect to* σ if

$$d(\sigma(x,y,z)) = \sigma(d(x),y,z) + \sigma(x,d(y),z) + \sigma(x,y,d(z)) \tag{2.3}$$

Remark 2.10. Let $\mathfrak{g}$ be a *gb*-triple system. Then by [7, Remark 3.9], the Lie algebra $Der(\mathfrak{g})$ of derivations of $\mathfrak{g}$ has a *gb*-triple system structure when endowed with the bracket

$$\{d_1,d_2,d_3\} = \left[d_2,\left[d_1,d_3\right]_{Der(\mathfrak{g})}\right]_{Der(\mathfrak{g})}.$$

Remark 2.11. For every derivation d of $\mathfrak{g}$ and $x,y,y',z\in\mathfrak{g}$, it follows by (2.3) and by setting $\sigma = [-,-,-]_{\mathfrak{g}}$ that

$$[x, y+d(y'), z]_{\mathfrak{g}} = [x,y,z]_{\mathfrak{g}} - [d(x),y',z]_{\mathfrak{g}} - [x,y',d(z)]_{\mathfrak{g}} + d([x,y',z]_{\mathfrak{g}}).$$

So if $\mathfrak{I}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{I}+d(\mathfrak{I})$ is also an ideal of $\mathfrak{g}$.

3. The Frattini subalgebra of *gb*-triple systems

This section is devoted to the introduction of the Frattini subalgebra and Frattini ideal of *gb*-triple systems.

Definition 3.1. A *maximal subalgebra* $\mathfrak{m}$ of a *gb*-triple system $\mathfrak{g}$ is a proper subalgebra of $\mathfrak{g}$ such that no proper subalgebra S strictly contains $\mathfrak{m}$.

Remark 3.2. Let $\mathfrak{m}$ be a *maximal left ideal* of a *gb*-triple system $\mathfrak{g}$. Then as $\mathfrak{m}$ is a left ideal of $\mathfrak{g}$, $\mathfrak{m} \subseteq \mathfrak{N}^r_{\mathfrak{g}}(\mathfrak{m})$. Now since $\mathfrak{m}$ is maximal, then $\mathfrak{N}^r_{\mathfrak{g}}(\mathfrak{m}) = \mathfrak{m}$ or $\mathfrak{N}^r_{\mathfrak{g}}(\mathfrak{m}) = \mathfrak{g}$.

Definition 3.3. The intersection of all maximal subalgebras of a *gb*-triple system $\mathfrak{g}$ is the subalgebra $F(\mathfrak{g})$ of $\mathfrak{g}$ called the *Frattini subalgebra*.

Definition 3.4. The largest ideal of a *gb*-triple system $\mathfrak{g}$ contained in $F(\mathfrak{g})$ is denoted $\phi(\mathfrak{g})$ and called the *Frattini ideal* of $\mathfrak{g}$.

Proposition 3.5. *Let $\mathfrak{g}$ be a non perfect gb-triple system. Then $F(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}$. In particular, $F(\mathfrak{g}) = 0$ if $\mathfrak{g}$ is abelian.*

Proof. By contradiction, let $x \in F(\mathfrak{g})$ with $x \notin [\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}$. Any subalgebra S of $\mathfrak{g}$ with dimension $dim\mathfrak{g} - 1$ containing $[\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}$ and with $x \notin S$ is a maximal subalgebra of $\mathfrak{g}$. A contradiction with $x \in F(\mathfrak{g})$. □

Following the proofs of [2, Propositions 2.1, 2.2, 2.4], it is easy to show that the following statements which hold for Leibniz 3-algebras also hold for *gb*-triple systems.

Proposition 3.6. *Let $\mathfrak{g}$ be a gb-triple system and $\mathfrak{I}$ an ideal of $\mathfrak{g}$. Then there are proper subalgebras $S$and S' of $\mathfrak{g}$ such*

(1) *$\mathfrak{g} = \mathfrak{I} + S$ iff $\mathfrak{I}$ is not contained in $F(\mathfrak{g})$.*

(2) *$\mathfrak{g} = \mathfrak{I} + S'$ iff $\mathfrak{I}$ is not contained in $\phi(\mathfrak{g})$.*

Proposition 3.7. *Let $\mathfrak{g}$ be a gb-triple system, $\mathfrak{I}$ an ideal of $\mathfrak{g}$ and S a subalgebra of $\mathfrak{g}$. Then the following statements hold:*

(1) *If $S + F(\mathfrak{g}) = \mathfrak{g}$, then $S = \mathfrak{g}$.*

(2) *If $S + \phi(\mathfrak{g}) = \mathfrak{g}$, then $S = \mathfrak{g}$.*

(3) *If $\mathfrak{I} \subseteq F(S)$, then $\mathfrak{I} \subseteq F(\mathfrak{g})$.*

(4) *If $\mathfrak{I} \subseteq \phi(S)$, then $\mathfrak{I} \subseteq \phi(\mathfrak{g})$.*

(5) *If $F(S)$ is an ideal of $\mathfrak{g}$, then $F(S) \subseteq F(\mathfrak{g})$.*

(6) *If $\phi(S)$ is an ideal of $\mathfrak{g}$, then $\phi(S) \subseteq \phi(\mathfrak{g})$.*

(7) $(F(\mathfrak{g}) + \mathfrak{I})/\mathfrak{I} \subseteq F(\mathfrak{g}/\mathfrak{I})$.

(8) $(\phi(\mathfrak{g}) + \mathfrak{I})/\mathfrak{I} \subseteq \phi(\mathfrak{g}/\mathfrak{I})$.

(9) *If $\mathfrak{I} \subseteq F(\mathfrak{g})$, then $F(\mathfrak{g})/\mathfrak{I} = F(\mathfrak{g}/\mathfrak{I})$.*

(10) *If $\mathfrak{I} \subseteq \phi(\mathfrak{g})$, then $\phi(\mathfrak{g})/\mathfrak{I} = \phi(\mathfrak{g}/\mathfrak{I})$.*

(11) *If* $F(\mathfrak{g}/\mathfrak{I}) = 0$, *then* $F(\mathfrak{g}) \subseteq \mathfrak{I}$.

(12) *If* $\phi(\mathfrak{g}/\mathfrak{I}) = 0$, *then* $\phi(\mathfrak{g}) \subseteq \mathfrak{I}$.

(13) *If* S *is minimal with respect to* $\mathfrak{g} = \mathfrak{I} + S$, *then* $\mathfrak{I} \cap S \subseteq \mathfrak{g}$.

(14) *If* $\mathfrak{I}$ *is abelian and* $\mathfrak{I} \cap \phi(\mathfrak{g}) = 0$, *then* $\mathfrak{g} = \mathfrak{I} + K$ *for some subalgebra* K *of* $\mathfrak{g}$.

Proof. The proof is similar to the case of Lie 3-algebras (see [2]). □

4. Nilpotency of *gb*-triple systems

4.1. Definition and Examples

Definition 4.1. The *lower central series* of a *gb*-triple system $\mathfrak{g}$ is the sequence of subalgebras defined by $\mathfrak{g}^{(s+1)} = [\mathfrak{g}, \mathfrak{g}^{(s)}, \mathfrak{g}]$ with $\mathfrak{g}^{(1)} = \mathfrak{g}$.

A *gb*-triple system $\mathfrak{g}$ is *nilpotent* if this sequence terminates, i.e., $\mathfrak{g}^{(s)} = 0$ for some positive integer s. The smallest of such values s is called *class of nilpotency* of $\mathfrak{g}$.

Remark 4.2. Let $\mathfrak{g}$ be a nontrivial nilpotent *gb*-triple system of class s. Then the following holds.

(1) $\mathfrak{g}$ *has a non trivial center.* Indeed, since there is some positive integer s such that $\mathfrak{g}^{(s)} = 0$ i.e. $[\mathfrak{g}, \mathfrak{g}^{(s-1)}, \mathfrak{g}]_{\mathfrak{g}} = 0$, it follows that $\mathfrak{g}^{(s-1)} \subseteq Z(\mathfrak{g})$.

(2) $\mathfrak{g}$ *is abelian if and only if its class is* $s = 2$.

Proposition 4.3. *Let* $\mathfrak{g}$ *be a gb-triple system. Then* $\mathfrak{g}$ *is nilpotent if and only if* $\mathfrak{g}/Z_f(\mathfrak{g})$ *is nilpotent*

Proof. If $\mathfrak{g}$ is nilpotent of class s, then $[\mathfrak{g}, \mathfrak{g}^{(s-1)}, \mathfrak{g}]_{\mathfrak{g}} = \mathfrak{g}^{(s)} = 0$. Using (2.2), it is easy to show that $\left(\mathfrak{g}/Z_f(\mathfrak{g})\right)^{(s)} = \mathfrak{g}^{(s)}/Z_f(\mathfrak{g}) = Z_f(\mathfrak{g})$. Therefore $\mathfrak{g}/Z_f(\mathfrak{g})$ is nilpotent. Conversely, if $\mathfrak{g}/Z_f(\mathfrak{g})$ is nilpotent of class s, then $\mathfrak{g}^{(s)}/Z_f(\mathfrak{g}) = \left(\mathfrak{g}/Z_f(\mathfrak{g})\right)^{(s)} = Z_f(\mathfrak{g})$. This implies that $\mathfrak{g}^{(s)} \subseteq Z_f(\mathfrak{g})$. So $\mathfrak{g}^{(s+1)} = [\mathfrak{g}, \mathfrak{g}^{(s)}, \mathfrak{g}]_{\mathfrak{g}} \subseteq [\mathfrak{g}, Z_f(\mathfrak{g}), \mathfrak{g}]_{\mathfrak{g}} = 0$. Hence $\mathfrak{g}$ is nilpotent. □

It is worth mentioning that the above definition of nilpotency appears to extend the definition of nilpotency for both left and right Leibniz algebras [11].

The following theorem classifies a subfamily of two dimensional nilpotent complex *gb*-triple systems.

Theorem 4.4. *Up to isomorphisms, there are three two-dimensional nilpotent complex gb-triple systems with one dimensional derived algebra.*

Proof. Among the seven two-dimensional complex *gb*-triple systems with one dimensional derived algebra established in the proof of [6, Theorem 11], only the following are nilpotent, all with class of nilotency $s = 3$.

$$\mathfrak{g}_2 : [a_i, a_j, a_k]_{\mathfrak{g}} = \begin{cases} \alpha a_1, & \text{if } i,j,k = 2 \\ 0, & \text{else} \end{cases},$$

$$\mathfrak{g}_3 : [a_i, a_j, a_k]_{\mathfrak{g}} = \begin{cases} a_1, & \text{if } i = 1,\ j,k = 2 \\ -a_1, & \text{if } i,j = 2,\ k = 1 \\ 0, & \text{else} \end{cases},$$

$$\mathfrak{g}_6 : [a_i, a_j, a_k]_{\mathfrak{g}} = \begin{cases} a_1, & \text{if } \ i = 1,\ j,k = 2 \\ -a_1, & \text{if } \ i,j = 2,\ k = 1 \\ \alpha a_1, & \text{if } \ i,j,k = 2 \\ 0, & \text{else} \end{cases},$$

with $\alpha \neq 0$. □

It was shown in [3] that every maximal subalgebra $\mathfrak{m}$ of a nilpotent 3-Lie algebra L is an ideal of L. The following example shows that this statement does not hold for gb-triple systems, and Corollary 4.7 shows that the result holds if $\mathfrak{m}$ is a maximal left ideal (or right ideal).

Example 4.5. Consider the nilpotent gb-triple system $\mathfrak{g}_3$ above with basis $\{a_1, a_2\}$. The one-dimensional subspace with basis $\{a_2\}$ is a maximal subalgebra of $\mathfrak{g}_3$, but not an ideal of $\mathfrak{g}_3$ since $[a_1, a_2, a_2]_{\mathfrak{g}} = a_1 \notin < a_2 >$.

As in Lie algebras, we say that a gb-triple system $\mathfrak{g}$ satisfies the right normalizer condition if there is no proper subalgebra S of $\mathfrak{g}$ such that $\mathfrak{N}^r_{\mathfrak{g}}(S) = S$. The following result which holds for groups and Leibniz algebras also holds gb-triple systems, and the proof is similar.

Proposition 4.6. *Nilpotent gb-triple systems satisfy the right normalizer condition.*

Corollary 4.7. *If* $\mathfrak{m}$ *is a maximal left or rihgt ideal of a nilpotent gb-triple system* $\mathfrak{g}$, *then* $\mathfrak{m}$ *is an ideal of* $\mathfrak{g}$.

Proof. Since $\mathfrak{g}$ is nilpotent, it follows from Proposition 4.6 that $\mathfrak{m} \neq \mathfrak{N}^r_{\mathfrak{g}}(\mathfrak{m})$. So by Remark 3.2, $\mathfrak{N}^r_{\mathfrak{g}}(\mathfrak{m}) = \mathfrak{g}$. Hence $\mathfrak{m}$ is an ideal of $\mathfrak{g}$. □

4.2. Engel's Theorem for gb-triple systems

Definition 4.8. (cf. [17]) A *Leibniz algebra* (sometimes called a *Loday algebra*, named after Jean-Louis Loday) is a $\mathfrak{K}$ vector space L with a bilinear product $[-,-]$ satisfying the *Leibniz identity*

$$[x, [y, z]] = [[x, y]\, z] + [y, [x, z]] \tag{4.1}$$

.

A Leibniz algebra L is *nilpotent* if $L^{<s>} = 0$ for some positive integer s, where $L^{<1>} = L$ and $L^{<s+1>} = [L, L^{<s>}]$. A 2-*sided ideal* of L is a subalgebra I of L satisfying $[I, L] \subseteq L$ and $[L, I] \subseteq L$.

Proposition 4.9. *Every Leibniz algebra L has a gb-triple system structure given by the bracket*

$$\{x, y, z\} = [[x, z], y].$$

Proof. To check that $\{-, -, -\}$ satisfies the identity (2.1), let $x, y, a, b, c \in L$; we have on one hand

$$\begin{aligned}\{x, y, \{a, b, c\}\} + \{\{a, x, c\}, y, b\} &= [[x, \{a, b, c\}], y] + [[\{a, x, c\}, b], y] \\ &= \big[[x, [[a, c], b]], y\big] + [[[a, c], x], b], y\big] \\ &= \big[[[a, c], [x, b]], y\big].\end{aligned}$$

On the other hand

$$\begin{aligned}\{a, \{x, y, b\}, c\} - \{x, \{a, y, c\}, b\} &= \big[[a, c], \{x, y, b\}\big] - \big[[x, b], \{a, y, c\}\big] \\ &= \big[[a, c], [[x, b], y]\big] - \big[[x, b], [[a, c], y]\big].\end{aligned}$$

The equality holds by the identity (4.1). □

Now recall that for a *gb*-triple system $\mathfrak{g}$, $\mathfrak{g}^{\otimes 2}$ is a Leibniz algebra (see [6, Proposition 2.1]) when endowed with the bracket

$$\big[a_1 \otimes a_2, b_1 \otimes b_2\big]_{\mathfrak{g}^{\otimes 2}} = \big[a_1, b_1, a_2\big]_{\mathfrak{g}} \otimes b_2 + b_1 \otimes \big[a_1, b_2, a_2\big]_{\mathfrak{g}}.$$

Lemma 4.10. *Let k be a positive integer such that $k \geqslant 2$. Then for all $a_1, b_1, a_2, b_2, \dots, a_k, b_k, g_1, g_2 \in \mathfrak{g}$ we have*

$$\begin{aligned}&\big[a_1 \otimes b_1, \big[a_2 \otimes b_2, \big[\dots, [a_k \otimes b_k, g_1 \otimes g_2]_{\mathfrak{g}^{\otimes 2}}\big]_{\mathfrak{g}^{\otimes 2}}\big]_{\mathfrak{g}^{\otimes 2}}\big]_{\mathfrak{g}^{\otimes 2}} \\ &= [a_1, [a_2, [\dots [a_k, g_1, b_k]_{\mathfrak{g}} \dots]_{\mathfrak{g}}, b_2]_{\mathfrak{g}}, b_1]_{\mathfrak{g}} \otimes g_2 \\ &\quad + g_1 \otimes [a_1, [a_2, [\dots [a_k, g_2, b_k]_{\mathfrak{g}} \dots]_{\mathfrak{g}}, b_2]_{\mathfrak{g}}, b_1]_{\mathfrak{g}} \\ &\quad + \sum_{i=1}^{k-1} [a_1, [a_2, \dots, [\widehat{a_i}, \dots [a_k, g_1, b_k]_{\mathfrak{g}} \dots, \widehat{b_i}]_{\mathfrak{g}}, \dots b_2]_{\mathfrak{g}}, b_1]_{\mathfrak{g}} \otimes [a_i, g_2, b_i]_{\mathfrak{g}} \\ &\quad + \sum_{i=1}^{k-1} [a_i, g_1, b_i]_{\mathfrak{g}} \otimes [a_1, [a_2, \dots, [\widehat{a_i}, \dots [a_k, g_2, b_k]_{\mathfrak{g}} \dots, \widehat{b_i}]_{\mathfrak{g}}, \dots b_2]_{\mathfrak{g}}, b_1]_{\mathfrak{g}},\end{aligned}$$

where $\widehat{g}$ means that the variable g is deleted.

Proof. The proof follows by induction and by the formula (2.1) in [6, Proposition 2.1]. □

Corollary 4.11. *If $\mathfrak{g}$ is a nilpotent gb-triple system of class s, then $\mathfrak{g}^{\otimes 2}$ is a nilpotent Leibniz algebra of class $s+1$.*

Proof. The proof follows directly by Lemma 4.10. □

Recall also that the map $A_{g_1\otimes g_2} : \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by $A_{g_1\otimes g_2}(z) = [g_1, z, g_2]_{\mathfrak{g}}$ is a derivation of $\mathfrak{g}$, and the subspace $\mathfrak{A}(\mathfrak{g}) = \{A_{g_1\otimes g_2} \mid g_1, g_2 \in \mathfrak{g}\}$ is a Lie algebra (see [6, Proposition 2.1]) with respect to the product

$$[A_{a_1\otimes a_2},\ A_{b_1\otimes b_2}]_{\mathfrak{A}(\mathfrak{g})} = A_{a_1\otimes a_2} \circ A_{b_1\otimes b_2} - A_{b_1\otimes b_2} \circ A_{a_1\otimes a_2}.$$

Proposition 4.12. *Let $\mathfrak{g}$ be a gb-triple system, $K = \{g_1 \otimes g_2 \in \mathfrak{g}^{\otimes 2} \mid A_{g_1\otimes g_2} = 0\}$. If $\mathfrak{h} := \mathfrak{g}^{\otimes 2}/K$ is a nilpotent Leibniz algebra of class s, then $\mathfrak{g}^{\otimes 2}$ is a nilpotent Leibniz algebra of class $s+1$.*

Proof. From the proof of [6, Proposition 2.4], we have

$$A_{[a_1\otimes a_2, b_1\otimes b_2]_{\mathfrak{g}^{\otimes 2}}} = [A_{a_1\otimes a_2},\ A_{b_1\otimes b_2}]_{\mathfrak{A}(\mathfrak{g})}$$

for all $a_1, a_2, b_1, b_2 \in \mathfrak{g}$. It follows that K is a 2-sided ideal of $\mathfrak{g}^{\otimes 2}$. Since $\mathfrak{h}$ is nilpotent of class s, $ad^{(s)}_{\mathfrak{h}}(\mathfrak{h}) = \{[h_1, [h_2, [\dots, [h_s, h]_{\mathfrak{h}}]_{\mathfrak{h}}]_{\mathfrak{h}}]_{\mathfrak{h}},\ h_1, h_2, \dots, h_s, h \in \mathfrak{h}\} = K$. This implies that $ad^{(s)}_{\mathfrak{g}^{\otimes 2}}(\mathfrak{g}^{\otimes 2}) \subseteq K$. Now for all $g_1 \otimes g_2 \in K$ and $a \otimes b \in \mathfrak{g}^{\otimes 2}$, we have

$$\begin{aligned} ad_{g_1\otimes g_2}(a\otimes b) &= [g_1\otimes g_2, a\otimes b]_{\mathfrak{g}^{\otimes 2}} \\ &= [g_1, a, g_2]_{\mathfrak{g}} \otimes b + a \otimes [g_1, b, g_2]_{\mathfrak{g}} \\ &= A_{g_1\otimes g_2}(a) \otimes b + a \otimes A_{g_1\otimes g_2}(b) = 0. \end{aligned}$$

So $[K, \mathfrak{g}^{\otimes 2}]_{\mathfrak{g}^{\otimes 2}} = ad_K(\mathfrak{g}^{\otimes 2}) = 0$. Therefore

$$ad^{(s+1)}_{\mathfrak{g}^{\otimes 2}}(\mathfrak{g}^{\otimes 2}) = [ad^{(s)}_{\mathfrak{g}^{\otimes 2}}(\mathfrak{g}^{\otimes 2}), \mathfrak{g}^{\otimes 2}]_{\mathfrak{g}^{\otimes 2}} \subseteq [K, \mathfrak{g}^{\otimes 2}]_{\mathfrak{g}^{\otimes 2}} = 0.$$

Hence $\mathfrak{g}^{\otimes 2}$ is nilpotent of class $s+1$. □

The following theorem is known as Engel's Theorem. It was extended to Leibniz algebras in [1].

Theorem 4.13. (cf. [13]) *A Lie algebra L is nilpotent if and only if ad_x is nilpotent for any $x \in L$, where $ad_x(y) := [x, y]$.*

Note that the Leibniz algebras version of Theorem 4.13 could be used to prove Proposition 4.12.

The following is a Engel-like Theorem for *gb*-triple system.

Theorem 4.14. *A gb-triple system $\mathfrak{g}$ is nilpotent if and only if $A_{g_1\otimes g_2}$ is nilpotent for every $g_1, g_2 \in \mathfrak{g}$.*

Proof. Assume that $\mathfrak{g}$ is nilpotent. Then $\mathfrak{g}^{(s)} = 0$ for some positive integer s. So for every $g, a_1, \ldots, a_{s-1}, b_1, \ldots, b_{s-1} \in \mathfrak{g}$,

$$[a_1, [a_2, [\ldots, [a_{s-1}, g, b_{s-1}]_g \ldots]_g, b_2]_g, b_1]_g = 0.$$

that is

$$A_{a_1 \otimes b_1} \circ A_{a_2 \otimes b_2} \circ \ldots \circ A_{a_{s-1} \otimes b_{s-1}}(g) = 0.$$

In particular,

$$\big(\underbrace{A_{g_1 \otimes g_2} \circ A_{g_1 \otimes g_2} \circ \ldots \circ A_{g_1 \otimes g_2}}_{\text{(s-1)-times}}\big)(g) = 0 \quad \text{for every} \quad g_1, g_2 \in \mathfrak{g}.$$

So for every $g_1, g_2 \in \mathfrak{g}$, $A_{g_1 \otimes g_2}$ is nilpotent.

Conversely, assume that $A_{g_1 \otimes g_2}$ is nilpotent for every $g_1, g_2 \in \mathfrak{g}$. So the Lie algebra $\mathfrak{A}(\mathfrak{g}) = \{A_{g_1 \otimes g_2} \mid g_1, g_2 \in \mathfrak{g}\}$ is a Lie algebra of nilpotent linear maps. Moreover, by the proof of [6, Proposition 3.6], $\mathfrak{A}(\mathfrak{g})$ is an ideal of $Der(\mathfrak{g})$, and thus a closed subset of $End(\mathfrak{g})$. It follows by [9, Theorem 3.5] that the associative subalgebra generated by $\mathfrak{A}(\mathfrak{g})$ is nilpotent. So there exists a positive integer s such that

$$\big(A_{a_1 \otimes b_1} \circ A_{a_2 \otimes b_2} \circ \ldots \circ A_{a_s \otimes b_s}\big)(g) = 0 \quad \text{for all} \quad g, a_1, \ldots, a_s, b_1, \ldots, b_s \in \mathfrak{g}.$$

This implies that

$$[a_1, [a_2, [\ldots, [a_s, g, b_s]_g \ldots]_g, b_2]_g, b_1]_g = 0 \quad \text{for all} \quad g, a_1, \ldots, a_s, b_1, \ldots, b_s \in \mathfrak{g}.$$

Hence $\mathfrak{g}^{(s+1)} = 0$. Therefore $\mathfrak{g}$ is nilpotent. □

Corollary 4.15. *Let $\mathfrak{g}$ be a gb-triple system. Then if $\mathfrak{g}$ is nilpotent, so is any subalgebra S of $\mathfrak{g}$.*

Proof. Let S be a subalgebra of $\mathfrak{g}$. If S is not nilpotent, then by Theorem 4.14 there exists $g_1, g_2 \in S$ such that the restriction $A^{(s)}_{g_1 \otimes g_2}|_S \neq 0$ for all positive integer s. But this implies that $A^{(s)}_{g_1 \otimes g_2} \neq 0$ for all positive integer s. So $A_{g_1 \otimes g_2}$ is not nilpotent, and thus $\mathfrak{g}$ is not nilpotent by Theorem 4.14. □

Corollary 4.16. *If L is nilpotent as a Leibniz algebra, then L is also nilpotent as a gb-triple system.*

Proof. For all $g_1, g_2 \in \mathfrak{g}$, $A_{g_1 \otimes g_2} = ad_{[g_1, g_2]}$ by Proposition 4.9. The result now follows by applying both Engel's theorems for Leibniz and gb-triple systems. □

4.3. Nilpotent ideals of gb-triple systems

For an ideal $\mathfrak{I}$ of a gb-triple system $\mathfrak{g}$, consider the series defined by $\mathfrak{I}^{(0)} = \mathfrak{g}$, and $\mathfrak{I}^{(s+1)} = [\mathfrak{I}, \mathfrak{I}^{(s)}, \mathfrak{g}]$ with $\mathfrak{I}^{(1)} = \mathfrak{I}$ where s is a positive integer, $s \geqslant 1$.

Proposition 4.17. *$\mathfrak{I}^{(s)}$ is an ideal of $\mathfrak{g}$ for every integer $s \geqslant 0$.*

Proof. The cases $s = 0, 1$ are trivial. By induction, assume that for $s \geqslant 2$, $\mathfrak{I}^{(s-1)}$ is an ideal of $\mathfrak{g}$ and let $x, y, z \in \mathfrak{g}, b \in \mathfrak{I}$ and $a \in \mathfrak{I}^{(s-1)}$. Then it follows from the identity (2.1) that

$$[x, [b, a, z]_{\mathfrak{g}}, y]_{\mathfrak{g}} = [b, a, [x, z, y]_{\mathfrak{g}}]_{\mathfrak{g}} + [\underbrace{[x, b, y]_{\mathfrak{g}}}_{\in\, \mathfrak{I}}, a, z]_{\mathfrak{g}} + [b, \underbrace{[x, a, y]_{\mathfrak{g}}}_{\in\, \mathfrak{I}^{(s-1)}}, z]_{\mathfrak{g}} \ \in \mathfrak{I}^{(s)}.$$

So $\mathfrak{I}^{(s)}$ is an ideal of $\mathfrak{g}$. □

Proposition 4.18. *Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two ideals of $\mathfrak{g}$ such that $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$. Then*

$$\mathfrak{I}_1^{(s)} \subseteq \mathfrak{I}_2^{(s)}$$

for all integer $s \geqslant 0$.

Proof. The cases $s = 0, 1$ are trivial. By induction, assume that the result is true for $s \geqslant 2$. Then $\mathfrak{I}_1^{(s+1)} = [\mathfrak{I}_1, \mathfrak{I}_1^{(s)}, \mathfrak{g}]_g \subseteq [\mathfrak{I}_2, \mathfrak{I}_2^{(s)}, \mathfrak{g}]_g = \mathfrak{I}_2^{(s+1)}$. □

Definition 4.19. *An ideal $\mathfrak{I}$ of $\mathfrak{g}$ is nilpotent if $\mathfrak{I}^{(s)} = 0$ for some positive integer s. The smallest of such values s is called class of nilpotency of $\mathfrak{I}$.*

The following lemma provides the fitting decomposition of a *gb*-triple system relative to a derivation in $\mathfrak{A}(\mathfrak{g})$.

Lemma 4.20. *Let $\mathfrak{g}$ be a finite dimensional gb-tiple system and $g_1, g_2 \in \mathfrak{g}$. Then*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid A^{(s)}_{g_1 \otimes g_2}(x) = 0$ for some integer $s > 0\}$ and $A_{g_1 \otimes g_2}(\mathfrak{g}) = \mathfrak{g}$

Proof. Apply the Fitting Lemma [15, Chapter 2] on the linear transformation $A_{g_1 \otimes g_2}$. □

The spaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are called the Fitting null and one-components of $\mathfrak{g}$ with respect to $A_{g_1 \otimes g_2}$.

The following theorem was proved in [4] for Lie algebras and in [18] for n-Lie algebras.

Theorem 4.21. *Let $\mathfrak{g}$ be a gb-triple system. If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are 3-sided ideals of $\mathfrak{g}$ such that $\mathfrak{I}_1 \subseteq \phi(\mathfrak{g}) \cap \mathfrak{I}_2$ and $\mathfrak{I}_2/\mathfrak{I}_1$ is nilpotent, then $\mathfrak{I}_2$ is nilpotent.*

Proof. We proceed by contradiction. Assume that $\mathfrak{I}_2$ is not nilpotent. Then by Theorem 4.14, there exists $g_1, g_2 \in \mathfrak{I}_2$ such that $A^{(s)}_{g_1 \otimes g_2}(x) \neq 0$ for all positive integer s. By Lemma 4.20 let $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be the Fitting null and one-components of $\mathfrak{g}$ with respect to $A_{g_1 \otimes g_2}$. Since $\mathfrak{I}_2$ is a 3-sided ideal, it follows that $\mathfrak{g}_1 \subseteq \mathfrak{I}_2$. Also, since $\mathfrak{I}_2/\mathfrak{I}_1$ is nilpotent we have $\mathfrak{I}_2^{(k)} \subseteq \mathfrak{I}_1$ for some positive integer k. It follows by definition of $\mathfrak{g}_1$ that $\mathfrak{g}_1 = A^{(k)}_{g_1 \otimes g_2}(\mathfrak{g}_1) \subseteq \mathfrak{I}_2^{(k)} \subseteq \mathfrak{I}_1$. Therefore $\mathfrak{g}_1 \subseteq \phi(\mathfrak{g})$. So

$\mathfrak{g} = \mathfrak{g}_0 + \phi(\mathfrak{g})$. Now since by [6, Proposition 2.6] $A_{g_1 \otimes g_2}$ is a derivation of $\mathfrak{g}$, $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. So there is a maximal subalgebra $\mathfrak{m}$ that contains $\mathfrak{g}_0$. Since by definition $\phi(\mathfrak{g}) \subseteq \mathfrak{m}$, it follows that $\mathfrak{m} = \mathfrak{g}$. This contradicts the maximality of $\mathfrak{m}$. Therefore $\mathfrak{I}_2$ is nilpotent. □

Corollary 4.22. *Every 3-sided ideal of a gb-tiple system $\mathfrak{g}$ contained in the Frattini ideal $\phi(\mathfrak{g})$ is nilpotent. In particular, if $\phi(\mathfrak{g})$ is a 3-sided ideal, then $\phi(\mathfrak{g})$ is a nilpotent ideal of $\mathfrak{g}$.*

Proof. Let $\mathfrak{I}$ be a 3-sided ideal of $\mathfrak{g}$ such that $\mathfrak{I} \subseteq \phi(\mathfrak{g})$. The result follows from Theorem 4.21 by setting $\mathfrak{I}_1 = \mathfrak{I}_2 = \mathfrak{I}$. In particular, take $\mathfrak{I} = \phi(\mathfrak{g})$ to show that $\phi(\mathfrak{g})$ is nilpotent. □

Corollary 4.23. *Let $\mathfrak{g}$ be a gb-triple system for which $\phi(\mathfrak{g})$ is a 3-sided ideal. Then $\mathfrak{g}/\phi(\mathfrak{g})$ is nilpotent if and only if $\mathfrak{g}$ is nilpotent.*

Proof. The first implication follows from Theorem 4.21 by setting $\mathfrak{I}_1 = \phi(\mathfrak{g})$ and $\mathfrak{I}_2 = \mathfrak{g}$. Conversely, if $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}^{(s)} = 0$ for some positive integer s. So $\big(\mathfrak{g}/\phi(\mathfrak{g})\big)^{(s)} = \mathfrak{g}^{(s)}/\phi(\mathfrak{g}) = \phi(\mathfrak{g})$. □

5. Invariance of the nilradical under derivations

The following Lemma which was proved (see [9, Lemma 3.3]) for Leibniz 3-algebras also holds for *gb*-triple systems, and the proof is identical.

Lemma 5.1. *For every derivation d of $\mathfrak{g}$ and every positive integer s,*

$$d^s([x,y,z]_g) = \sum_{i+j+k=s} \frac{s!}{i!j!k!} \big[d^i(x), d^j(y), d^k(z)\big]_g. \tag{5.1}$$

Analogues of the following results were established in [16] for Leibniz 3-algebras.

Proposition 5.2. *Let $\mathfrak{I}$ be an ideal of a gb-triple system $\mathfrak{g}$ and d a derivation of $\mathfrak{g}$. Then*

$$\big(d(\mathfrak{I})\big)^{(s)} \subseteq d^s(\mathfrak{I}^{(s)}) \tag{5.2}$$

for all positive integer s.

Proof. Notice that the assertion is trivial for $s = 1$. Now assume by induction that the result holds for any positive integer s, then

$$\big(d(\mathfrak{I})\big)^{(s+1)} = \big[d(\mathfrak{I}), \big(d(\mathfrak{I})\big)^{(s)}, \mathfrak{g}\big]_{\mathfrak{g}} \subseteq \big[d(\mathfrak{I}), \big(d^s(\mathfrak{I}^{(s)}\big), \mathfrak{g}\big]_{\mathfrak{g}} \subseteq d^{s+1}\big(\mathfrak{I}^{(s+1)}\big) \text{ by (5.1)}.$$

□

Proposition 5.3. *Let $\mathfrak{I}$ be an ideal of a gb-triple system $\mathfrak{g}$ and d a derivation of $\mathfrak{g}$. Then for all $s \geqslant 2$,*

$$\big(\mathfrak{I} + d(\mathfrak{I})\big)^{(s)} \subseteq \mathfrak{I} + \big(d(\mathfrak{I})\big)^{(s)} + \sum_{i=1}^{s-1} d^i(\mathfrak{I}^{(s)}). \tag{5.3}$$

Proof. We verify the assertion for $s = 2$.

$$\begin{aligned}\big(\mathfrak{I} + d(\mathfrak{I})\big)^{(2)} &= [\mathfrak{I} + d(\mathfrak{I}), \mathfrak{I} + d(\mathfrak{I}), \mathfrak{g}]_{\mathfrak{g}}\\ &\subseteq \mathfrak{I} + [\mathfrak{I}, d(\mathfrak{I}), \mathfrak{g}]_{\mathfrak{g}} + [d(\mathfrak{I}), d(\mathfrak{I}), \mathfrak{g}]_{\mathfrak{g}}\\ &\subseteq \mathfrak{I} + d(\mathfrak{I}^{(2)}) + (d(\mathfrak{I}))^{(2)} \text{ by } (5.1).\end{aligned}$$

Now assume by induction that the result holds for any positive integer s, then

$$\begin{aligned}\big(\mathfrak{I} + d(\mathfrak{I})\big)^{(s+1)} &= \big[\mathfrak{I} + d(\mathfrak{I}), \big(\mathfrak{I} + d(\mathfrak{I})\big)^{(s)}, \mathfrak{g}\big]_{\mathfrak{g}}\\ &\subseteq \Big[\mathfrak{I} + d(\mathfrak{I}), \mathfrak{I} + \big(d(\mathfrak{I})\big)^{(s)} + \sum_{i=1}^{s-1} d^i(\mathfrak{I}^{(s)}), \mathfrak{g}\Big]_{\mathfrak{g}}\\ &\subseteq \mathfrak{I} + [\mathfrak{I}, (d(\mathfrak{I}))^{(s)}, \mathfrak{g}]_{\mathfrak{g}} + \sum_{i=1}^{s-1} \big[\mathfrak{I}, d^i(\mathfrak{I}^{(s)}), \mathfrak{g}\big]_{\mathfrak{g}}\\ &\quad + [d(\mathfrak{I}), (d(\mathfrak{I}))^{(s)}, \mathfrak{g}]_{\mathfrak{g}} + \sum_{i=1}^{s-1} \big[d(\mathfrak{I}), d^i(\mathfrak{I}^{(s)}), \mathfrak{g}\big]_{\mathfrak{g}}\\ &\subseteq \mathfrak{I} + \underbrace{d^s(\mathfrak{I}^{(s+1)})}_{\text{by } (5.2) \text{ and } (5.1)} + \sum_{i=1}^{s-1} \underbrace{d^i(\mathfrak{I}^{(s+1)})}_{\text{by } (5.1)} + (d(\mathfrak{I}))^{(s+1)}\\ &\quad + \sum_{i=1}^{s-1} d^{i+1}(\mathfrak{I}^{(s+1)}) \text{ by } (5.1)\\ &\subseteq \mathfrak{I} + 2\sum_{i=1}^{s} d^i(\mathfrak{I}^{(s+1)}) + (d(\mathfrak{I}))^{(s+1)}\\ &\subseteq \mathfrak{I} + \sum_{i=1}^{s} d^i(\mathfrak{I}^{(s+1)}) + (d(\mathfrak{I}))^{(s+1)}.\end{aligned}$$

□

For the remainder of this paper, we assume that all ideals are also right ideals. We call them 2-*sided* (*right*) *ideals.*

Lemma 5.4. *If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are 2-sided (right) ideals of a gb-triple system $\mathfrak{g}$, then*

$$\big(\mathfrak{I}_1 + \mathfrak{I}_2\big)^{(s)} \subseteq \sum_{\substack{i+j=s,\\ 0\leqslant i,j\leqslant s}} \mathfrak{I}_1^{(i)} \bigcap \mathfrak{I}_2^{(j)}.$$

Proof. By definition, $\big(\mathfrak{I}_1+\mathfrak{I}_2\big)^{(1)}=\mathfrak{I}_1+\mathfrak{I}_2=\mathfrak{I}_1^{(1)}\bigcap\mathfrak{I}_2^{(0)}+\mathfrak{I}_1^{(0)}\bigcap\mathfrak{I}_2^{(1)}$ since $\mathfrak{I}_1^{(0)}=\mathfrak{I}_2^{(0)}=\mathfrak{g}$. By induction, assume the result holds for $\big(\mathfrak{I}_1+\mathfrak{I}_2\big)^{(s-1)}$. Then

$$\begin{aligned}\big(\mathfrak{I}_1+\mathfrak{I}_2\big)^{(s)}&=[\mathfrak{I}_1+\mathfrak{I}_2,\big(\mathfrak{I}_1+\mathfrak{I}_2\big)^{(s-1)},\mathfrak{g}]_{\mathfrak{g}}\\&=[\mathfrak{I}_1,\big(\mathfrak{I}_1+\mathfrak{I}_2\big)^{(s-1)},\mathfrak{g}]_{\mathfrak{g}}+[\mathfrak{I}_2,\big(\mathfrak{I}_1+\mathfrak{I}_2\big)^{(s-1)},\mathfrak{g}]_{\mathfrak{g}}\\&\subseteq[\mathfrak{I}_1,\mathfrak{I}_1^{(s-1)},\mathfrak{g}]_{\mathfrak{g}}+\sum_{r=1}^{s-2}\big[\mathfrak{I}_1,\mathfrak{I}_1^{(s-r-1)}\bigcap\mathfrak{I}_2^{(r)},\mathfrak{g}\big]_{\mathfrak{g}}+[\mathfrak{I}_1,\mathfrak{I}_2^{(s-1)},\mathfrak{g}]_{\mathfrak{g}}\\&\quad+[\mathfrak{I}_2,\mathfrak{I}_1^{(s-1)},\mathfrak{g}]_{\mathfrak{g}}+\sum_{r=1}^{s-2}\big[\mathfrak{I}_2,\mathfrak{I}_1^{(s-r-1)}\bigcap\mathfrak{I}_2^{(r)},\mathfrak{g}\big]_{\mathfrak{g}}+[\mathfrak{I}_2,\mathfrak{I}_2^{(s-1)},\mathfrak{g}]_{\mathfrak{g}}\\&\subseteq\sum_{\substack{i+j=s,\\0\leqslant i,j\leqslant s}}\mathfrak{I}_1^{(i)}\bigcap\mathfrak{I}_2^{(j)}\end{aligned}$$

because

$$\big[\mathfrak{I}_1,\mathfrak{I}_1^{(s-r-1)}\cap\mathfrak{I}_2^{(r)},\mathfrak{g}\big]_{\mathfrak{g}}\subseteq\mathfrak{I}_1^{(s-r)}\cap\mathfrak{I}_2^{(r)},$$
$$\big[\mathfrak{I}_2,\mathfrak{I}_1^{(s-r-1)}\cap\mathfrak{I}_2^{(r)},\mathfrak{g}\big]_{\mathfrak{g}}\subseteq\mathfrak{I}_1^{(s-r-1)}\cap\mathfrak{I}_2^{(r+1)},$$

and

$$\big[\mathfrak{I}_1,\mathfrak{I}_2^{(s)},\mathfrak{g}\big]_{\mathfrak{g}}\subseteq\mathfrak{I}_1^{(1)}\bigcap\mathfrak{I}_2^{(s)},\qquad\big[\mathfrak{I}_2,\mathfrak{I}_1^{(s)},\mathfrak{g}\big]_{\mathfrak{g}}\subseteq\mathfrak{I}_1^{(s)}\bigcap\mathfrak{I}_2^{(1)}$$

as $\mathfrak{I}_1^{(s-r-1)},\mathfrak{I}_2^{(r)}$, $\mathfrak{I}_1^{(s)},\mathfrak{I}_2^{(s)}$ are ideals and $\mathfrak{I}_1,\mathfrak{I}_2$ are right ideals. □

Proposition 5.5. *If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are nilpotent 2-sided (right) ideals of $\mathfrak{g}$, then $\mathfrak{I}_1+\mathfrak{I}_2$ is also a nilpotent 2-sided (right) ideal of $\mathfrak{g}$.*

Proof. This follows by definition and using Lemma 5.4. More precisely one shows that if $\mathfrak{I}_1$ is nilpotent of class s_1 and $\mathfrak{I}_2$ is nilpotent of class s_2, then $\mathfrak{I}_1+\mathfrak{I}_2$ is nilpotent of class s_1+s_2. □

As a consequence of Proposition 5.5, the sum of all nilpotent 2-sided (right) ideals of $\mathfrak{g}$ is also nilpotent and contains all nilpotent 2-sided (right) ideals of $\mathfrak{g}$. It is the unique maximal nilpotent 2-sided (right) ideal called *nilradical 2-sided (right) ideal* of $\mathfrak{g}$ and denoted $\mathfrak{N}$.

The following result shows that $\mathfrak{N}$ is invariant under derivations of $\mathfrak{g}$.

Corollary 5.6. *For every derivation d of $\mathfrak{g}$, we have $d(\mathfrak{N})\subseteq\mathfrak{N}$.*

Proof. Since $\mathfrak{N}$ is nilpotent, $\mathfrak{N}^{(s)}=0$ for some positive integer s. Then by (5.2) and (5.3), it follows that $\big(\mathfrak{N}+d(\mathfrak{N})\big)^{(s)}\subseteq\mathfrak{N}+\big(d(\mathfrak{N})\big)^{(s)}\subseteq\mathfrak{N}+d^s(\mathfrak{N}^{(s)})\subseteq\mathfrak{N}$. Now by Proposition 4.18, $\big(\mathfrak{N}+d(\mathfrak{N})\big)^{(2s)}\subseteq\mathfrak{N}^{(s)}=0$. Thus $\mathfrak{N}+d(\mathfrak{N})$ is nilpotent. Therefore $\mathfrak{N}+d(\mathfrak{N})\subseteq\mathfrak{N}$ as $\mathfrak{N}$ is maximal. Hence $d(\mathfrak{N})\subseteq\mathfrak{N}$. □

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