Quasigroups and Related Systems **24** (2016), 219 − 230

On (i, j)-commutativity in Menger algebras of n-place functions

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Abstract. We present an abstract characterization of Menger $(2,n)$-semigroups of n-place functions containing the operation $\pi_{ij}\colon f(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n)\mapsto f(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n)$.

1. Introduction

On the set $\mathcal{F}(A^n,A)$ of all partial n-place functions $f\colon A^n\to A$ $(n\geqslant 2)$ one can consider the following operations:

- the $(n+1)$-ary *Menger's superposition* $\mathcal{O}\colon (f,g_1,\ldots,g_n)\mapsto f[g_1\ldots g_n]$ such that

$$f[g_1\ldots g_n](a_1^n)=f(g_1(a_1^n),g_2(a_1^n),\ldots,g_n(a_1^n)),$$

- the binary *Mann's superpositions* $\underset{1}{\oplus},\underset{2}{\oplus},\ldots,\underset{n}{\oplus}$ defined by

$$(f\underset{i}{\oplus}g)(a_1^n)=f(a_1^{i-1},g(a_1^n),a_{i+1}^n),$$

where a_i^j denotes the sequence $a_i,a_{i+1},\ldots,a_{j-1},a_j$ if $i\leqslant j$, and the empty symbol if $i>j$.

Let Φ be a nonempty subset of $\mathcal{F}(A^n,A)$. If Φ is closed with respect to the Menger superposition, then the algebra $(\Phi,\mathcal{O})$ is called a *Menger algebra of n-place functions*. Since each Mann's superposition is an associative operation, the algebra $(\Phi,\underset{1}{\oplus},\underset{2}{\oplus},\ldots,\underset{n}{\oplus})$ is called a *$(2,n)$-semigroup of n-place functions*. Consequently, the algebra $(\Phi,\mathcal{O},\underset{1}{\oplus},\underset{2}{\oplus},\ldots,\underset{n}{\oplus})$ is called a *Menger $(2,n)$-semigroup of n-place functions*.

One can prove (cf. [3] or [8]) that an abstract $(n+1)$-ary algebra (G,o) is isomorphic to some algebra $(\Phi,\mathcal{O})$ of n-place functions if and only if it satisfies the *superassociative law*:

$$x[y_1\ldots y_n][z_1\ldots z_n]=x[y_1[z_1\ldots z_n]\ldots y_n[z_1\ldots z_n]],\qquad(1)$$

2010 Mathematics Subject Classification: 20N15
Keywords: Menger algebra, $(2,n)$-semigroup, n-place function, (i,j)-commutativity.

where $x[y_1 \ldots y_n]$ denotes $o(x, y_1, \ldots, y_n)$. An $(n+1)$-ary algebra (G, o) satisfying this law is called a *Menger algebra of rank* n. An algebra $(G, \underset{1}{\oplus}, \underset{2}{\oplus}, \ldots, \underset{n}{\oplus})$ with n binary associative operations $\underset{1}{\oplus}, \underset{2}{\oplus}, \ldots, \underset{n}{\oplus}$ is called a $(2,n)$-*semigroup*. A $(2,n)$-semigroup closed with respect to an $(n+1)$-ary operation o satisfying (1) is called a *Menger* $(2,n)$-*semigroup* and is denoted by $(G, o, \underset{1}{\oplus}, \underset{2}{\oplus}, \ldots, \underset{n}{\oplus})$.

For simplicity, all expressions of the form $(\cdots((x \underset{i_1}{\oplus} y_1) \underset{i_2}{\oplus} y_2)\cdots) \underset{i_k}{\oplus} y_k$ will be denoted by $x \overset{i_k}{\underset{i_1}{\oplus}} y_1^k$. In the case when $i = i_k$ and $i \notin \{i_1, \ldots, i_{k-1}\}$ for some $k \in \{1, \ldots, s\}$, the expression $x_k \overset{i_s}{\underset{i_{k+1}}{\oplus}} x_{k+1}^s$ will be written in the form $\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s)$. In any other case $\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s)$ is the empty symbol. For example, $\mu_1(\underset{2}{\oplus} x \underset{1}{\oplus} y \underset{3}{\oplus} z) = y \underset{3}{\oplus} z$, $\mu_2(\underset{2}{\oplus} x \underset{1}{\oplus} y \underset{3}{\oplus} z) = x \underset{1}{\oplus} y \underset{3}{\oplus} z$, $\mu_3(\underset{2}{\oplus} x \underset{1}{\oplus} y \underset{3}{\oplus} z) = z$. The symbol $\mu_4(\underset{2}{\oplus} x \underset{1}{\oplus} y \underset{3}{\oplus} z)$ is empty.

It is known (cf. [7] or [8]) that *an algebra* $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ *with* n *binary operations is isomorphic to some algebra* $(\Phi, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ *of* n*-place functions if and only if for all* $g, x_i, y_j \in G$, $i = 1, \ldots, s$, $j = 1, \ldots, k$, *it satisfies the implication*

$$\bigwedge_{i=1}^{n} \Big(\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s) = \mu_i(\overset{j_k}{\underset{j_1}{\oplus}} y_1^k)\Big) \longrightarrow g \overset{i_s}{\underset{i_1}{\oplus}} x_1^s = g \overset{j_k}{\underset{j_1}{\oplus}} y_1^k, \qquad (2)$$

where $i_1, \ldots, i_s, j_1, \ldots, j_k \in \{1, \ldots, n\}$.

Note that the condition (2) implies the associativity of all binary operations $\underset{1}{\oplus}, \underset{2}{\oplus}, \ldots, \underset{n}{\oplus}$. Indeed, for two expressions $\underset{i}{\oplus} y \underset{i}{\oplus} z$ and $\underset{i}{\oplus}(y \underset{i}{\oplus} z)$, where $y, z \in G$, we have $\mu_i(\underset{i}{\oplus} y \underset{i}{\oplus} z) = y \underset{i}{\oplus} z = \mu_i(\underset{i}{\oplus}(y \underset{i}{\oplus} z))$. For $k \neq i$ the symbols $\mu_k(\underset{i}{\oplus} y \underset{i}{\oplus} z)$ and $\mu_k(\underset{i}{\oplus}(y \underset{i}{\oplus} z))$ are empty. So, the premise of (2) is satisfied. Therefore for all $x \in G$ we have $x \underset{i}{\oplus} y \underset{i}{\oplus} z = x \underset{i}{\oplus}(y \underset{i}{\oplus} z)$, i.e., $(x \underset{i}{\oplus} y) \underset{i}{\oplus} z = x \underset{i}{\oplus}(y \underset{i}{\oplus} z)$.

An abstract characterization of Menger $(2,n)$-semigroup of n-place functions is more difficult. For such characterization we need to use the implication (2) and several identities. Namely, as it is proved in [6] (cf. also [8]), *an algebra* $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ *of type* $(n+1, 2, \ldots, 2)$ *is isomorphic to some algebra* $(\Phi, \mathcal{O}, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ *of* n*-place functions if and only if it satisfies* (1), (2) *and*

$$(x \underset{i}{\oplus} y)[z_1 \ldots z_n] = x[z_1 \ldots z_{i-1}\, y[z_1 \ldots z_n]\, z_{i+1}, \ldots, z_n]\,, \qquad (3)$$

$$x[y_1 \ldots y_n] \underset{i}{\oplus} z = x[(y_1 \underset{i}{\oplus} z) \ldots (y_n \underset{i}{\oplus} z)]\,, \qquad (4)$$

$$x \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = x[\mu_1(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\ \ldots\ \mu_n(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)]\,, \qquad (5)$$

where $i = 1, 2, \ldots, n$ *and* $\{i_1, \ldots, i_s\} = \{1, \ldots, n\}$.

2. Menger $(2,n)$-semigroups

Algebras with one n-ary operation allowing certain permutations of variables were investigated by various authors (cf. for example [2, 9, 10, 12, 13]). Such n-ary algebras also are used to study the properties of some affine geometries (cf. [11]).

In this section we describe algebras on n-place functions allowing an exchange of variables at two fixed places. Namely, on the set $\mathcal{F}(A^n, A)$ we will consider the unary operation π_{ij} defined in the following way:

$$(\pi_{ij} f)(a_1^n) = f(a_1^{i-1}, a_j, a_{i+1}^{j-1}, a_i, a_{j+1}^n),$$

where $1 \leqslant i < j \leqslant n$ are fixed and the left and right hand sides are defined or not defined simultaneously. The operation π_{ij} is called the *operation of* (i,j)-*commutativity.* Functions with the property $\pi_{ij} f = f$ are called (i,j)-*commutative*; functions with the property $\pi_{1n} f = f$ – *semicommutative.* Some n-ary algebras (G, f) in which the operation f is semicommutative are strongly connected with medial (entropic) algebras (cf. [4], [9]) and abelian groups (cf. [1] and [5]).

Let $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \pi_{ij})$ be an arbitrary algebra of type $(n+1, \overbrace{2, \ldots, 2}^{n}, 1)$, where, for simplicity, the unary operation is denoted by π_{ij}.

Theorem 1. *An algebra* $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \pi_{ij})$ *of type* $(n+1, 2, \ldots, 2, 1)$ *is isomorphic to the algebra* $(\Phi, \mathcal{O}, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \pi_{ij})$ *of partial* n*-place functions if and only if* $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ *is a Menger* $(2,n)$*-semigroup satisfying the following identities:*

$$(\pi_{ij}\, x)[y_1 \ldots y_n] = x[y_1^{i-1}\, y_j\, y_{i+1}^{j-1}\, y_i\, y_{j+1}^n], \tag{6}$$

$$\pi_{ij}(x[y_1 \ldots y_n]) = x[\pi_{ij} y_1 \ldots \pi_{ij} y_n], \tag{7}$$

$$\pi_{ij}(x \underset{k}{\oplus} y) = \begin{cases} (\pi_{ij} x) \underset{k}{\oplus} (\pi_{ij} y), & \text{if } k \in \{1, \ldots, n\} - \{i, j\}, \\ (\pi_{ij} x) \underset{j}{\oplus} (\pi_{ij} y), & \text{if } k = i, \\ (\pi_{ij} x) \underset{i}{\oplus} (\pi_{ij} y), & \text{if } k = j, \end{cases} \tag{8}$$

$$\pi_{ij}^2 x = x. \tag{9}$$

Proof. Let $(\Phi, \mathcal{O}, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \pi_{ij})$ be an arbitrary algebra of partial n-place functions $f\colon A^n \to A$. Then obviously $(\Phi, \mathcal{O}, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ is a Menger $(2,n)$-semigroup. To prove that π_{ij} satisfies the conditions (6) – (9) consider $f, g_1, \ldots, g_n \in \Phi$ and $a_1, \ldots, a_n \in A$. Then

$$(\pi_{ij}f)[g_1\dots g_n](a_1^n)=(\pi_{ij}f)(g_1(a_1^n),\dots,g_n(a_1^n))$$
$$=f(g_1(a_1^n),\dots,g_{i-1}(a_1^n),g_j(a_1^n),g_{i+1}(a_1^n),\dots,g_{j-1}(a_1^n),g_i(a_1^n),g_{j+1}(a_1^n),\dots,g_n(a_1^n))$$
$$=f[g_1\dots g_{i-1}g_jg_{i+1}\dots g_{j-1}g_ig_{j+1}\dots g_n](a_1^n)=f[g_1^{i-1}g_jg_{i+1}^{j-1}g_ig_{j+1}^n](a_1^n),$$

which proves (6).

Similarly, we can see that

$$\pi_{ij}(f[g_1\dots g_n])(a_1^n)=f[g_1\dots g_n](a_1^{i-1},a_j,a_{i+1}^{j-1},a_i,a_{j+1}^n)$$
$$=f(g_1(a_1^{i-1},a_j,a_{i+1}^{j-1},a_i,a_{j+1}^n),\dots,g_n(a_1^{i-1},a_j,a_{i+1}^{j-1},a_i,a_{j+1}^n))$$
$$=f(\pi_{ij}g_1(a_1^n),\dots,\pi_{ij}g_n(a_1^n))=f[\pi_{ij}g_1\dots\pi_{ij}g_n](a_1^n).$$

This proves (7).

To prove (8) we must consider three cases: $k\in\{1,\dots,n\}-\{i,j\}$, $k=i$ and $k=j$. In the first case we have three subcases:

$$1\leqslant k<i<j\leqslant n,\quad 1\leqslant i<k<j\leqslant n,\quad 1\leqslant i<j<k\leqslant n.$$

In the first subcase:

$$\pi_{ij}(f\underset{k}{\oplus}g)(a_1^n)=f\underset{k}{\oplus}g(a_1^{i-1},a_j,a_{i+1}^{j-1},a_i,a_{j+1}^n)$$
$$=f(a_1^{k-1},g(a_1^{i-1},a_j,a_{i+1}^{j-1},a_i,a_{j+1}^n),a_{k+1}^{i-1},a_j,a_{i+1}^{j-1},a_i,a_{j+1}^n)$$
$$=\pi_{ij}f(a_1^{k-1},\pi_{ij}g(a_1^n),a_{k+1}^n)=(\pi_{ij}f)\underset{k}{\oplus}(\pi_{ij}g)(a_1^n).$$

The remaining two subcases can be verified analogously.

In the case $k=i$ we have

$$\pi_{ij}(f\underset{i}{\oplus}g)(a_1^n)=f\underset{i}{\oplus}g(a_1^{i-1},a_j,a_{i+1}^{j-1},a_i,a_{j+1}^n)$$
$$=f(a_1^{i-1},g(a_1^{i-1},a_j,a_{i+1}^{j-1},a_i,a_{j+1}^n),a_{i+1}^{j-1},a_i,a_{j+1}^n)$$
$$=\pi_{ij}f(a_1^{j-1},\pi_{ij}g(a_1^n),a_{j+1}^n)=(\pi_{ij}f)\underset{j}{\oplus}(\pi_{ij}g)(a_1^n).$$

In a similar way we can verify the case $k=j$.

So the condition (8) is valid.

The condition (9) is obvious.

So, the algebra $(\Phi,\mathcal{O},\underset{1}{\oplus},\dots,\underset{n}{\oplus},\pi_{ij})$ satisfies all the conditions mentioned in the theorem.

Conversely, let $(G,o,\underset{1}{\oplus},\dots,\underset{n}{\oplus},\pi_{ij})$ be an arbitrary Menger $(2,n)$-semigroup with the unary operation π_{ij} satisfying the conditions (6) – (9). We will show that there exists an algebra of n-place functions λ_g and the mapping $P\colon g\mapsto\lambda_g$ such that $P\colon G\to\Phi=\{\lambda_g : g\in G\}$ is an isomorphism.

Consider the set $G^* = G \cup \{e_1, \ldots, e_n\}$, where $e_1, \ldots, e_n$ are different elements not belonging to G. For every $g \in G$ we define on G^* an n-place function λ_g putting

$$\lambda_g(x_1^n) = \begin{cases} g[x_1 \ldots x_n], & \text{if } x_1, \ldots, x_n \in G, \\ g, & \text{if } (x_1, \ldots, x_n) = (e_1, \ldots, e_n), \\ \pi_{ij} g, & \text{if } (x_1, \ldots, x_n) = (e_1^{i-1}, e_j, e_{i+1}^{j-1}, e_i, e_{j+1}^n), \\ g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, & \text{if } x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ i = 1, \ldots, n, \\ & \text{for some } y_1, \ldots, y_s \in G,\ \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}, \\ (\pi_{ij} g) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, & \text{if } x_i = \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_j = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \\ & x_k = \mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ k \in \{1, \ldots, n\} - \{i, j\}, \\ & y_1, \ldots, y_s \in G \text{ and } \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}, \end{cases}$$

where by $\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s)$ we denote the element of G^* such that

$$\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s) = \begin{cases} \mu_i(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s) & \text{if } i \in \{i_1, \ldots, i_s\}, \\ e_i & \text{if } i \notin \{i_1, \ldots, i_s\}. \end{cases}$$

In other cases $\lambda_g(x_1^n)$ is not defined.

Note that, according to (2), in the above definition the value of $g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$ does not depends on $y_1, \ldots, y_s \in G$.

We shall prove that algebras $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \pi_{ij})$ and $(\Phi, \mathcal{O}, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \pi_{ij})$, where $\Phi = \{\lambda_g \mid g \in G\}$, are isomorphic. For this consider the map $P\colon g \mapsto \lambda_g$.

• First we check that such defined P is a homomorphism of (G, o) onto $(\Phi, \mathcal{O})$, i.e., $P(g[g_1 \ldots g_n]) = P(g)[P(g_1) \ldots P(g_n)]$, or equivalently,

$$\lambda_{g[g_1 \ldots g_n]}(x_1^n) = \lambda_g[\lambda_{g_1} \ldots \lambda_{g_n}](x_1^n)$$

for all $g, g_1, \ldots, g_n \in G$ and $x_1, \ldots, x_n \in G^*$.

1) Let $x_1, \ldots, x_n \in G$. Then, in view of (3), we obtain:

$$\begin{aligned} \lambda_{g[g_1 \ldots g_n]}(x_1^n) &= g[g_1 \ldots g_n][x_1 \ldots x_n] = g[g_1[x_1 \ldots x_n] \ldots g_n[x_1 \ldots x_n]] \\ &= \lambda_g(g_1[x_1 \ldots x_n], \ldots, g_n[x_1 \ldots x_n]) = \lambda_g(\lambda_{g_1}(x_1^n), \ldots, \lambda_{g_n}(x_1^n)) \\ &= \lambda_g[\lambda_{g_1} \ldots \lambda_{g_n}](x_1^n). \end{aligned}$$

2) For $(x_1, \ldots, x_n) = (e_1, \ldots, e_n)$, we have

$$\begin{aligned} \lambda_{g[g_1 \ldots g_n]}(e_1^n) &= g[g_1 \ldots g_n] = \lambda_g(g_1^n) \\ &= \lambda_g(\lambda_{g_1}(e_1^n), \ldots, \lambda_{g_n}(e_1^n)) = \lambda_g[\lambda_{g_1} \ldots \lambda_{g_n}](e_1^n). \end{aligned}$$

3) If $(x_1,\ldots,x_n)=(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n)$, then

$$\begin{aligned}\lambda_{g[g_1\ldots g_n]}(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n)&=\pi_{ij}(g[g_1\ldots g_n])=g[\pi_{ij}g_1\ldots\pi_{ij}g_n]\\&=\lambda_g(\lambda_{g_1}(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n),\ldots,\lambda_{g_n}(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n))\\&=\lambda_g[\lambda_{g_1}\ldots\lambda_{g_n}](e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n).\end{aligned}$$

4) Now let $(x_1,\ldots,x_n)=(\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),\ldots,\mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s))$. If $\{i_1,\ldots,i_s\}=\{1,\ldots,n\}$, then, according to (5), this case is reduced to the case when $x_1,\ldots,x_n\in G$. For $\{i_1,\ldots,i_s\}\neq\{1,\ldots,n\}$, we have

$$\begin{aligned}\lambda_{g[g_1\ldots g_n]}(x_1^n)&=g[g_1\ldots g_n]\overset{i_s}{\underset{i_1}{\oplus}}y_1^s=g[g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\ldots g_n\overset{i_s}{\underset{i_1}{\oplus}}y_1^s]\\&=\lambda_g\left(\lambda_{g_1}(x_1^n),\ldots,\lambda_{g_n}(x_1^n)\right)=\lambda_g[\lambda_{g_1}\ldots\lambda_{g_n}](x_1^n).\end{aligned}$$

5) In the case $(x_1,\ldots,x_n)=(x_1^{i-1},\mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),x_{i+1}^{j-1},\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),x_{j+1}^n)$, where $x_k=\mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$ and $k\in\{1,\ldots,n\}-\{i,j\}$, we obtain

$$\begin{aligned}\lambda_{g[g_1\ldots g_n]}(x_1^{i-1},\mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),x_{i+1}^{j-1},\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),x_{j+1}^n)&=\pi_{ij}(g[g_1\ldots g_n])\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\\&\overset{(7)}{=}g[\pi_{ij}g_1\ldots\pi_{ij}g_n]\overset{i_s}{\underset{i_1}{\oplus}}y_1^s=g[(\pi_{ij}g_1)\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\ldots(\pi_{ij}g_n)\overset{i_s}{\underset{i_1}{\oplus}}y_1^s]\\&=\lambda_g(\lambda_{g_1}(x_1^n),\ldots,\lambda_{g_n}(x_1^n))\\&=\lambda_g[\lambda_{g_1}\ldots\lambda_{g_n}](x_1^{i-1},\mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),x_{i+1}^{j-1},\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),x_{j+1}^n).\end{aligned}$$

This completes the proof that P is a homomorphism of (G,o) onto $(\Phi,\mathcal{O})$.

• Now we check that P a homomorphism of a $(2,n)$-semigroup $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ onto a $(2,n)$-semigroup $(\Phi,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$, i.e., $P(g_1\underset{i}{\oplus}g_2)=P(g_1)\underset{i}{\oplus}P(g_2)$, or equivalently,

$$\lambda_{g_1\underset{i}{\oplus}g_2}(x_1^n)=\lambda_{g_1}\underset{i}{\oplus}\lambda_{g_2}(x_1^n)$$

for all $i=1,2,\ldots,n$, $g_1,g_2\in G$ and $x_1,\ldots,x_n\in G^*$. Similarly as in previous case we must verify several cases.

1) If $x_1,\ldots,x_n\in G$, then, applying (3), we obtain

$$\begin{aligned}\lambda_{g_1\underset{i}{\oplus}g_2}(x_1^n)&=(g_1\underset{i}{\oplus}g_2)[x_1\ldots x_n]=g_1[x_1\ldots x_{i-1}\,g_2[x_1\ldots x_n]\,x_{i+1}\ldots x_n]\\&=\lambda_{g_1}\left(x_1^{i-1},\lambda_{g_2}(x_1^n),x_{i+1}^n\right)=\lambda_{g_1}\underset{i}{\oplus}\lambda_{g_2}(x_1^n).\end{aligned}$$

2) For $(x_1,\ldots,x_n)=(e_1,\ldots,e_n)$ we have $\lambda_{g_1\underset{i}{\oplus}g_2}(e_1^n)=g_1\underset{i}{\oplus}g_2$. Consequently,

$$\lambda_{g_1}\underset{i}{\oplus}\lambda_{g_2}(e_1^n)=\lambda_{g_1}\left(e_1^{i-1},\lambda_{g_2}(e_1^n),e_{i+1}^n\right)=\lambda_{g_1}\left(e_1^{i-1},g_2,e_{i+1}^n\right)=g_1\underset{i}{\oplus}g_2,$$

because $\mu_i^*(\underset{i}{\oplus}g_2)=g_2$ and $\mu_k^*(\underset{i}{\oplus}g_2)=e_k$ for $k\neq i$, $k=1,\ldots,n$.

Thus, $\lambda_{g_1\underset{i}{\oplus}g_2}(e_1^n)=\lambda_{g_1}\underset{i}{\oplus}\lambda_{g_2}(e_1^n)$.

3) In the case $(x_1,\ldots,x_n)=(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n)$, for $k=1,\ldots,n$, according to (8), we have

$$\lambda_{g_1\underset{k}{\oplus}g_2}(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n)=\pi_{ij}(g_1\underset{k}{\oplus}g_2)=\begin{cases}(\pi_{ij}g_1)\underset{k}{\oplus}(\pi_{ij}g_2), & \text{if } k\notin\{i,j\},\\ (\pi_{ij}g_1)\underset{j}{\oplus}(\pi_{ij}g_2), & \text{if } k=i,\\ (\pi_{ij}g_1)\underset{i}{\oplus}(\pi_{ij}g_2), & \text{if } k=j.\end{cases}$$

For $k\notin\{i,j\}$ we have three possibilities:

$$1\leqslant k<i<j\leqslant n \quad\text{or}\quad 1\leqslant i<k<j\leqslant n \quad\text{or}\quad 1\leqslant i<j<k\leqslant n.$$

In the case $1\leqslant k<i<j\leqslant n$ we get

$$\begin{aligned}\lambda_{g_1}\underset{k}{\oplus}\lambda_{g_2}&(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n)\\ &=\lambda_{g_1}(e_1^{k-1},\lambda_{g_2}(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n),e_{k+1}^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n)\\ &=\lambda_{g_1}(e_1^{k-1},\pi_{ij}g_2,e_{k+1}^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n)=(\pi_{ij}g_1)\underset{k}{\oplus}(\pi_{ij}g_2)\\ &=\lambda_{g_1}\underset{k}{\oplus}\lambda_{g_2}(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n),\end{aligned}$$

since $\mu_k^*(\underset{k}{\oplus}\pi_{ij}g_2)=\pi_{ij}g_2$ and $\mu_s^*(\underset{k}{\oplus}\pi_{ij}g_2)=e_s$ for $s\neq k$, $s=1,\ldots,n$.

In the remaining two cases the proof is analogous.

If $k=i$, then

$$\begin{aligned}\lambda_{g_1}\underset{i}{\oplus}\lambda_{g_2}(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n)&=\lambda_{g_1}(e_1^{i-1},\lambda_{g_2}(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n),e_{i+1}^{j-1},e_i,e_{j+1}^n)\\ &=\lambda_{g_1}(e_1^{i-1},\pi_{ij}g_2,e_{i+1}^{j-1},e_i,e_{j+1}^n)=(\pi_{ij}g_1)\underset{j}{\oplus}(\pi_{ij}g_2)\\ &=\lambda_{g_1}\underset{i}{\oplus}\lambda_{g_2}(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n),\end{aligned}$$

since $\mu_j^*(\underset{j}{\oplus}\pi_{ij}g_2)=\pi_{ij}g_2$ and $\mu_s^*(\underset{j}{\oplus}\pi_{ij}g_2)=e_s$ for $s\neq j$, $s=1,\ldots,n$.

In the same manner we can verity the case $k=j$.

4) Now let $(x_1,\ldots,x_n)=(\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),\ldots,\mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s))$. Then, according to the definition, $\lambda_{g_1\underset{i}{\oplus}g_2}(x_1^n)=g_1\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$. On the other side, we have $\mu_i^*(\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)=$

$g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = \lambda_{g_2}(x_1^n)$ and $\mu_k^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = x_k$ for all $k \neq i$, $k = 1, \ldots, n$. Hence

$$\lambda_{g_1} \underset{i}{\oplus} \lambda_{g_2}(x_1^n) = \lambda_{g_1}\left(x_1^{i-1}, \lambda_{g_2}(x_1^n), x_{i+1}^n\right) = g_1 \underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s.$$

Thus $\lambda_{g_1 \underset{i}{\oplus} g_2}(\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \ldots, \mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)) = \lambda_{g_1} \underset{i}{\oplus} \lambda_{g_2}(\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \ldots, \mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s))$.

5) In the case when $(x_1, \ldots, x_n) = (x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n)$, where $x_k = \mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$ for $k \in \{1, \ldots, n\} - \{i, j\}$, we get

$$\lambda_{g_1 \underset{k}{\oplus} g_2}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n)$$
$$= \pi_{ij}(g_1 \underset{k}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = \begin{cases} (\pi_{ij} g_1) \underset{k}{\oplus} (\pi_{ij} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, & \text{if } k \in \{1, \ldots, n\} - \{i, j\}, \\ (\pi_{ij} g_1) \underset{j}{\oplus} (\pi_{ij} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, & \text{if } k = i, \\ (\pi_{ij} g_1) \underset{i}{\oplus} (\pi_{ij} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, & \text{if } k = j. \end{cases}$$

For $k \in \{1, \ldots, n\} - \{i, j\}$ we get three cases:

$$1 \leqslant k < i < j \leqslant n, \quad 1 \leqslant i < k < j \leqslant n, \quad 1 \leqslant i < j < k \leqslant n.$$

We verify only the last case. Other cases can be verified analogously.

To prove this case let $x_1, \ldots, x_n, y_1, \ldots, y_s \in G^*$ and $1 \leqslant i < j < k \leqslant n$. Then

$$\lambda_{g_1} \underset{k}{\oplus} \lambda_{g_2}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n)$$
$$= \lambda_{g_1}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^{k-1}, \lambda_{g_2}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n), x_{k+1}^n)$$
$$= \lambda_{g_1}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^{k-1}, (\pi_{ij} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, x_{k+1}^n)$$
$$= (\pi_{ij} g_1) \underset{k}{\oplus} (\pi_{ij} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s,$$

since $\mu_k^*(\underset{k}{\oplus}(\pi_{ij} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = (\pi_{ij} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$ and $\mu_s^*(\underset{k}{\oplus}(\pi_{ij} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_s^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$ for $s \neq k$, $s = 1, \ldots, n$. Thus,

$$\lambda_{g_1 \underset{k}{\oplus} g_2}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n) = \lambda_{g_1} \underset{k}{\oplus} \lambda_{g_2}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n).$$

So in this case $P(g_1 \underset{k}{\oplus} g_2) = P(g_1) \underset{k}{\oplus} P(g_2)$.

In the case $k = i$, we get

$$\lambda_{g_1} \underset{i}{\oplus} \lambda_{g_2}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n)$$
$$= \lambda_{g_1}(x_1^{i-1}, \lambda_{g_2}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n)$$

$$= \lambda_{g_1}(x_1^{i-1}, (\pi_{ij}g_2)\overset{i_s}{\underset{i_1}{\oplus}} y_1^s, x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n) = (\pi_{ij}g_1)\underset{j}{\oplus}(\pi_{ij}g_2)\overset{i_s}{\underset{i_1}{\oplus}} y_1^s,$$

since $\mu_j^*(\underset{j}{\oplus}(\pi_{ij}g_2)\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = (\pi_{ij}g_2)\overset{i_s}{\underset{i_1}{\oplus}} y_1^s$ and $\mu_s^*(\underset{j}{\oplus}(\pi_{ij}g_2)\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_s^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$ for $s \neq j$, $s = 1,\ldots,n$. Thus,

$$\lambda_{g_1\underset{i}{\oplus}g_2}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n) = \lambda_{g_1}\underset{i}{\oplus}\lambda_{g_2}(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n).$$

So, $P(g_1 \underset{i}{\oplus} g_2) = P(g_1)\underset{i}{\oplus} P(g_2)$.

For $k = j$ the proof is very similar.

In this way we have proved that P a homomorphism of a $(2,n)$-semigroup $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ onto a $(2,n)$-semigroup $(\Phi, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$.

• This homomorphism also saves the operation π_{ij}, i.e., $P(\pi_{ij}g) = \pi_{ij}P(g)$ or, in other words, $\lambda_{\pi_{ij}g}(x_1^n) = \pi_{ij}\lambda_g(x_1^n)$ for all $g \in G$ and $x_1,\ldots,x_n \in G^*$.

1) If $x_1,\ldots,x_n \in G$, then

$$\begin{aligned}\lambda_{\pi_{ij}g}(x_1^n) = (\pi_{ij}g)[x_1\ldots x_n] &\overset{(6)}{=} g[x_1^{i-1}x_jx_{i+1}^{j-1}x_ix_{j+1}^n] \\ &= \lambda_g(x_1^{i-1}, x_j, x_{i+1}^{j-1}, x_i, x_{j+1}^n) = \pi_{ij}\lambda_g(x_1^n).\end{aligned}$$

2) For $(x_1,\ldots,x_n) = (e_1,\ldots,e_n)$ we have

$$\lambda_{\pi_{ij}g}(e_1^n) = \pi_{ij}g = \lambda_g(e_1^{i-1}, e_j, e_{i+1}^{j-1}, e_i, e_{j+1}^n) = \pi_{ij}\lambda_g(e_1^n).$$

3) In the case $(x_1,\ldots,x_n) = (e_1^{i-1}, e_j, e_{i+1}^{j-1}, e_i, e_{j+1}^n)$, we obtain

$$\begin{aligned}\lambda_{\pi_{ij}g}(e_1^{i-1}, e_j, e_{i+1}^{j-1}, e_i, e_{j+1}^n) = \pi_{ij}(\pi_{ij}g) = \pi_{ij}^2 g &\overset{(9)}{=} g \\ &= \lambda_g(e_1^n) = \pi_{ij}\lambda_g(e_1^{i-1}, e_j, e_{i+1}^{j-1}, e_i, e_{j+1}^n).\end{aligned}$$

4) Now, if $(x_1,\ldots,x_n) = (\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ldots,\mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s))$, then

$$\begin{aligned}\lambda_{\pi_{ij}g}(\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ldots,\mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)) &= (\pi_{ij}g)\overset{i_s}{\underset{i_1}{\oplus}} y_1^s \\ &= \lambda_g(x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n) \\ &= \pi_{ij}\lambda_g(x_1^{i-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n) \\ &= \pi_{ij}\lambda_g(\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ldots,\mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)).\end{aligned}$$

5) In the last case when $(x_1,\ldots,x_n) = (x_1^{i-1}, \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{i+1}^{j-1}, \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), x_{j+1}^n)$, where $x_k = \mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$ and $k \in \{1,\ldots,n\} - \{i,j\}$, we get

$$\lambda_{\pi_{ij}g}(x_1^{i-1},\mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),x_{i+1}^{j-1},\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),x_{j+1}^n)=(\pi_{ij}^2g)\overset{i_s}{\underset{i_1}{\oplus}}y_1^s=g\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$$
$$=\lambda_g(\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),\ldots,\mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s))=\pi_{ij}\lambda_g(x_1^{i-1},\mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),x_{i+1}^{j-1},\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),x_{j+1}^n).$$

This completes the proof that $P\colon g\mapsto\lambda_g$ is an epimorphism of $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus},\pi_{ij})$ onto $(\Phi,\underset{1}{\oplus},\ldots,\underset{n}{\oplus},\pi_{ij})$. Since $P(g_1)=P(g_2)$ implies $\lambda_{g_1}(e_1^n)=\lambda_{g_2}(e_1^n)$, which gives $g_1=g_2$, we see that $P\colon g\mapsto\lambda_g$ is an isomorphism. □

From the above theorem we can deduce the following two corollaries.

Corollary 1. *An algebra (G,o,π_{ij}) of type $(n+1,1)$ is isomorphic to an algebra $(\Phi,\mathcal{O},\pi_{ij})$ of partial n-place functions if and only if it satisfies* (1)*,* (6)*,* (7) *and* (9)*.*

Proof. From the first part of the proof of Theorem 1 it follows that $(\Phi,\mathcal{O},\pi_{ij})$ is a Menger algebra with the operation π_{ij} satisfying the conditions (6), (7) and (9).

To prove the converse statement consider an arbitrary algebra (G,o,π_{ij}) of type $(n+1,1)$ satisfying all the conditions mentioned in the corollary and define on the set $G^*=G\cup\{e_1,\ldots,e_n\}$, where $e_1,\ldots,e_n$ are different elements not belonging to G, the function λ_g putting:

$$\lambda_g(x_1^n)=\begin{cases} g[x_1\ldots x_n], & \text{if } x_1,\ldots,x_n\in G,\\ g, & \text{if } (x_1,\ldots,x_n)=(e_1,\ldots,e_n),\\ \pi_{ij}g, & \text{if } (x_1,\ldots,x_n)=(e_1^{i-1},e_j,e_{i+1}^{j-1},e_i,e_{j+1}^n).\end{cases}$$

In other cases $\lambda_g(x_1^n)$ is not defined.

Then in the same way as in the second part of the proof of Theorem 1, we can prove that the algebras (G,o,π_{ij}) and $(\Phi,\mathcal{O},\pi_{ij})$, where $\Phi=\{\lambda_g\,|\,g\in G\}$, are isomorphic. This isomorphism has the form $P\colon g\mapsto\lambda_g$. □

Corollary 2. *An algebra $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus},\pi_{ij})$ of type $(2,\ldots,2,1)$ is isomorphic to the algebra $(\Phi,\underset{1}{\oplus},\ldots,\underset{n}{\oplus},\pi_{ij})$ of partial n-place functions if and only if it satisfies the identities* (8)*,* (9)*, and the implication* (2)*.*

Proof. Clearly, the algebra $(\Phi,\underset{1}{\oplus},\ldots,\underset{n}{\oplus},\pi_{ij})$ of partial n-place functions satisfies (2), (8) and (9).

Conversely, if an algebra $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus},\pi_{ij})$ of type $(2,\ldots,2,1)$ satisfies the conditions (2), (8) and (9), then for each element $g\in G$ we define on the set $G^*=G\cup\{e_1,\ldots,e_n\}$, where $e_1,\ldots,e_n$ are different elements not belonging to G, the n-place function $\lambda_g\colon(G^*)^n\to G^*$ putting

$$\lambda_g(x_1^n) = \begin{cases} g, & \text{if } (x_1,\ldots,x_n) = (e_1,\ldots,e_n), \\ \pi_{ij}g, & \text{if } (x_1,\ldots,x_n) = (e_1^{i-1}, e_j, e_{i+1}^{j-1}, e_i, e_{j+1}^n), \\ g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, & \text{if } x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ i = 1,\ldots,n, \\ & \text{for some } y_1,\ldots,y_s \in G, \\ & \{i_1,\ldots,i_s\} \subset \{1,\ldots,n\}, \\ (\pi_{ij}g) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, & \text{if } x_i = \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ x_j = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \\ & x_k = \mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ k \in \{1,\ldots,n\} - \{i,j\}, \\ & \text{where } y_1,\ldots,y_s \in G,\ \{i_1,\ldots,i_s\} \subset \{1,\ldots,n\}. \end{cases}$$

In other cases $\lambda_g(x_1^n)$ is not defined.

In the same way as in the second part of the proof of Theorem 1, we can see that $P\colon g \mapsto \lambda_g$ is an isomorphism between $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \pi_{ij})$ and $(\Phi, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \pi_{ij})$, where $\Phi = \{\lambda_g \mid g \in G\}$. $\square$

From Theorem 1 we deduce the following characterizations of algebras of (i,j)-commutative functions.

Corollary 3. *An algebra $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ of type $(n+1, 2, \ldots, 2)$ is isomorphic to the algebra $(\Phi, \mathcal{O}, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ of partial (i,j)-commutative n-place functions if and only if it satisfies the condition* (1), (2), (3), (4), (5) *and*

$$x[y_1 \ldots y_n] = x[y_1^{i-1}\, y_j\, y_{i+1}^{j-1}\, y_i\, y_{j+1}^n], \tag{10}$$

$$x \underset{k}{\oplus} y = \begin{cases} x \underset{j}{\oplus} y, & \textit{if } k = i, \\ x \underset{i}{\oplus} y, & \textit{if } k = j. \end{cases} \tag{11}$$

Corollary 4. *An $(n+1)$-ary algebra (G, o) is isomorphic to the algebra $(\Phi, \mathcal{O})$ of partial (i,j)-commutative n-place functions if and only if it satisfies the conditions* (1) *and* (10).

Corollary 5. *An algebra $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ of type $(2, \ldots, 2)$ is isomorphic to the algebra $(\Phi, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ of partial (i,j)-commutative n-place functions if and only if it satisfies the condition* (11) *and the implication* (2).

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Received April 26, 2016

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