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Centralizers on semiprime MA-semirings

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Abstract. Let T_1, T_2 be left centralizers on 2-torsion free non-commutative semiprime MA-semiring S such that $[T_2(x), T_1(x)]T_2(x) + T_2(x)[T_2(x), T_1(x)] = 0$ holds for all $x \in S$, then $[T_1(x), T_2(x)] = 0$.

1. Introduction

By a *semiring* we mean a nonempty set S in which two binary operations '+' (addition) and '$\cdot$' (multiplication) are defined in this way that $(S, \cdot)$ is a semigroup and $(S, +)$ is a commmutative semigroup with an absorbing zero 0 (i.e., $a + 0 = 0 + a = a$, $a0 = 0 = 0a$, for all $a \in S$) and both right and left distributive laws holds in S. A semiring S is called an *inverse semiring* [4] if for every $a \in S$ there exists an element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$, where a' is called the *pseudo inverse* of a. Throughout this paper S will denote MA-semiring which is an inverse semiring that satisfies the Bandlet's and Petrich's condition A_2, i.e., $a + a'$ is in center $Z(S)$ of S. For example, commutative inverse semirings and distributive lattices are MA-semirings. For more examples (non-commutative) we refer reader to [2]. According to [2], a *commutator* $[.,.]$ is defined as $[x, y] = xy + y'x$. We will make use the following commutator identities:

$$[x, yz] = [x, y]z + y[x, z], \quad [xy, z] = [x, z]y + x[y, z], \quad [xy, x] = x[y, x],$$
$$[x, xy] = x[x, y], \quad [xy, y] = [x, y]y, \quad [y, xy] = [y, x]y$$

(see [2], for their proofs). One can see that these fundamental identities including jacobian identity are useful tools to explore and extend various Lie type results of rings in the structure of inverse semirings (see [2] and [3]).

A semiring S is *prime* if $aSb = (0)$ implies $a = 0$ or $b = 0$ and it is *semiprime* if $aSa = 0$ implies $a = 0$. S is *n-torsion free* if $nx = 0$, $x \in S$ implies $x = 0$. Following [7], an additive mapping $T : S \to S$ is called a *left* (*right*) *centralizer* if for all $x, y \in S$, $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) and T is a *centralizer* if it is both right and left centralizer.

Motivated by the work of Zalar [7] on centralizers, J. Vukman investigated the identities satisfied by centralizers on semiprime rings.

In this paper, we explore these identities and extend J. Vukman's results [6] to MA-semirings.

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2. Main results

To prove our results we need the following lemma (Lemm 1.1 in [5]).

Lemma 2.1. *Let S be an inverse semiring. Then $a+b=0$ implies $a=b'$, for all $a,b\in S$.*

Theorem 2.2. *Let S be a 2-torsion free non-commutative semiprime MA-semiring and T_1,T_2 be left centralizers on S. If*

$$[T_2(x),T_1(x)]T_2(x)+T_2(x)[T_2(x),T_1(x)]=0 \tag{1}$$

holds for all $x\in S$, then $[T_2(x),T_1(x)]=0$.

Proof. Linearize (1), we get

$$\begin{aligned}&[T_2(x),T_1(x)]T_2(y)+T_2(y)[T_2(x),T_1(x)]+[T_2(y),T_1(y)]T_2(x)+\\&\quad T_2(x)[T_2(y),T_1(y)]+[T_2(y),T_1(x)]T_2(x)+T_2(x)[T_2(y),T_1(x)]+\\&\quad [T_2(x),T_1(y)]T_2(x)+T_2(x)[T_2(x),T_1(y)]+[T_2(x),T_1(y)]T_2(y)+\\&\quad T_2(y)[T_2(x),T_1(y)]+[T_2(y),T_1(x)]T_2(y)+T_2(y)[T_2(y),T_1(x)]=0.\end{aligned} \tag{2}$$

Replacing x by x' in (2), and using the fact that $T_i(x')=T_i'(x)=(T_i(x))'$, $i=1,2$, we have

$$\begin{aligned}&[T_2(x),T_1(x)]T_2(y)+T_2(y)[T_2(x),T_1(x)]+[T_2(y),T_1(y)]T_2(x')+\\&\quad T_2(x')[T_2(y),T_1(y)]+[T_2(y),T_1(x)]T_2(x)+T_2(x)[T_2(y),T_1(x)]+\\&\quad [T_2(x),T_1(y)]T_2(x)+T_2(x)[T_2(x),T_1(y)]+[T_2(x'),T_1(y)]T_2(y)+\\&\quad T_2(y)[T_2(x'),T_1(y)]+[T_2(y),T_1(x')]T_2(y)+T_2(y)[T_2(y),T_1(x')]=0.\end{aligned} \tag{3}$$

Multiplying (2) by 2 and adding the result in (3), we have

$$\begin{aligned}&3[T_2(x),T_1(x)]T_2(y)+3T_2(y)[T_2(x),T_1(x)]+[T_2(y),T_1(y)]T_2(x'+2x)+\\&\quad T_2(x'+2x)[T_2(y),T_1(y)]+3[T_2(y),T_1(x)]T_2(x)+3T_2(x)[T_2(y),T_1(x)]+\\&\quad 3[T_2(x),T_1(y)]T_2(x)+3T_2(x)[T_2(x),T_1(y)]+[T_2(x'+2x),T_1(y)]T_2(y)+\\&T_2(y)[T_2(x'+2x),T_1(y)]+[T_2(y),T_1(x'+2x)]T_2(y)+T_2(y)[T_2(y),T_1(x'+2x)]=0.\end{aligned}$$

Using (2) and the fact that S is a 2-torsion free inverse semiring, we get

$$\begin{aligned}&[T_2(x),T_1(x)]T_2(y)+T_2(y)[T_2(x),T_1(x)]+[T_2(y),T_1(x)]T_2(x)+\\&\quad T_2(x)[T_2(y),T_1(x)]+[T_2(x),T_1(y)]T_2(x)+T_2(x)[T_2(x),T_1(y)]=0.\end{aligned} \tag{4}$$

Replacing y by xy in (4) and then using (1) we obtain

$$T_2(x)y[T_2(x),T_1(x)]+2[T_2(x),T_1(x)]yT_2(x)+T_2(x)[y,T_1(x)]T_2(x)+ (T_2(x))^2[y,T_1(x)]+T_1(x)[T_2(x),y]T_2(x)+T_2(x)[T_2(x),T_1(x)]y+ T_2(x)T_1(x)[T_2(x),y]=0. \quad (5)$$

Replacing y by $yT_2(x)$ in last equation, we get

$$T_2(x)yT_2(x)[T_2(x),T_1(x)]+T_2(x)y[T_2(x),T_1(x)]T_2(x)+ 2[T_2(x),T_1(x)]y(T_2(x))^2+T_2(x)[y,T_1(x)](T_2(x))^2+ (T_2(x))^2y[T_2(x),T_1(x)]+(T_2(x))^2[y,T_1(x)]T_2(x)+ T_1(x)[T_2(x),y](T_2(x))^2+T_2(x)[T_2(x),T_1(x)]yT_2(x)+ T_2(x)T_1(x)[T_2(x),y]T_2(x)=0. \quad (6)$$

Using (5) in (6), we get

$$T_2(x)yT_2(x)[T_2(x),T_1(x)]+(T_2(x))^2y[T_2(x),T_1(x)]=0 \quad (7)$$

Replacing y by $T_1(x)y$ in (7), we have

$$T_2(x)T_1(x)yT_2(x)[T_2(x),T_1(x)]+(T_2(x))^2T_1(x)y[T_2(x),T_1(x)]=0. \quad (8)$$

Pre-multiplying (7) by $T_1(x)$, we have

$$T_1(x)T_2(x)yT_2(x)[T_2(x),T_1(x)]+T_1(x)(T_2(x))^2y[T_2(x),T_1(x)]=0. \quad (9)$$

Adding pseudo inverse of (9) in (8) and then using (1), we obtain

$$[T_2(x),T_1(x)]yT_2(x)[T_2(x),T_1(x)]=0, \quad (10)$$

which implies

$$T_2(x)[T_2(x),T_1(x)]=0. \quad (11)$$

From (1), we get

$$[T_2(x),T_1(x)]T_2(x)=0. \quad (12)$$

As (4) obtained from (1), from (12) we get

$$[T_2(y),T_1(x)]T_2(x)+[T_2(x),T_1(y)]T_2(x)+[T_2(x),T_1(x)]T_2(y)=0. \quad (13)$$

Replacing y by xy in (13) and then using (12), we get

$$[T_2(x),T_1(x)](2y+y')T_2(x)+T_2(x)yT_1(x)T_2(x)+T_1(x)y'(T_2(x))^2=0$$

or

$$[T_2(x),T_1(x)]yT_2(x)+T_2(x)yT_1(x)T_2(x)+T_1(x)y'(T_2(x))^2=0. \quad (14)$$

Post-multiplying (14) by $T_1(x)$

$$[T_2(x), T_1(x)]yT_2(x)T_1(x) + T_2(x)yT_1(x)T_2(x)T_1(x) + T_1(x)y'(T_2(x))^2T_1(x) = 0. \tag{15}$$

Replacing y by $yT_1(x)$ in (14), we get

$$[T_2(x), T_1(x)]yT_1(x)T_2(x) + T_2(x)y(T_1(x))^2T_2(x) + T_1(x)y'T_1(x)(T_2(x))^2 = 0. \tag{16}$$

Adding pseudo inverse of (16) in (15) and then using (1), we have

$$[T_2(x), T_1(x)]y[T_2(x), T_1(x)] + T_2(x)yT_1(x)[T_2(x), T_1(x)] + T_1(x)y[(T_2)^2, T_1(x)] = 0$$

or

$$[T_2(x), T_1(x)]y[T_2(x), T_1(x)] + T_2(x)yT_1(x)[T_2(x), T_1(x)] = 0. \tag{17}$$

Replacing y by $zT_2(x)y$ in (17), we have

$$[T_2(x), T_1(x)]zT_2(x)y[T_2(x), T_1(x)] + T_2(x)zT_2(x)yT_1(x)[T_2(x), T_1(x)]. \tag{18}$$

Pre-multiplying (17) by $T_2(x)z$

$$T_2(x)z[T_2(x), T_1(x)]y[T_2(x), T_1(x)] + T_2(x)zT_2(x)yT_1(x)[T_2(x), T_1(x)]. \tag{19}$$

Applying Lemma 2.1 to (19) and using it in (18), we get

$$F(x, z)y[T_2(x), T_1(x)] = 0, \tag{20}$$

where $F(x, z) = [T_2(x), T_1(x)]zT_2(x) + T_2(x)z'[T_2(x), T_1(x)]$.
Replacing y by $yT_2(x)z$ in (20)

$$F(x, z)yT_2(x)z[T_2(x), T_1(x)] = 0. \tag{21}$$

Post-multiplying (20) by $zT_2(x)$

$$F(x, z)y[T_2(x), T_1(x)]zT_2(x) = 0. \tag{22}$$

Adding pseudo inverse of (21) in (22), we get

$$F(x, z)yF(x, z) = 0.$$

Semiprimness of S implies

$$F(x, z) = [T_2(x), T_1(x)]zT_2(x) + T_2(x)z'[T_2(x), T_1(x)] = 0. \tag{23}$$

Applying Lemma 2.1 to (23), we get

$$[T_2(x), T_1(x)]zT_2(x) = T_2(x)z[T_2(x), T_1(x)]. \tag{24}$$

Replacing z by $yT_1(x)$ in last equation, we have

$$[T_2(x), T_1(x)]yT_1(x)T_2(x) = T_2(x)yT_1(x)[T_2(x), T_1(x)]. \tag{25}$$

Using (25) to (17), we get

$$[T_2(x), T_1(x)]yT_2(x)T_1(x) + [T_2(x), T_1(x)](y + y')(T_1(x))'T_2(x) = 0$$

or

$$[T_2(x), T_1(x)]yT_2(x)T_1(x) + [T_2(x), T_1(x)]y(T_1(x) + T_1(x'))T_2(x) = 0,$$

since $T_1(x) + T_1(x') \in Z(S)$, so we have

$$[T_2(x), T_1(x)]yT_2(x)T_1(x) + [T_2(x), T_1(x)]yT_2(x)(T_1(x) + T_1(x')) = 0$$

or

$$[T_2(x), T_1(x)]yT_2(x)T_1(x) = 0. \tag{26}$$

Replacing y by $yT_1(x)$ in above equation

$$[T_2(x), T_1(x)]yT_1(x)T_2(x)T_1(x) = 0. \tag{27}$$

Post-multiplying (26) by $T_1(x)$

$$[T_2(x), T_1(x)]yT_2(x)(T_1(x))^2 = 0. \tag{28}$$

Adding pseudo inverse of (27) in (28), we have

$$[T_2(x), T_1(x)]y[T_2(x), T_1(x)]T_1(x) = 0.$$

Replacing y by $T_1(x)y$ and using semiprimeness of S, we have

$$[T_2(x), T_1(x)]T_1(x) = 0. \tag{29}$$

Replacing z by $T_1(x)y$ in (24) and using (29), we have

$$T_2(x)T_1(x)y[T_2(x), T_1(x)] = 0. \tag{30}$$

As (14) obtained from (12), from (11), we obtain

$$T_2(x)y[T_2(x), T_1(x)] + T_2(x)T_1(x)y'T_2(x) + (T_2(x))^2yT_1(x) = 0. \tag{31}$$

By (24), we have

$$T_2(x)T_1(x)(y + y')T_2(x) + (T_1(x))'T_2(x))yT_2(x) + (T_2(x))^2yT_1(x) = 0$$

or

$$T_2(x)(T_1(x) + T_1(x'))yT_2(x) + (T_1(x))'T_2(x))yT_2(x) + (T_2(x))^2yT_1(x) = 0$$

or

$$(T_1(x)+T_1(x'))T_2(x)yT_2(x)+(T_1(x))'T_2(x))yT_2(x)+(T_2(x))^2yT_1(x)=0$$

$$(T_1(x))'T_2(x)yT_2(x)+(T_2(x))^2yT_1(x)=0. \quad (32)$$

Replacing y by $yT_1(x)$ in last equation

$$(T_1(x))'T_2(x)yT_1(x)T_2(x)+(T_2(x))^2y(T_1(x))^2=0. \quad (33)$$

Post-multiplying (32) by $T_1(x)$

$$(T_1(x))'T_2(x)yT_2(x)T_1(x)+(T_2(x))^2y(T_1(x))^2=0. \quad (34)$$

Applying Lemma 2.1 to (34) and using the result in (33) we get

$$T_1(x)T_2(x)y[T_2(x),T_1(x)]=0. \quad (35)$$

Adding pseudo inverse of (35) in (30) we get

$$[T_2(x),T_1(x)]y[T_2(x),T_1(x)].$$

This implies

$$[T_2(x),T_1(x)]=0,$$

which completes the proof. □

Theorem 2.3. *Let S be 2-torsion free non-commutative semiprime MA-semiring. If T_1 and T_2 are left centralizers on S satisfying*

$$[[T_2(x),T_1(x)],T_2(x)]=0 \quad (36)$$

for all $x\in S$. then $[T_2(x),T_1(x)]=0$.

Proof. As (4) obtained from (1), from (36), we get

$$[[T_2(x),T_1(x)],T_2(y)]+[[T_2(x),T_1(y)],T_2(x)]+[[T_2(y),T_1(x)],T_2(x)]=0. \quad (37)$$

Replacing y by xy in last equation and using (36), we obtain

$$T_2(x)[[T_2(x),T_1(x)],y]+3[T_2(x),T_1(x)][y,T_2(x)]+ T_1(x)[[T_2(x),y],T_2(x)]+T_2(x)[[y,T_1(x)],T_2(x)]=0. \quad (38)$$

Replacing y by $yT_2(x)$ in last equation, we get

$$T_2(x)[[T_2(x),T_1(x)],y]T_2(x)+T_2(x)y[[T_2(x),T_1(x)],T_2(x)]+ 3[T_2(x),T_1(x)][y,T_2(x)]T_2(x)+T_1(x)[y[T_2(x),T_2(x)],T_2(x)]+ T_1(x)[[T_2(x),y]T_2(x),T_2(x)]+T_2(x)[y[T_2(x),T_1(x)],T_2(x)]+ T_2(x)[[y,T_1(x)],T_2(x)]T_2(x)=0. \quad (39)$$

Now as $T_2(x)+T_2(x')\in Z(S)$, so we have

$$\begin{aligned}&T_1(x)[y[T_2(x),T_2(x)],T_2(x)]+T_1(x)[[T_2(x),y]T_2(x),T_2(x)]\\&=T_1(x)[y(T_2(x)+T_2(x'))T_2(x),T_2(x)]+T_1(x)[(T_2(x)yT_2(x)+y'T_2(x)T_2(x)),T_2(x)]\\&=T_1(x)[(T_2(x)+T_2(x')+T_2(x))yT_2(x)+y'T_2(x)T_2(x),T_2(x)]\\&=T_1(x)[[T_2(x),y]T_2(x),T_2(x)].\qquad(40)\end{aligned}$$

From (40), (39) and (38), we obtain

$$T_2(x)[y,T_2(x)][T_2(x),T_1(x)]=0$$

or

$$T_2(x)yT_2(x)[T_2(x),T_1(x)]+(T_2(x))^2y'[T_2(x),T_1(x)]=0.\qquad(41)$$

Replacing y by $T_1(x)y$ in last equation, we obtain

$$T_2(x)T_1(x)yT_2(x)[T_2(x),T_1(x)]+(T_2(x))^2T_1(x)y'[T_2(x),T_1(x)]=0.\qquad(42)$$

Pre-multiplying (41) by $T_1(x)$, we get

$$T_1(x)T_2(x)yT_2(x)[T_2(x),T_1(x)]+T_1(x)(T_2(x))^2y'[T_2(x),T_1(x)]=0.\qquad(43)$$

Adding pseudo inverse of (43) in (42)

$$\begin{aligned}&[T_2(x),T_1(x)]yT_2(x)[T_2(x),T_1(x)]+T_2(x)[T_2(x),T_1(x)]y'[T_2(x),T_1(x)]+\\&[T_2(x),T_1(x)]T_2(x)y'[T_2(x),T_1(x)]=0.\qquad(44)\end{aligned}$$

Applying Lemma 2.1 in (36) and using the result in the last equation, we have

$$[T_2(x),T_1(x)]yT_2(x)[T_2(x),T_1(x)]+2T_2(x)[T_2(x),T_1(x)]y'[T_2(x),T_1(x)]=0.\qquad(45)$$

Pre-multiplying (45) by $T_2(x)$, we have

$$\begin{aligned}&2(T_2(x))^2[T_2(x),T_1(x)]y'[T_2(x),T_1(x)]+\\&T_2(x)[T_2(x),T_1(x)]yT_2(x)[T_2(x),T_1(x)]=0.\qquad(46)\end{aligned}$$

Replacing y by $[T_2(x),T_1(x)]y$ in (41), we have

$$\begin{aligned}&T_2(x)[T_2(x),T_1(x)]yT_2(x)[T_2(x),T_1(x)]+\\&(T_2(x))^2[T_2(x),T_1(x)]y'[T_2(x),T_1(x)]=0.\qquad(47)\end{aligned}$$

Applying Lemma 2.1 to (46) and using the result in (47), we have

$$T_2(x)[T_2(x),T_1(x)]yT_2(x)[T_2(x),T_1(x)]=0,\qquad(48)$$

and semiprimness of S implies

$$T_2(x)[T_2(x),T_1(x)]=0,\qquad(49)$$

and from (36) and Lemma 2.1, we obtain

$$[T_2(x),T_1(x)]T_2(x),\qquad(50)$$

which gives $[T_2(x),T_1(x)]=0$, as in the proof of Theorem 2.2. □

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