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Some results on multigroups

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Abstract. The theory of multisets is an extension of the set theory. In this paper, we have studied some new results on multigroups following [11].

1. Introduction

A mathematical structure known as multiset (mset, for short) is obtained if the restriction of distinctness on the nature of the objects forming a set is relaxed. Unlike classical set theory which assumes that mathematical objects occur without repetition. However, the situation in science and in ordinary life is not like that. It is observed that there is enormous repetition in the physical world. For example, consideration of repeated roots of polynomial equation, repeated observations in statistical sample, repeated hydrogen atoms in a water molecule H_2O, etc., do play a significant role. The challenging task of formulating sufficiently rich mathematics of multiset started receiving serious attention from beginning of the 1970s. An updated exposition on both historical and mathematical perspective of the development of theory of multisets can be found in [3, 4, 5, 8, 9, 10, 13, 14, 15].

The theory of groups is an important algebraic structure in modern mathematics. Several authors have studied the algebraic structure of set theories dealing with uncertainties such as the concept of group in fuzzy sets [12], soft sets [1], smooth sets [6], rough sets [2] etc.

2. Preliminaries

In this section, we present fundamental definitions of multisets that will be used in the subsequent sections of this paper.

Definition 2.1. Let X be a set. A *multiset* (mset) A drawn from X is represented by a count function C_A defined as $C_A : X \to \mathcal{D} = \{0, 1, 2, \ldots\}$. For each $x \in X$, $C_A(x)$ denotes the number of occurrences of the element x in the mset A. The representation of the mset A drawn from $X = \{x_1, x_2, \ldots, x_n\}$ will be as $A = [x_1, x_2, \ldots, x_n]_{m_1, m_2, \ldots, m_n}$ such that x_i appears m_i times, $i = 1, 2, \ldots, n$ in the mset A.

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Definition 2.2. A domain X is defined as a set of elements from which msets are constructed. For any positive integer n, the mset space $[X]^n$ is the set of all msets whose elements are in X such that no element in the mset occurs more than n times. The set $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of times an element in an mset occurs.

Definition 2.3. Let $A_1, A_2, A_i \in [X]^n$, $i \in I$. Then

(i) $A_1 \subseteq A_2 \Leftrightarrow C_{A_1}(x) \leqslant C_{A_2}(x)$, $\forall\, x \in X$.

(ii) $A_1 = A_2 \Leftrightarrow C_{A_1}(x) = C_{A_2}(x)$, $\forall\, x \in X$.

(iii) $\bigcap_{i\in I} A_i = \bigwedge_{i\in I} C_{A_i}(x)$, $\forall\, x \in X$ (where $\bigwedge$ is the minimum operation).

(iv) $\bigcup_{i\in I} A_i = \bigvee_{i\in I} C_{A_i}(x)$, $\forall\, x \in X$ (where $\bigvee$ is the maximum operation).

(v) $A_i^c = n - C_{A_i}(x)$, $\forall x \in X$, $n \in \mathbb{Z}^+$.

Definition 2.4. Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Then the *image* $f(A)$ of an mset $A \in [X]^n$ is defined as

$$C_{f(A)}(y) = \begin{cases} \bigvee_{f(x)=y} C_A(x), & f^{-1}(y) \neq \emptyset, \\ 0, & f^{-1}(y) = \emptyset. \end{cases}$$

Definition 2.5. Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Then the *inverse image* $f^{-1}(B)$ of an mset $B \in [Y]^n$ is defined as $C_{f^{-1}B}(x) = C_B\,(f(x))$.

3. Multigroup

In this section, we briefly give the definition of multigroup, some remarks and present some existing results given by [11], and $MS(X)$ is denoted as the set all msets over X (which is assumed to be an initial universal set unless it is stated otherwise).

Definition 3.1. Let X be a group. A multiset A over X is called a *multigroup* over X if the count function A or C_A satisfies the following conditions:

(i) $C_A(xy) \geqslant [C_A(x) \wedge C_A(y)]$, $\forall\, x, y \in X$,

(ii) $C_A(x^{-1}) \geqslant C_A(x)$, $\forall\, x \in X$.

We denote the set of all multigroups over X by $MG(X)$.

Example 3.2. Let the subset $X = \{1, -1, i, -i\}$ of complex numbers be a group and $A = [1, -1, i, -i]_{3,2,2,2}$ be a multiset over X. Then, as it is not difficult to verify, A is a multigroup over X.

Definition 3.3. Let $A, B \in MG(X)$, we have the following definitions:

(i) $C_{A\circ B}(x) = \bigvee \{C_A(y) \wedge C_B(z) : y, z \in X, yz = x\}$
$= \max [\min \{C_A(y), C_B(z)\} : y, z \in X, yz = x]$,

(ii) $C_{A^{-1}}(x) = C_A(x^{-1})$.

We call $A \circ B$ the product of A and B, and A^{-1} the inverse of A.

Definition 3.4. (cf. [11]) Let $A \in MG(X)$. Then A is called an *abelian multigroup* over X if $C_A(xy) = C_A(yx)$, $\forall\ x, y \in X$. The set of all abelian multigroups is denoted by $AMG(X)$.

Definition 3.5. (cf. [11]) Let $A, B \in MG(X)$. Then A is said to be a *submultigroup* of B if $A \subseteq B$.

Definition 3.6. (cf. [11]) Let $H \in MG(X)$. For any $x \in X$, xH and Hx defined by $C_{xH}(y) = C_H(x^{-1}y)$ and $C_{Hx}(y) = C_H(yx^{-1})$, $\forall\ y \in X$ are respectively called the *left* and *right mcosets* of H in X.

The following results have been given by [11] as related to this paper except for Remark 3.25 and 3.25.

Proposition 3.7. *Let $A \in MG(X)$. Then*

(i) $C_A(x^n) \geqslant C_A(x),\ \ \forall\ x \in X,$

(ii) $C_A(x^{-1}) = C_A(x), \forall\ x \in X,$

(iii) $C_A(e) \geqslant C_A(x),\ \ \forall\ x \in X.$

Proposition 3.8. *Let $A, B, C, A_i \in MG(X)$, then the following hold:*

(i) $C_{A\circ B}(x) = \bigvee_{y\in X} \left[C_A(y) \wedge C_B(y^{-1}x)\right] = \bigvee_{y\in X} \left[C_A\left(xy^{-1}\right) \wedge C_B(y)\right], \forall x \in X,$

(ii) $A^{-1} = A,$

(iii) $\left(A^{-1}\right)^{-1} = A,$

(iv) $A \subseteq B \Longrightarrow A^{-1} \subseteq B^{-1},$

(v) $\left(\bigcup_{i\in I} A_i\right)^{-1} = \bigcup_{i\in I} \left(A^{-1}\right),$

(vi) $\left(\bigcap_{i\in I} A_i\right)^{-1} = \bigcap_{i\in I} \left(A^{-1}\right)\ ,$

(vii) $(A \circ B)^{-1} = B^{-1} \circ A^{-1},$

(viii) $(A \circ B) \circ C = A \circ (B \circ C).$

Proposition 3.9. *Let $A, B \in AMG(X)$. Then $A \circ B = B \circ A$.*

Proposition 3.10. *If $A, B \in MG(X)$, then $C_{A\circ B}(x^{-1}) = C_{A\circ B}(x)$.*

Proposition 3.11. *Let $A \in [X]^n$. Then $A \in MG(X)$ if and only if $A \circ A \leqslant A$ and $A^{-1} = A$.*

Proposition 3.12. *Let $A \in [X]^n$. Then $A \in MG(X)$ if and only if $C_A\left(xy^{-1}\right) \geqslant [C_A(x) \wedge C_A(y)]$, $\forall\ x, y \in X$.*

Proposition 3.13. *Let $A, B \in MG(X)$. Then $A \cap B \in MG(X)$.*

Remark 3.14. If $\{A_i\}_{i\in I}$ is a family of multigroups over X, then their intersection $\bigcap_{i\in I} A_i$ is a multigroup over X.

Remark 3.15. If $\{A_i\}_{i\in I}$ is a family of multigroups over X, then their union $\bigcup_{i\in I} A_i$ need not be a multigroup over X.

Proposition 3.16. *Let $A \in MG(X)$. Then the non-empty sets of the form*
$$A_n = \{x \in X : C_A(x) \geqslant n,\ n \in \mathbb{N}\}$$
are subgroups of X.

Proposition 3.17. *Let $A \in MG(X)$. Then the non-empty sets defined as*
$$A^* = \{x \in X : C_A(x) > 0\} \text{ and } A_* = \{x \in X : C_A(x) = C_A(e)\}$$
are subgroups of X.

Proposition 3.18. *Let $A \in MS(X)$. Then the following assertions are equivalent:*

(a) $C_A(xy) = C_A(yx)$, $\forall\ x, y \in X$,

(b) $C_A\left(xyx^{-1}\right) = C_A(y)$, $\forall\ x, y \in X$,

(c) $C_A\left(xyx^{-1}\right) \geqslant C_A(y)$, $\forall\ x, y \in X$,

(d) $C_A\left(xyx^{-1}\right) \leqslant C_A(y)$, $\forall\ x, y \in X$.

Proposition 3.19. *Let $A \in AMG(X)$. Then A_*, A^* and A_n, $n \in \mathbb{N}$ are normal subgroups of X.*

Proposition 3.20. *Let $H \in MG(X)$, then $xH = yH$ if and only if $x^{-1}y \in H_*$.*

Remark 3.21. If $H \in AMG(X)$, then $xH = Hx$, $\forall\ x \in X$.

Proposition 3.22. *Let X and Y be two groups and $f : X \to Y$ be a homomorphism. If $A \in MG(X)$, then $f(A) \in MG(Y)$.*

Remark 3.23. Let X and Y be two groups and $f : X \to Y$ be a homomorphism. If $A_i \in MG(X)$, $i \in I$, then $f\left(\cap_{i\in I} A_i\right) \in MG(Y)$.

Proposition 3.24. *Let X and Y be two groups and $f : X \to Y$ be a homomorphism. If $B \in MG(Y)$, then $f^{-1}(B) \in MG(X)$.*

Remark 3.25. Let X and Y be two groups and $f : X \to Y$ be a homomorphism. If $B_i \in MG(Y)$, $i \in I$, then $f^{-1}\left(\bigcap_{i\in I} B_i\right) \in MG(X)$.

We now present some results to broaden the theoretical aspect of multigroup theory.

Proposition 3.26. *Let $A \in MG(X)$. Then*

(i) $C_A(xy)^{-1} \geqslant C_A(x) \wedge C_A(y)$, $\forall\ x, y \in X$,

(ii) $C_A(xy)^n \geqslant C_A(xy)$, $\forall\ x, y \in X$.

Proof. The proofs are straightforward. □

Proposition 3.27. *Let $A \in MG(X)$. If $C_A(x) < C_A(y)$ for some $x, y \in X$, then $C_A(xy) = C_A(x) = C_A(yx)$.*

Proof. Given that $C_A(x) < C_A(y)$ for some $x, y \in X$. Since $A \in MG(X)$, then $C_A(xy) \geqslant C_A(x) \wedge C_A(y) = C_A(x)$. Now, $C_A(x) = C_A\left(xyy^{-1}\right) \geqslant C_A(xy) \wedge C_A(y) = C_A(xy)$, since $C_A(x) < C_A(y)$, $C_A(xy) < C_A(y)$. Therefore, $C_A(xy) = C_A(x)$. Similarly, $C_A(yx) = C_A(x)$. □

Proposition 3.28. *Let $A \in MG(X)$. Then $C_A(xy^{-1}) = C_A(e)$ implies $C_A(x) = C_A(y)$.*

Proof. Given $A \in MG(X)$ and $C_A(xy^{-1}) = C_A(e)\ \ \forall\ x, y \in X$. Then

$$C_A(x)=C_A(x(y^{-1}y))=C_A((xy^{-1})y)\geqslant C_A(xy^{-1})\wedge C_A(y)=C_A(e)\wedge C_A(y)=C_A(y),$$

i.e., $C_A(x) \geqslant C_A(y)$.

Also, $C_A(y) = C_A(y^{-1}) = C_A(ey^{-1}) = C_A((x^{-1}x)y^{-1}) \geqslant C_A(x^{-1}) \wedge C_A(xy^{-1}) = C_A(x) \wedge C_A(e) = C_A(x)$, i.e., $C_A(y) \geqslant C_A(x)$. Hence, $C_A(x) = C_A(y)$. □

Proposition 3.29. *Let $A, B, C, D \in MG(X)$. If $A \subseteq B$ and $C \subseteq D$, then $A \circ C \subseteq B \circ D$.*

Proof. Since $A \subseteq B$ and $C \subseteq D$, it follows that $C_A(x) \geqslant C_B(x)$, $\forall\ x \in X$ and $C_C(x) \leqslant C_D(x)$, $\forall\ x \in X$. So,

$$\begin{aligned} C_{(A\circ C)}(x) &= \bigvee \{C_A(y) \wedge C_C(z) : y, z \in X, yz = x\} \\ &\leqslant \bigvee \{C_B(y) \wedge C_D(z) : y, z \in X, yz = x\} = C_{(B\circ D)}(x). \end{aligned}$$

Hence, $A \circ C \subseteq B \circ D$. □

Proposition 3.30. *Let $A, B \in MG(X)$ and $A \subseteq B$ or $B \subseteq A$. Then $A \cup B \in MG(X)$.*

Proof. The proof is straightforward. □

Remark 3.31. Let $A \in MG(X)$, then A^c need not be a multigroup over X. Indeed, if $X = (V_4, +) = \{0, a, b, c\}$ is the Klein's 4-group, then for $A = [0, a]_{2,1}$ we have $A^c = [0, a]_{2,3} \neq MG(X)$ because $\exists\ C_A(a) > C_A(0)$.

Proposition 3.32. *If $A \in MG(X)$, then $A^c \in MG(X)$ if and only if $C_A(x) = C_A(e)$, $\forall\ x \in X$.*

Proposition 3.33. *Let $A \in MG(X)$ and $x \in X$. Then $C_A(xy) = C_A(y)\ \forall\ y \in X$ if and only if $C_A(x) = C_A(e)$.*

Proof. Let $C_A(xy) = C_A(y)$, $\forall y \in X$. Then $C_A(x) = C_A(xe) = C_A(e)$.

Conversely, let $C_A(x) = C_A(e)$. Since $C_A(e) \geqslant C_A(y)\ \forall y \in X$, we have $C_A(x) \geqslant C_A(y)$. Thus, $C_A(xy) \geqslant C_A(x) \wedge C_A(y) = C_A(e) \wedge C_A(y) = C_A(y)$, i.e., $C_A(xy) \geqslant C_A(y)$, $\forall y \in X$.

But $C_A(y) = C_A(x^{-1}xy) \geqslant C_A(x) \wedge C_A(xy)$ and $C_A(x) \geqslant C_A(xy)$, $\forall y \in X$, imply $C_A(x) \wedge C_A(xy) = C_A(xy) \leqslant C_A(y)$, $\forall y \in X$. So, $C_A(y) \geqslant C_A(xy)$, $\forall y \in X$. Hence, $C_A(xy) = C_A(y)\ \ \forall y \in X$. □

Proposition 3.34. *If $A \in MG(X)$ and $H \leqslant X$, then $A|_H \in MG(H)$.*

Proof. Let $x, y \in H$. Then $xy^{-1} \in H$. Since $A \in MG(X)$, then $C_A(xy^{-1}) \geqslant C_A(x) \wedge C_A(y)\ \ \forall x, y \in X$. Moreover, $C_{A|_H}(xy^{-1}) \geqslant C_{A|H}(x) \wedge C_{A|H}(y)\ \ \forall x, y \in X$. Hence, $A|_H \in MG(H)$. □

4. Multigroup homomorphism

Proposition 4.1. *Let $f : X \longrightarrow Y$ be an epimorphism and $B \in MS(Y)$. If $f^{-1}(B) \in MG(X)$, then $B \in MG(Y)$.*

Proof. Let $x, y \in Y$ then $\exists\ a, b \in X$ such that $f(a) = x$ and $f(b) = y$. It follows that

$$\begin{aligned} C_B(xy) = C_B(f(a)f(b)) = C_B(f(ab)) = C_{f^{-1}(B)}(ab) &\geqslant C_{f^{-1}(B)}(a) \wedge C_{f^{-1}(B)}(b) \\ &= C_B(f(a)) \wedge C_B(f(b)) = C_B(x) \wedge C_B(y). \end{aligned}$$

Again,

$$\begin{aligned} C_B(x^{-1}) &= C_B(f(a)^{-1}) = C_B(f(a^{-1})) = C_{(f^{-1}(B))}(a^{-1}) = C_{f^{-1}(B)}(a) \\ &= C_B(f(a)) = C_B(x). \end{aligned}$$

Therefore, $B \in MG(Y)$. □

Proposition 4.2. *Let X be a group and $f : X \longrightarrow X$ is an automorphism. If $A \in MG(X)$, then $f(A) = A$ if and only if $f^{-1}(A) = A$.*

Proof. Let $x \in X$. Then $f(x) = x$. Now, $C_{(f^{-1}(A))}(x) = C_A(f(x)) = C_A(x)$ implies $f^{-1}(A) = A$.

Conversely, let $f^{-1}(A) = A$. Since f is an automorphism, then

$$\begin{aligned} C_{f(A)}(x) &= \bigvee \{C_A(x') : x' \in X,\ f(x') = f(x) = x\} \\ &= C_A(f(x)) = C_{(f^{-1}(A))}(x) = C_A(x). \end{aligned}$$

Hence, the proof. □

Proposition 4.3. *Let $f : X \to Y$ be a homomorphism of groups, $A \in MG(X)$ and $B \in MG(Y)$. If A is a constant on $Kerf$, then $f^{-1}(f(A)) = A$.*

Proof. Let $f(x) = y$. Then

$$C_{f^{-1}(f(A))}(x) = C_{f(A)}f(x) = C_{f(A)}(y) = \bigvee\{C_A(x) : x \in X,\ f(x) = y\}.$$

Since $f(x^{-1}z) = f(x^{-1})f(z) = (f(x))^{-1}f(z) = y^{-1}y = e'$, $\forall z \in X$ such that $f(z) = y$, which implies $x^{-1}z \in Kerf$. Also, since A is constant on $Kerf$, then $C_A(x^{-1}z) = C_A(e)$. Therefore, $C_A(x) = C_A(z)\ \ \forall\ z \in X$ such that $f(z) = y$ by Proposition 3.28 . Hence, the proof. □

Proposition 4.4. *Let $H \in AMG(X)$. Then the map $f : X \to X/H$ defined by $f(x) = xH$ is a homomorphism $Kerf = \{x \in X : C_H(x) = C_H(e)\}$, where e is the identity of X.*

Proof. Let $x, y \in X$. Then $f(xy) = (xy)H = xHyH = f(x)f(y)$. Hence, f is a homomorphism. Further,

$$\begin{aligned} Kerf &= \{x \in X : f(x) = eH\} = \{x \in X : xH = eH\} \\ &= \big\{x \in X : C_H(x^{-1}y) = C_H(y)\ \forall y \in X\big\} \\ &= \big\{x \in X : C_H(x^{-1}) = C_H(e)\big\} = \{x \in X : C_H(x) = C_H(e)\} = H_*, \end{aligned}$$

which completes the proof. □

Remark 4.5. By Propositions 4.4 and 3.19, $Ker\,f$ is a normal subgroup of X.

Proposition 4.6. (First Isomorphism Theorem) *Let $f : X \to Y$ be an epimorphism of groups and $H \in AMG(X)$, then $X/H_* \cong Y$, where $H_* = Kerf$.*

Proof. Define $\Theta : X/H_* \to Y$ by $\theta(xH_*) = f(x)\ \ \forall x \in X$. Let $xH = yH$ such that $C_H(x^{-1}y) = C_H(e)$. Since $x^{-1}y \in H_*$, then $f(x^{-1}y) = f(e) \Longrightarrow f(x) = f(y)$. Hence, Θ is well-defined. Obviously it is an epimorphism. Moreover, $f(x) = f(y)$ implies $f(x)^{-1}f(y) = f(e)$. So, $f(x^{-1})f(y) = f(x^{-1}y) = f(e)$, i.e., $x^{-1}y \in H_*$ and consequantly, $C_H(x^{-1}y) = C_H(e)$. Thus, $xH = yH$, which shows Θ is an isomorphism. □

Proposition 4.7. (Second Isomorphism Theorem) *If* $H, N \in AMG(X)$ *such that* $C_H(e) = C_N(e)$, *then* $H_*N_*/N \cong H_*/H \cap N$.

Proof. Clearly, for any $x \in H_*N_*$, $x = hn$ where $h \in H_*$ and $n \in N_*$. Define $\varphi : H_*N_*/N \to H_*/H \cap N$ by $\varphi(xN) = h(H \cap N)$.

If $xN = yN$, where $y = h_1n_1$, $h_1 \in H_*$ and $n_1 \in N_*$, then

$$C_N(x^{-1}y) = C_N((hn)^{-1}h_1n_1) = C_N(n^{-1}h^{-1}h_1n_1) = C_N(h^{-1}h_1n^{-1}n_1) = C_N(e).$$

Hence, $C_N(h^{-1}h_1) = C_N(n^{-1}n_1) = C_N(e)$. Thus,

$$C_{H\cap N}(h^{-1}h_1) = C_H(h^{-1}h_1) \wedge C_N(h^{-1}h_1) = C_H(e) \wedge C_N(e) = C_{H\cap N}(e),$$

i.e., $h(H \cap N) = h_1(H \cap N)$. Hence, φ is well-defined.

If $xN, yN \in H_*N_*\ N$, then $xy = hnh_1n_1$. Since $H \in AMG(X)$, then $C_H(nh_1n_1) = C_H(h_1)$ gives $nh_1n_1 \in H_*$. Hence,

$\varphi(xNyN) = \varphi(xyN) = h(nh_1n1)(H \cap N) = h(H \cap N)nh_1n_1(H \cap N)$ and

$$\begin{aligned} C_{H\cap N}(h_1^{-1}(nh_1n_1)) &\geqslant C_H(h_1^{-1}nh_1n_1) \wedge C_N(h_1^{-1}nh_1n_1) \\ &= C_H(h_1^{-1}(nh_1n_1)) \wedge C_N(n(h_1^{-1}h_1n_1)) \\ &= C_H(e) \wedge C_N(e) \\ &= C_{H\cap N}(e). \end{aligned}$$

Hence, $nh_1n_1(H \cap N) = h_1(H \cap N)$, i.e., $\varphi(xNyN) = h(H \cap N)h_1(H \cap N) = \varphi(xN)\varphi(yN)$, which shows that φ is a homomorphism.

φ is also an epimorphism, since for $h(H \cap N) \in H_*/H \cap N$ and $n \in N_*$, we have $x = hn \in H_*N_*$ and $\varphi(xN) = h(H \cap N)$.

Moreover, if $x, y \in H_*N_*$, where $x = hn$ and $y = h_1n_1$, $h, h_1 \in H_*$ and $n, n_1 \in N_*$ and $h(H \cap N) = h_1(H \cap N)$, then $C_{H\cap N}(h^{-1}h_1) = C_{H\cap N}(e)$, i.e., $C_H(h^{-1}h_1) \wedge C_N(h^{-1}h_1) = C_H(e) \wedge C_N(e)$. But $C_H(e) = C_N(e)$ and $C_H(h^{-1}h_1) = C_H(e)$, so $C_N(h^{-1}h_1) = C_N(e)$. Therefore,

$$\begin{aligned} C_N(x^{-1}y) &= C_N((hn)^{-1}h_1n_1) \\ &= C_N(n^{-1}h^{-1}h_1n_1) = C_N(h^{-1}h_1n^{-1}n_1) \\ &\geqslant C_N(h^{-1}h_1) \wedge C_N(n^{-1}n_1) = C_N(e) \wedge C_N(e) = C_N(e). \end{aligned}$$

Thus, $C_N(x^{-1}y) = C_N(e)$, and consequently, $xN = yN$.

Hence, $H_*N_*/N \cong H_*/H \cap N$. □

Proposition 4.8. (Third Isomorphism Theorem) *Let* $H, N \in AMG(X)$ *with* $H \subseteq N$ *and* $C_H(e) = C_N(e)$. *Then* $X/N \cong (X/H)/(N_*/H)$.

Proof. Define $f : X/H \to X/N$ by $f(xH) = xN \ \ \forall x \in X$ such that $C_H(x^{-1}y) = C_H(e) = C_N(e) \ \ \forall xH = yH$. Because $H \subseteq N$, we have $C_N(x^{-1}y) \geqslant C_H(x^{-1}y) = C_N(e)$ and so $C_N(x^{-1}y) = C_N(e)$, i.e., $xN = yN$, which means that f is well-defined. Obviously f is an epimorphism.

Moreover,

$$\begin{aligned} Kerf &= \{xH \in X/H : f(xH) = eN\} \\ &= \{xH \in X/Hx : N = eN\} \\ &= \{xH \in X/H : C_N(x) = C_N(e)\} \\ &= \{xH \in X/H : x \in N_*\} = N_*/H. \end{aligned}$$

Thus, $Kerf = N_*/H$ and $X/N \cong (X/H)/(N_*/H)$. □

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