Quasigroups and Related Systems **24** (2016), 157 − 168

Structure of the finite groups with $4p$ elements of maximal order

Bahareh Asadian and Neda Ahanjideh

Abstract. Let G be a finite group and $p > 3$ be a prime number. We determine the structure of the finite group G with $4p$ elements of maximal order. In particular, we show that if G is a finite group with 20 elements of maximal order, then G is a non-abelian 2-group of order 32 with $\exp(G) = 4$, $G \cong C_6 \times \mathbb{S}_3$ or $G \cong \mathbb{S}_5$, where $\mathbb{S}_n$ denotes the symmetric group of degree n, $G \cong C_{44} \rtimes (C_u \times C_l)$, where $u|10$ and $l|2$, $G \cong C_{25} \rtimes C_l$ or $G \cong C_{50} \rtimes C_l$, where $l|4$.

1. Introduction

Throughout this paper, we use the following notations: For a finite group G, we denote by $\pi(G)$ the set of prime divisors of $|G|$ and by $\pi_e(G)$ the set of element orders of G. By $m_i(G)$, we denote the number of elements of order i, where $i \in \pi_e(G)$. Set $\mathrm{nse}(G) := \{m_i(G) : i \in \pi_e(G)\}$.

One of the interesting topics in the group theory is to determine the solvability of a group with the given particular properties. For example, one of the problems which is proposed by Thompson is:

Thompson's Problem. *Let $T(G) = \{(n, m_n) : n \in \pi_e(G) \ \ and \ \ m_n \in \mathrm{nse}(G)\}$, where m_n is the number of elements of order n in G. Suppose that $T(G) = T(S)$. If S is a solvable group, is it true that G is also necessarily solvable?*

Up to now, nobody can solve this problem and it remains as an open problem. In order to approach to this problem, some authors have examined the solvability of a group with a given number of elements of maximal order. For instance, in [2, 9, 10], the authors have examined the structure of the groups which have a given number of the elements of maximal order. Also, in [4], some groups with exactly $4p$ elements of maximal order have been studied. The purpose of this paper is to study the structure of a group containing exactly $4p$ elements of maximal order. Then as an example, we find the structure of finite groups with exactly 20 elements of maximal order.

From now on, we use $\mathrm{Syl}_p(G)$ for the set of the p-Sylow subgroups of G, where $p \in \pi(G)$. Also, G_p denotes a p-Sylow subgroup of G and $n_p(G) = |\mathrm{Syl}_p(G)|$. We denote by ϕ the Euler's totient function. For every $x \in G$, $o(x)$ denotes the order

2010 Mathematics Subject Classification: 20D05, 20D60
Keywords: group, maximal order,Thompson's problem.

of x and $\langle x\rangle$ denotes the generated subgroup by x in G. $C_G(\langle x\rangle)$ and $N_G(\langle x\rangle)$ are used as centralizer and normalizer of $\langle x\rangle$ in G, respectively. Let A and N be finite groups. The action of A on N is Frobenius if and only if $C_N(a)=1$, for all nonidentity elements $a\in A$. We use $a|n$ when a is a divisor of n and use $|n|_a=a^e$, when $a^e||n$, i.e., $a^e|n$ but $a^{e+1}\nmid n$. By C_n, we denote a cyclic group of order n. Throughout this paper, k denotes the maximal order of elements in G, $M(G)$ is the number of elements of order k and $n,l\in\mathbb{N}$. Also, $Z(G)$ denotes the center of group G. We apply symbol $(*)$ instead of assumption $M(G)=4p$, where p is a prime number. All unexplained notations are standard and can be found in [7]. In this paper, we will prove that:

Main Theorem. *Suppose that G is a finite group with $M(G)=4p$, where $p>3$ is a prime number. Then G is one of the following groups:*

(1) *If $k=4$, then G is a non-abelian 2-group with $|G|<16p$ and* $\exp(\mathrm{G})=4$*;*

(2) *if $k=5$,* $\exp(\mathrm{G})=5$ *and $p=(5^u-1)/4$, then either G is a 5-group of order 5^u or $G\cong \mathrm{G}_5\rtimes C_{2^t}$, where $t\in\{1,2\}$ and G_5 denotes 5-Sylow subgroup of G;*

(3) *if $k=6$, then either $G\cong\mathbb{S}_5$, where $\mathbb{S}_5$ denotes the symmetric group of degree 5 or G is a $\{2,3\}$-group;*

(4) *if $k=10$, then G is a $\{2,5\}$-group;*

(5) *if $k=12$, then G is a $\{2,3\}$-group;*

(6) *if $4p+1$ is a prime number and $k=4p+1$, then $G\cong \mathrm{C}_{4\mathrm{p}+1}\rtimes\mathrm{C}_\mathrm{l}$, where $l|4p$;*

(7) *if $2p+1$ is a prime number and $k=4(2p+1)$, then $G\cong \mathrm{C}_{4(2\mathrm{p}+1)}\rtimes(\mathrm{C}_u\times C_l)$, where $u|2p$ and $l|2$;*

(8) *if $4p+1$ is a prime number and $k=2(4p+1)$, then $G\cong \mathrm{C}_{2(4\mathrm{p}+1)}\rtimes\mathrm{C}_u$, where $u|4p$;*

(9) *if $k=25$, then $G\cong C_{25}\rtimes C_l$, where $l|4$;*

(10) *if $k=50$, then then $G\cong C_{50}\rtimes C_l$, where $l|4$.*

As a consequent of the main theorem, we will prove that:

Corollary. *Suppose that G is a finite group with $M(G)=20$. Then G is one of the following groups:*

(1) *If $k=4$, then G is a non-abelian 2-group of order 32;*

(2) *if $k=6$, then either $G\cong\mathbb{S}_5$ or $G\cong C_6\times\mathbb{S}_3$;*

(3) *if $k=25$, then $G\cong C_{25}\rtimes C_l$, where $l|4$;*

(4) *if $k=44$, then $G\cong C_{44}\rtimes(C_u\times C_l)$, where $u|10$ and $l|2$;*

(5) *if $k=50$, then $G\cong C_{50}\rtimes C_l$, where $l|4$.*

2. Preliminary results

Throughout this paper, we assume that $p > 3$ is a prime number. In the following lemmas, we bring some facts which will be used during the proof of the main theorem:

Lemma 2.1. [3, Lemma 2.2] *Suppose that G has exactly n cyclic subgroups of order l, then $m_l(G) = n \cdot \phi(l)$. In particular, if n denotes the number of cyclic subgroups of G of order k, then $M(G) = n \cdot \phi(k)$.*

The following lemma is concluded from Lemma 2.1:

Lemma 2.2. *If $M(G) = 4p$, then the possible values of k and $\phi(k)$ are given in the following table:*

$\phi(k)$	k	Condition
1	2	–
2	3, 4, 6	–
4	5, 10, 12	–
p	null	–
$2p$	$2p+1$, $2(2p+1)$	$2p+1$ is prime
$4p$	25, 50	$p = 5$
$4p$	$4(2p+1)$	$2p+1$ is prime
$4p$	$4p+1$, $2(4p+1)$	$4p+1$ is prime

Lemma 2.3. [2, Lemma 6] *If k is a prime number, then $k|M(G)+1$.*

Corollary 2.4. *Let $M(G) = 4p$. Then $k \neq 2$ and $k \neq 5$ except when $p = 5t+1$, where $t \in \mathbb{N}$. Also, if $2p+1$ is prime, then $k \neq 2p+1$.*

Proof. It follows from Lemma 2.3. □

Lemma 2.5. [2, Lemma 7] *If there exists a prime divisor p of k with $p(p-1) > M(G)$, then G contains a unique normal p-Sylow subgroup $\mathrm{G_p}$ and $|\mathrm{G_p}| = p$.*

Lemma 2.6. *Let G be a finite group such that $[C_G(x) : \langle x \rangle]$ is a prime power number. Then $C_G(x)$ is direct product of its sylow subgroups.*

Proof. The proof is straightforward. □

Lemma 2.7. [2, Lemma 8] *There exists a positive integer α such that $|G|$ divides $M(G)k^{\alpha}$.*

Lemma 2.8. *For every element $x \in G$ of order k, $[G : N_G(\langle x \rangle)] \cdot \phi(o(x)) \leqslant M(G)$.*

Proof. The proof is straightforward. □

Lemma 2.9. *For every element $x \in G$ of order k, if $\pi_e(C_G(x)) = \pi_e(\langle x \rangle)$, then $[C_G(x) : \langle x \rangle] \cdot \phi(o(x)) \leqslant M(G)$.*

Proof. Fix $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant o(x)$, where $t = [C_G(x) : \langle x\rangle]$. Suppose that $\mathcal{A} = \{y_i\langle x\rangle : y_i \in C_G(x)\}$ is the distinct left coset of $\langle x\rangle$ in $C_G(x)$. It is easily seen that if $y_i\langle x\rangle \neq \langle x\rangle$ and $o(x^j) = k$, then $o(y_i x^j) = o(x^j)$. Also, for every element $y_i\langle x\rangle \in \mathcal{A}$, there exist exactly $\phi(x)$ elements $y_i x^j$ of order k. So, we have:

$$[C_G(x) : \langle x\rangle] \cdot \phi(o(x)) = |\{y_i x^j : o(x^j) = k\}|.$$

It is evident that $|\{y_i x^j : o(x^j) = k\}| \leqslant |\{g \in G : o(g) = k\}| = M(G)$. Hence, the lemma follows. □

Lemma 2.10. [9, Lemma 2.5] *Let P be a p-group of order p^t, where t is a positive integer. Suppose that $b \in Z(P)$, where for $u \in \mathbb{N}$, $o(b) = p^u = k$. Then P has at least $(p-1)p^{t-1}$ elements of order k.*

Lemma 2.11. [10, Lemma 4] *Let G be a non-abelian finite group with $\exp(G) = 4$. If $x \in G \setminus Z(G)$ is an element of order 2, then G has at least $\frac{|G|-|C_G(x)|}{2} = \frac{|C_G(x)|.([G:C_G(x)]-1)}{2}$ elements of order 4.*

Lemma 2.12. [5] *Let $p \in \pi(G)$ be odd. Let $\mathrm{G}_p \in \mathrm{Syl}_p(G)$ and $n = p^s m$ with* $\gcd(p, m) = 1$. *If G_p is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Lemma 2.13. [13, Theorem 3] *Let G be a finite group. Then the number of elements whose orders are multiples of n is either zero, or a multiple of the greatest divisor of $|G|$ that is prime to n.*

Lemma 2.14. [1] *Let $L = U_n(q)$, where $n > 3$, $q = p^\alpha$, and let $d = (n, q+1)$. Then $\pi_e(L)$ consists of all divisors of m, where $m = p^\gamma \frac{q^{n_1}-(-1)^{n_1}}{d}$, where γ, $n_1 > 0$ satisfying $p^{\gamma-1} + 1 + n_1 = n$.*

Lemma 2.15. [5] *Let t be a positive integer dividing $|G|$. If $M_t(G) = \{g \in G | g^t = 1\}$, then $t || M_t(G)|$.*

Corollary 2.16. *For a finite group G:*

- (i) *if $d \in \pi_e(G)$, then $d | \sum_{s|d} m_s$;*
- (ii) *if $P \in \mathrm{Syl}_p(G)$ is a cyclic group of prime order p and $r \in \pi(G) - \{p\}$, then $m_{rp} = n_p(G)(p-1)(r-1)t$, where t is the number of cyclic subgroups of order r in $C_G(P)$.*

Proof. (i) follows from Lemma 2.15. For proving (ii), let $P \in \mathrm{Syl}_p(G)$. Since $m_p(P) = p - 1$ and every element of order rp is in $C_G(P^g)$, for some $g \in G$, we deduce that $m_{rp}(G) = m_p(G) \cdot n_p(G) \cdot m_r(C_G(P)) = (p-1) \cdot n_p(G) \cdot (\phi(r) \cdot t) = n_p(G) \cdot (p-1) \cdot (r-1) \cdot t$, where t is the number of the cyclic subgroups of order r in $C_G(P)$. □

Lemma 2.17. *If p is a prime number, then $4p + 1 \neq 3^t$.*

Proof. Suppose on the contrary that $4p+1=3^t$. Then $4|3^t-1$ and hence, t is a even number. Thus $3^2-1|3^t-1=4p$, which is a contradiction. □

Lemma 2.18. [12] *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\frac{K}{H}$ is a direct product of isomorphic non-abelian simple groups and $|\frac{G}{K}||| \mathrm{Out}(\frac{K}{H})|$.*

Lemma 2.19. *Let $H \trianglelefteq G$ and let $r \in \pi(H)$. If $p \in \pi(\frac{G}{H})$, $p \notin \pi(H)$ and $pr \notin \pi_e(G)$, then $p||H|_r-1$.*

Proof. By Frattini's argument, $G = HN_G(R)$, where $R \in \mathrm{Syl}_r(H)$. Thus we can see that $\mathrm{G_p}^g \leqslant N_G(R)$ for some $g \in G$. But $pr \notin \pi_e(G)$ and hence, the action of $\mathrm{G_p}^g$ on R is Frobenius. Therefore, $|G|_p||H|_r-1$ and the result follows. □

A finite group G is called a *simple K_n-group*, if G is a simple group with $|\pi(G)|=n$. So, a simple K_3-group is a simple group with $|\pi(G)|=3$. In the following lemma, the simple K_3-groups and their orders are recognized:

Lemma 2.20. [8] *Let G be a simple K_3-group. Then G is isomorphic to one of following simple groups: $A_5(2^2\cdot3\cdot5)$, $A_6(2^3\cdot3^2\cdot5)$, $L_2(7)(2^3\cdot3\cdot7)$, $L_2(8)(2^3\cdot3^2\cdot7)$, $L_2(17)(2^4\cdot3^2\cdot17)$, $L_3(3)(2^4\cdot3^3\cdot13)$, $U_3(3)(2^5\cdot3^3\cdot7)$, $U_4(2)(2^6\cdot3^4\cdot5)$.*

Theorem 2.21. *If G is a non-solvable group with $M(G)=4p$, where p is a prime number, then $p=5$ and $G \cong \mathbb{S}_5$.*

Proof. Since G is a non-solvable group, Lemma 2.18 shows that there exists a chief series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\frac{K}{H}$ is a direct product of isomorphic non-abelian simple groups and $|\frac{G}{K}|||\mathrm{Out}(\frac{K}{H})|$. Lemmas 2.2 and 2.7 show that $|\pi(G)| \leqslant 3$ and every finite group such that its order is divisible by exactly two prime numbers is solvable. Thus $|\pi(\frac{K}{H})|=3$ and $p||\frac{K}{H}|$. Therefore, $\frac{K}{H}$ is a simple K_3-group and $p \in \pi(\frac{K}{H})$. Also, by virtu of Lemma 2.20, we can see that for every simple K_3-group S, $3 \in \pi(S)$. Since our assumption forces $p>3$, $k \neq 3$. Therefore, Lemmas 2.2 and 2.7 imply that the non-solvability of G can be occurred when $k \in \{6,12\}$.

We continue the proof in the following cases:

1. If $k=6$, then Lemma 2.7 and the above statements show that $|K/H|=2^\alpha3^\beta p$, where $\alpha,\beta>0$. Then $\frac{K}{H}$ is a simple K_3-group and hence, Lemma 2.20 shows that one of the following subcases holds:

(i). If $\frac{K}{H} \cong A_5$ or A_6, then $p=5$. Let z be an element of G with $o(z)=6$. By Lemma 2.9, we have $[C_G(z):\langle z\rangle] \leqslant 10$. Since $k=6$, $5 \nmid |C_G(z)|$. If $[C_G(z):\langle z\rangle] \in \{8,9\}$, then Lemma 2.6 implies that $C_G(z)$ is direct product of its sylow subgroups. Hence, it is easy to see that $m_6(C_G(z))>20$. So, we get a contradiction with $M(G)=20$. Therefore, $[C_G(z):\langle z\rangle] \in \{1,2,3,4,6\}$. We have,

$$|G| = 6\cdot[C_G(z):\langle z\rangle]\cdot[N_G(\langle z\rangle):C_G(z)]\cdot[G:N_G(\langle z\rangle)].$$

By virtue of Lemma 2.8, we can see that $[G:N_G(\langle z\rangle)] \leqslant 2p=10$.

Since $[N_G(\langle z\rangle) : C_G(\langle z\rangle)]||\mathrm{Aut}(\langle z\rangle)| = 2$, we deduce that $5|[G : N_G(\langle z\rangle)]$. Hence, $|G||2^5 \cdot 3^2 \cdot 5$ and $2^2 \cdot 3 \cdot 5||\frac{K}{H}|$. Therefore, $|H||2^3 \cdot 3$. But $2 \cdot 5, 3 \cdot 5 \notin \pi_e(G)$. So, Lemma 2.19 shows that $5|3^t - 1$ or $5|2^u - 1$, where $t < 2$ and $u \leqslant 3$. Thus, $t = u = 0$ and hence, $|H| = 1$. Thus $K \cong A_5$ or A_6. Since $|\frac{G}{K}|||\mathrm{Out}(K)|$, we deduce that $G \cong \mathbb{S}_5$ or $\mathbb{S}_6$. But $M(\mathbb{S}_5) = 20$ and $M(\mathbb{S}_6) = 240$. Thus $G \cong \mathbb{S}_5$.

(ii). If $\frac{K}{H} \cong L_2(7), L_2(8), L_2(17), L_3(3), U_3(3)$, then there exists $p \in \pi(G)$ such that $p > 6$, which is a contradiction.

(iii). If $\frac{K}{H} \cong U_4(2)$, then Lemma 2.14 implies that $12 \in \pi_e(\frac{K}{H})$ and hence, we arrive at a contradiction.

2. Let $k = 12$. Then applying Lemma 2.7 shows that $\pi(G) = \{2, 3, p\}$. Since every finite group such that its order is divisible by exactly two prime numbers is solvable and $|\pi(G)| = 3$, we deduce that $|\pi(\frac{K}{H})| = 3$ and $p||\frac{K}{H}|$. Since $k = 12$, we deduce that $p \leqslant 11$ and for every $x \in G$ with $o(x) = 12$, $C_G(\langle x\rangle)$ is a $\{2,3\}$-group. Since $\frac{K}{H}$ is a simple K_3-group, Lemma 2.20 shows that one of the following subcases holds:

(i). If $\frac{K}{H} \cong A_5$ or A_6, then $p = 5$. In the following, we show that this case is impossible with our assumption. It is easy to see that $|C_G(\langle x\rangle)| = 2^u \cdot 3^v$ such that $2 \leqslant u \leqslant 4$ and $1 \leqslant v \leqslant 2$. Applying Lemma 2.8 to this case shows that $[G : N_G(\langle x\rangle)] \in \{1, 2, 3, 4, 5\}$. Note that for every $x \in G$ with $o(x) = 12$, $[N_G(\langle x\rangle) : C_G(\langle x\rangle)]||\mathrm{Aut}(\langle x\rangle)| = 4$. But $5||G|$ and

$$|G| = [G : N_G(\langle x\rangle)] \cdot [N_G(\langle x\rangle) : C_G(\langle x\rangle)] \cdot |C_G(\langle x\rangle)|. \tag{1}$$

Thus $[G : N_G(\langle x\rangle)] = 5$ and $|G||2^6 \cdot 3^2 \cdot 5$. Since $|\mathrm{Aut}(\langle x\rangle)| = 4$, we conclude that $\mathrm{G}_3 \leqslant C_G(\langle x\rangle)$. Set $C = C_G(\langle x\rangle)$. We examine two possibilities for v:

(a). Let $v = 2$. Applying Lemma 2.9 shows that $|C_G(\langle x\rangle)| = 2^2 \cdot 3^2$. Since $\langle x\rangle \leqslant Z(C)$, $12||Z(C)|$. Thus C is abelian and hence,

$$C = C_4 \times (C_3 \times C_3). \tag{2}$$

Therefore, $m_{12}(C) = 16$. If there exists $y \in G$ of order 12 such that $9 \nmid |C_G(y)|$, then obviously $y \notin C$ and (1) leads us to see that $3|[G : N_G(\langle y\rangle)] = 5$, which is a contradiction. This shows that for every $y \in G$ of order 12, $|C_G(y)|_3 = 9$, so for some $g \in G$,

$$\mathrm{G}_3^g \leqslant C_G(y). \tag{3}$$

Also, (2) shows that $C \leqslant C_G(\mathrm{G}_3)$. So, $C_G(\mathrm{G}_3)$ contains at least 16 elements of order 12. Thus for every $g \in G$ with $C_G(\mathrm{G}_3) \neq C_G(\mathrm{G}_3^g)$, $C_G(\mathrm{G}_3) \cap C_G(\mathrm{G}_3^g)$ contains at least 12 elements of order 12. Let y be an element of order 12 in $C_G(\mathrm{G}_3) \cap C_G(\mathrm{G}_3^g)$, then $\mathrm{G}_3, \mathrm{G}_3^g \trianglelefteq C_G(y)$. Thus $\mathrm{G}_3 = \mathrm{G}_3^g$ and hence $\mathrm{G}_3 \trianglelefteq G$. Therefore, (3) shows that for every $y \in G$ of order 12, $y \in C_G(\mathrm{G}_3) = \mathrm{G}_3 \times \mathrm{G}_2(C_G(\mathrm{G}_3))$. Hence $20 = m_{12}(G) = m_{12}(C_G(\mathrm{G}_3)) = m_3(\mathrm{G}_3) \cdot m_4(\mathrm{G}_2(C_G(\mathrm{G}_3))) = 8 \cdot m_4(\mathrm{G}_2(C_G(\mathrm{G}_3)))$, which is a contradiction.

(b). Let $v = 1$. Then $K/H \cong A_5$ and $|C_G(\langle x\rangle)| = 2^u \cdot 3$. Since $[N_G(\langle x\rangle) : C_G(\langle x\rangle)]$ divides 4 and $[G : N_G(\langle x\rangle)] = 5$, $|G|_3 = 3$. Also, Lemma 2.9 forces $u \leqslant 4$. Thus

$|G| | 2^6 \cdot 3 \cdot 5$ and hence, $n_3(G) = 2^\alpha \cdot 5^\beta$, where $\beta \in \{0,1\}$. On the other hand, $n_3(K/H) = 10 | n_3(G)$. But Corollary 2.16(ii) shows that $m_{12}(G) = n_3(G) \cdot \phi(3) \cdot t = 20$, where $t = m_4(C_G(\mathrm{G}_3)) \geqslant 2$, which is impossible.

(ii). If $\frac{K}{H} \cong L_3(3)$ or $L_2(17)$, then there exists $p \in \pi(\frac{K}{H})$ such that $p > 11$, which is contradiction.

(iii). If $\frac{K}{H} \cong U_4(2)$ or $U_3(3)$, then $p = 5$ or 7, respectively. Applying Lemma 2.14 and GAP program [6] imply that $12 \in \pi_e(\frac{K}{H})$. Since $m_{12}(U_3(3)) = 1008$ and $m_{12}(U_4(2)) = 4320$, we arrive at a contradiction;

(iv). If $\frac{K}{H} \cong L_2(7)$ or $L_2(8)$, then $p = 7$ and $|G| = 7 \cdot 2^u \cdot 3^v$, where $1 \leqslant u \leqslant 7$ and $1 \leqslant v \leqslant 2$. If $v = 2$, then we can see at once that either $K/H \cong L_2(8)$ or $|H|_3 = 3$. If $|H|_3 = 3$, then since $21 \notin \pi_e(G)$, Lemma 2.19 shows that $7 | 3-1$, which is a contradiction. Thus let $K/H \cong L_2(8)$. Then since for every $y \in G$ of order 12, y is central in $C_G(y)$, we deduce that $y \in C_G(\mathrm{G}_3)$. Thus we can see at once that $C_G(\mathrm{G}_3)$ contains at least 16 elements of order 12. So, for every $g \in G$ with $C_G(\mathrm{G}_3) \neq C_G(\mathrm{G}_3^g)$, $C_G(\mathrm{G}_3) \cap C_G(\mathrm{G}_3^g)$ contains at least 12 elements of order 12. Let y be an element of order 12 in $C_G(\mathrm{G}_3) \cap C_G(\mathrm{G}_3^g)$. Then $\mathrm{G}_3, \mathrm{G}_3^g \leqslant C_G(y)$. On the other hand, applying the argument in Subcase (i) shows that $|C_G(y)| \leq 3^2 \cdot 2^3$. Thus $\mathrm{G}_3 \times \langle y^3 \rangle$, $\mathrm{G}_3^g \times \langle y^3 \rangle \trianglelefteq C_G(y)$ and hence, $\mathrm{G}_3, \mathrm{G}_3^g \trianglelefteq C_G(y)$. Thus $\mathrm{G}_3 = \mathrm{G}_3^g$ which is a contradiction. Therefore, $C_G(\mathrm{G}_3) \trianglelefteq G$ and hence, $\mathrm{G}_3 \trianglelefteq G$. Thus the same argument as that of used in (2) shows that for every $y \in G$ of order 12, $y \in C_G(\mathrm{G}_3) = \mathrm{G}_3 \times \mathrm{G}_2(C_G(\mathrm{G}_3))$ and hence, $28 = m_{12}(G) = m_3(\mathrm{G}_3) \cdot m_4(\mathrm{G}_2(C_G(\mathrm{G}_3))) = 8 \cdot m_4(\mathrm{G}_2(C_G(\mathrm{G}_3)))$, which is impossible. Thus $v = 1$ and hence, $K/H \cong L_2(7)$ and $m_{12}(G) = 2 \cdot n_3(G) \cdot t = 28$, where $t = m_4(C_G(\mathrm{G}_3)) \geqslant 2$. But $n_3(L_2(7)) = 28 | n_3(G)$, which is impossible. □

3. Proof of the main theorem

In this section, we prove the main theorem by considering the eight values for k obtained in Lemma 2.2:

1) $k = 3$. By virtue of Lemma 2.7, we have $|G| | 4 \cdot 3^\alpha \cdot p$, where $\alpha > 0$. But $k = 3$ and according to our assumption $p > 3$. Thus $|G| | 2^2 \cdot 3^\alpha$. Since $k = 3$, two possibilities can be occurred for $|G|$:

(i). If $|G| = 3^u$, where $u \in \mathbb{N}$, then since $k = 3$, $\exp(G) = 3$ and hence, $|G| - 1 = M(G)$. Thus $3^u - 1 = 4p$, which is a contradiction with Lemma 2.17.

(ii). If $2 \in \pi(G)$, then $|G| = 2^{\alpha_1} \cdot 3^{\alpha_2}$ such that $0 \leqslant \alpha_1 \leqslant 2$ and $\alpha_2 > 0$. Thus G is solvable. Let N be a normal minimal subgroup of G. Then N is t-elementary abelian, where $t \in \{2,3\}$. Since $6 \notin \pi_e(G)$, we deduce that for $u \in \{2,3\} - \{t\}$, the action of G_u on N is Frobenius. Thus if $t = 2$, then G_3 is cyclic and since $k = 3$, we deduce that by Corollary 2.16(ii), $2 \cdot n_3(G) = 4p$. This forces $n_3(G) = 2p | |G|_2$, which is a contradiction. Now let $t = 3$. Then G_2 is a cyclic group or a quaternion group. But $4 \notin \pi_e(G)$ and hence, $|\mathrm{G}_2| = 2$. This guarantees that $\mathrm{G}_3 \trianglelefteq G$. Thus $m_3(G) = m_3(\mathrm{G}_3)$ and hence, applying the previous argument leads us to get a contradiction.

2) $k = 4$. Applying Lemma 2.7 shows that either $p = 3$ and $\pi(G) = \{2, 3\}$ or G is a 2-group. According to our assumption, $p > 3$ and hence, G is a 2-group. Let $|G| = 2^{\alpha}$, where $\alpha \in \mathbb{N}$. Then by $(*)$, we can see $|G| > 4p + 1$. If G is an abelian group such that $|G| = 2^{\alpha}$, then $\{x \in G : o(x)|2\} \leqslant G$ and hence, $1+m_2(G) = 2^u$ and $1+m_2(G)+m_4(G) = |G|$ gives that $2^u+4p = 2^{\alpha}$. This forces $2^u(2^{\alpha-u} - 1) = 4p$ and hence, $u = 2$. Thus $m_2(G) = 3$ and hence, $G \cong C_4 \times C_4$ or $C_2 \times C_4$. So, $m_4(G) \leqslant 12$, which is a contradiction. If G is a non-abelian 2-group, then we claim that there exists an element y in G such that $y \notin Z(G)$ and $o(y) = 2$. If not, then $Z(G)$ contains all elements of order 2 in G. If $2^{\alpha-3} \leqslant p$, then since our assumption shows that $|Z(G)| \geq |G| - 4p$, we have $|G/Z(G)| \leqslant 2$. Thus G is abelian, which is a contradiction. If $2^{\alpha-3} > p$, then Lemma 2.10 shows that there is no element of order 4 in $Z(G)$, so $|G| = |Z(G)| + M(G)$ and hence, $2^{\alpha} = 2^m + M(G)$, where $|Z(G)| = 2^m$. Thus $m = 2$ and $p = 2^{\alpha-2} - 1 > 2^{\alpha-3} > p$, which is a contradiction. So, there exists $y \in G \setminus Z(G)$ with $o(y) = 2$. Therefore, Lemma 2.11 and $(*)$ show that $\frac{|G|}{4} \leq \frac{|G|-|C_G(x)|}{2} \leqslant 4p$ and hence, we can conclude that $|G| < 16p$.

3) $k = 5$ and $p = 5t+1$. Then by virtue of Lemma 2.7, $|G||2^2 \cdot p \cdot 5^{\alpha}$, where $\alpha > 0$. Since $p = 5t + 1$ is a prime number which is greater than 5, $p \notin \pi(G)$. If G is a 5-group, then $\exp(G) = 5$, so $4p = |G| - 1 = 5^u - 1$ and hence, $p = (5^u - 1)/4$. If G is a $\{2, 5\}$-group, then G is solvable. Let N be a normal minimal subgroup of G. In the following, we examine two possibilities for order of N:

(i). If $|N| = 2^t$, where $t \in \mathbb{N}$, then the action of G_5 on N is Frobenius. Hence G_5 is cyclic. Since $25 \notin \pi_e(G)$, $|\mathrm{G}_5| = 5$. Corollary 2.16(ii) shows that $m_5(G) = n_5(G) \cdot 4$ $= 4p$ which follows that $p = n_5(G)||G|$. Hence, we arrive at a contradiction.

(ii). If $|N| = 5^u$, then $|\mathrm{G}_2| \in \{2, 4\}$. Thus $\mathrm{G}_5 \trianglelefteq G$ and hence, $G \cong \mathrm{G}_5 \rtimes C_{2^t}$, where $t \in \{1, 2\}$. Therefore, $5^u - 1 = |\mathrm{G}_5| - 1 = 4p$ and hence, $p = (5^u - 1)/4$, as claimed.

4) $k = 6$. By virtue of Lemma 2.7, we deduce that $|G||2^{\alpha_1} \cdot 3^{\alpha_2} \cdot p$, where for $i \in \{1, 2\}$, $\alpha_i > 0$. If $\pi(G) = \{2, 3, p\}$, then since $k = 6$, $p \leqslant 5$. But $p \neq 3$ and hence, $p = 5$, thus by Lemma 2.8, there is no element such as z in G with $o(z) = 6$ such that $[G : N_G(\langle z \rangle)] \in \{15, 20\}$. We claim that there exists z' in G such that $o(z') = 6$ and $5|[G : N_G(\langle z' \rangle)]$. If not, then since $|\mathrm{Aut}(\langle z' \rangle)| = 2$, it is concluded that $5||C_G(\langle z' \rangle)|$. So, G contains an element of order 30, which is contradiction with $k = 6$. Thus $5|[G : N_G(\langle z' \rangle)]$ and hence, Lemma 2.8 shows that $[G : N_G(\langle z' \rangle)] \in \{5, 10\}$. Since $[G : N_G(\langle z' \rangle)]|10$ and $[N_G(\langle z' \rangle) : C_G(\langle z' \rangle)]|2$, we deduce that $\mathrm{G}_3 \leqslant C_G(\langle z' \rangle)$. By our assumption, we can conclude $\exp(\mathrm{G}_3) = 3$ and hence, $|C_G(\langle z' \rangle)|_3 \leqslant 20$. So, we have $|\mathrm{G}_3| \in \{3, 9\}$. First let G be a solvable group and let H be a $\{3, 5\}$-Hall subgroup of G. Therefore, $n_3(H) = 3s + 1|5$ and hence, $s = 0$. So, $5||N_H(\mathrm{G}_3)|$ and hence, $5||N_G(\mathrm{G}_3)|$. But, $[N_G(\mathrm{G}_3) : C_G(\mathrm{G}_3)]||\mathrm{Aut}(\mathrm{G}_3)|$ and $\mathrm{Aut}(\mathrm{G}_3) \cong C_2$ or $GL_2(3)$. Therefore, $5||C_G(\mathrm{G}_3)|$ and hence, G contains an element of order 15, which is a contradiction with $k = 6$. Hence, G is a $\{2, 3\}$-group. Also, if G is a non-solvable group, then Theorem 2.21 shows that $G \cong \mathbb{S}_5$.

5) $k = 10$. In this case, Lemma 2.7 shows that $10||G|$ and $|G||2^{\alpha_1} \cdot 5^{\alpha_2} \cdot p^{\alpha_3}$, where for $i \in \{1,2\}$, $\alpha_i > 0$ and $\alpha_3 \in \{0,1\}$. If $p \neq 5$ and $\pi(G) = \{2,5,p\}$, then since $k = 10$, $p < 10$. Since $3 < p$, $p = 7$. Hence, Lemma 2.8 forces $[G : N_G(\langle z \rangle)] \leqslant 7$, where z is an element in G with $o(z) = 10$. We claim that 7 divides $[G : N_G(\langle z \rangle)]$. If not, then since $[N_G(\langle z \rangle) : C_G(\langle z \rangle)]|4$, (1) shows that $7||C_G(\langle z \rangle)|$, which is a contradiction with $k = 10$. Hence, $[G : N_G(\langle z \rangle)] = 7$, so (1) implies that $G_5 \leqslant C_G(\langle z \rangle)$. Thus $\exp(G_5) = 5$ and hence, $|C_G(\langle z \rangle)|_5 \leqslant 28$. Thus $|G_5| \in \{5, 25\}$. By virtue of Theorem 2.21, G is solvable. Let H be a $\{5,7\}$-Hall subgroup of G. Therefore, $n_5(H) = 5v + 1|7$ and hence, $v = 0$. So, $7||N_H(G_5)|$ and hence, $7||N_G(G_5)|$. But $[N_G(G_5) : C_G(G_5)]||\mathrm{Aut}(G_5)|$. Since $\mathrm{Aut}(G_5) \cong C_4$ or $GL_2(5)$, $7||C_G(G_5)|$, by (1). Hence, G contains an element of order 35, which is a contradiction with $k = 10$. Therefore, G is a $\{2,5\}$-group.

6) $k = 12$. Then applying Lemma 2.7 shows that $|G||2^{\alpha_1} \cdot 3^{\alpha_2} \cdot .p^{\alpha_3}$, where for $i \in \{1,2\}$, $\alpha_i > 0$ and $\alpha_3 \in \{0,1\}$. By our assumption, we have $p \neq 3$. If $\pi(G) = \{2,3,p\}$, then since $k = 12$, we deduce that $p \leqslant 11$. If $p = 5$, then repeating the argument given in the proof of Case (2-i) of Theorem 2.21 shows that $|G|_3 = 3$ and $n_3(G) \in \{1, 5, 10, 40, 160\}$. But Corollary 2.16($ii$) shows that $m_{12}(G) = n_3(G) \cdot \phi(3) \cdot t = 20$, where $t = m_4(C_G(G_3)) \geq 2$ and also, $n_3(G) = 3s+1 \neq 5$. Thus $n_3(G) \notin \{5, 10, 40, 160\}$ and hence, $n_3(G) = 1$. Also, $15 \notin \pi_e(G)$. So, the action of G_5 on G_3 is Frobenius and hence, $|G_5| = 5|3 - 1$, which is a contradiction. If $p = 7$, then repeating the argument given in the proof of Case (2-iv) of Theorem 2.21 shows that $|G|_3 = 3$ and $|G|_7 = 7$. Let N be a normal minimal subgroup of G. If $|N| = 7$, then since $14 \notin \pi_e(G)$, the action of G_2 on N is Frobenius and hence, $|G_2||7 - 1$. Thus $|G_2| = 2$, which is a contradiction. Also, since $21 \notin \pi_e(G)$ and $7 \nmid 3 - 1$, we can see that $3 \nmid |N|$. Thus $n_3(G) \neq 1$. But $28 = m_{12}(G) = 2 \cdot n_3(G) \cdot m_4(C_G(G_3))$. Therefore, $n_3(G) = 7$ and $m_4(C_G(G_3)) = 2$. Also, this allows us to assume that $G_2 \leqslant N_G(G_3)$. If $|N| = 2^t$, then since $14 \notin \pi_e(G)$, the action of G_7 on N is Frobenius and hence, G_2 is abelian and $7|2^t - 1$. Also, applying Lemmas 2.10 and 2.11 guarantee that $4 \leqslant |G_2(C_G(G_3))| \leqslant 8$. On the other hand, $|N_G(G_3)|/|C_G(G_3)|||\mathrm{Aut}(C_3)| = 2$ and $G_2 \leqslant N_G(G_3)$. So, $8 \leqslant |G_2| \leqslant 16$. Therefore, $t = 3$ and hence, we can see at once that N is a 2-elementary abelian group of order 8 and $C_G(N) = N$. Thus $|G_2| = 16$, because, $4 \in \pi_e(G)$. On the other hand, $12 \in \pi_e(G)$ and hence, we can see that $C_6 \lesssim N_G(N)/C_G(N) \cong GL_3(2)$, which is a contradiction, because $6 \notin \pi_e(GL_3(2))$. If $p = 11$, then Lemma 2.8 forces $[G : N_G(\langle z \rangle)] \leqslant 11$, where z is an element in G with $o(z) = 12$. We claim that $11|[G : N_G(\langle z \rangle)]$. If not, then since $[N_G(\langle z \rangle) : C_G(\langle z \rangle)]|4$, (1) shows that $11||C_G(\langle z \rangle)|$, which is a contradiction with $k = 12$. Hence, $[G : N_G(\langle z \rangle)] = 11$ and so, (1) implies that $G_3 \leqslant C_G(\langle z \rangle)$. Thus $\exp(G_3) = 3$ and hence, $|C_G(\langle z \rangle)|_3 \times 2 \leqslant 44$. Therefore, $|G_3| \in \{3, 9\}$. By virtue of Theorem 2.21, G is solvable. Let H be a $\{3, 11\}$-Hall subgroup of G. Therefore, $n_3(H) = 3v + 1|11$ and hence, $v = 0$. So, $11||N_H(G_3)|$ and hence, $11||N_G(G_3)|$. But $[N_G(G_3) : C_G(G_3)]||\mathrm{Aut}(G_3)|$ and $\mathrm{Aut}(G_3) \cong C_2$ or $GL_2(3)$, thus $11||C_G(G_3)|$, by (1). Hence, G contains an element of order 33, which is a contradiction with $k = 12$. Therefore, G is a $\{2,3\}$-group.

7) Let $2p + 1$ be a prime number and $k \in \{2(2p + 1), 4(2p + 1)\}$ or let $4p + 1$ be a prime number and $k \in \{4p+1, 2(4p+1)\}$. In the following, we examine the structure of G for every value of k:

(i). If $k = 4p+1$, then since $(4p+1)4p > 4p$, Lemma 2.5 implies that $n_{4p+1} = 1$ and $|\mathrm{G}_{4p+1}| = 4p+1$ and hence, G_{4p+1} is a cyclic normal subgroup of G. Since by Lemma 2.9, $|C_G(\mathrm{G}_{4p+1})| = 4p+1$, we have $G/C_G(\mathrm{G}_{4p+1}) \hookrightarrow \mathrm{Aut}(\mathrm{G}_{4p+1}) \cong C_{4p}$ and hence, $G \cong C_{4p+1} \rtimes C_l$, where $l|4p$.

(ii). If $k = 2(2p+1)$, then by virtue of Lemma 2.7, we deduce that $|G||2^{\alpha_1} \cdot p \cdot (2p+1)^{\alpha_2}$, where for $i \in \{1,2\}$, $\alpha_i > 0$. Since $(2p+1)2p > 4p$, Lemma 2.5 implies that $\mathrm{G}_{2p+1} \trianglelefteq G$ and $|\mathrm{G}_{2p+1}| = 2p+1$. Hence, G_{2p+1} is a cyclic subgroup of G. Thus Corollary 2.16(ii) shows that $m_{2(2p+1)}(G) = n_{2p+1}(G) \cdot 2p \cdot t$, where $t = m_2(C_G(\mathrm{G}_{2p+1}))$ and hence, $m_{2(2p+1)}(G) = 2p \cdot t = 4p$ which shows that $t = 2$. It is a contradiction with Corollary 2.16(i).

(iii). If $k = 4(2p+1)$ and x is an element G of order k, then by Lemmas 2.8 and 2.9, we can see that $C_G(x) = \langle x \rangle$ and $[G : N_G(\langle x \rangle)] = 1$. Thus $\langle x \rangle \trianglelefteq G$ and $G/\langle x \rangle \hookrightarrow \mathrm{Aut}(\langle x \rangle) \cong C_{2p} \times C_2$. Therefore, $G \cong C_{4(2p+1)} \rtimes (C_u \times C_l)$, where $u|2p$ and $l|2$.

(iv). If $k = 2(4p+1)$ and x is an element G of order k, then by Lemmas 2.8 and 2.9, we can see that $C_G(x) = \langle x \rangle$ and $[G : N_G(\langle x \rangle)] = 1$. Thus $\langle x \rangle \trianglelefteq G$ and $G/\langle x \rangle \hookrightarrow \mathrm{Aut}(\langle x \rangle) \cong C_{4p}$. Therefore, $G \cong C_{2(4p+1)} \rtimes C_l$, where $l|4p$.

8) Let $k = 25$ and let x be an element of order 25 in G. According to Lemma 2.2, in this case $p = 5$. Hence, Lemma 2.7 shows that $|G||2^2 \cdot 5^\alpha$, where $\alpha > 0$. It follows by Lemmas 2.8 and 2.9 that $C_G(x) = \langle x \rangle$ and $\langle x \rangle$ is a normal subgroup of G. Therefore, $G/\langle x \rangle \lesssim \mathrm{Aut}(C_{25}) \cong C_{20}$. If $5^3||G|$, then since $25 \in \pi_e(G)$ and $\mathrm{G}_5 \trianglelefteq G$, we deduce that $m_{25}(G) = m_{25}(\mathrm{G}_5)$. Since there is not any group of order 125 with the unique cyclic subgroup of order 25, we deduce that $|\mathrm{G}_5| = 25$. Thus $\frac{G}{C_G(x)} \lesssim C_4$ and hence, $G \cong C_{25} \rtimes C_l$, where $l|4$.

9) $k = 50$. Let $x \in G$ such that $o(x) = 50$. By virtue of Lemma 2.2, $p = 5$. Similar to the previous argument, we have $\langle x \rangle = C_G(x)$. Since $k = 50$, $5^2||G|$ and hence, we conclude that $5^2 \leqslant |G|_5$. We claim that $|G|_5 = 5^2$. If not, then $|\mathrm{G}_5| = 5^s$, where $s \geqslant 3$. Then it is evident that G_5 can not be a cyclic group and hence, Lemma 2.12 shows that $5^2|M(G) = 20$, which is impossible. So, we deduce that $|\mathrm{G}_5| = 5^2$ and hence, $\frac{G}{C_G(x)} \lesssim C_4$. Thus $G \cong C_{50} \rtimes C_l$, where $l|4$. □

In the following, as a consequent of the main theorem, we examine the structure of finite group G with $M(G) = 20$:

Corollary 3.1. *Let G be a finite group with $M(G) = 20$. Then G is one of the following groups:*

(1) *If $k = 4$, then G is a non-abelian 2-group of order 32;*
(2) *if $k = 6$, then either $G \cong \mathbb{S}_5$ or $G \cong C_6 \times \mathbb{S}_3$;*
(3) *if $k = 25$, then $G \cong C_{25} \rtimes C_l$, where $l|4$;*
(4) *if $k = 44$, then $G \cong C_{44} \rtimes (C_u \times C_l)$, where $u|10$ and $l|2$;*
(5) *if $k = 50$, then $G \cong C_{50} \rtimes C_l$, where $l|4$.*

Proof. In Lemma 2.1 and Lemma 2.2, the possible values for k, are recognized. On the other hand, according to Lemma 2.3, $k \neq 2, 5, 11$. Also, Theorem 2.21 implies that $k \neq 3$.

In the following, the other values of k are examined:

(1). Let $k = 4$. According to case (2) of the proof of the main theorem, G is a non-abelian 2-group with $|G| < 16 \cdot 5 = 80$. According to the classification of non-abelian groups of order 64, there is no group of order 64 with $\exp(G) = 4$ and $M(G) = 20$. So, G is a non-abelian 2-group of order 32.

(2). If $k = 6$ and G is a non-solvable group, then $G \cong \mathbb{S}_5$, by Theorem 2.21. In the following, we examine the structure of G, when G is a solvable group and $k = 6$. According to our main theorem, G is a $\{2, 3\}$-group. We have $|C_G(x)| = 2^u \cdot 3^v$, where $u, v \leqslant 2$ and $x \in G$ such that $o(x) = 6$. Since $M(G) = 20$, then Lemma 2.8 shows that $[G : N_G(\langle x\rangle)] \in \{1, 2, 3, 4, 6, 8, 9\}$. If there exists an element y of order 6 in G such that $[G : N_G(\langle y\rangle)] \in \{3, 6, 8, 9\}$, then our assumption, $M(G) = 20$, guarantees the existence of another element z of order 6 in G such that $[G : N_G(\langle z\rangle)] \in \{1, 2, 4\}$. In fact, without loss of generality, we can assume that G always has an element x such that $[G : N_G(\langle x\rangle)] \in \{1, 2, 4\}$. Also, $[N_G(\langle x\rangle) : C_G(\langle x\rangle)] | |\mathrm{Aut}(\langle x\rangle)| = 2$. So, (1) forces $|G| | 2^5 \cdot 3^2$.

Since $[G : N_G(\langle x\rangle)]|4$ and $[N_G(\langle x\rangle) : C_G(\langle x\rangle)]|2$, we deduce that $\mathrm{G}_3 \leqslant C_G(\langle x\rangle)$. Applying the third Sylow's theorem implies that $n_3(G) \in \{1, 4, 16\}$. In the following, we examine two possibilities for v:

(i). If $v = 1$, then $|G|_3 = 3$. Therefore, Corollary 2.16(ii) forces $m_6(G) = n_3(G) \cdot 2t$, where $t = m_2(C_G(\mathrm{G}_3))$. Thus $m_6(G) \in \{2t, 8t, 32t\}$. If $m_6(G) = 2t$, then $t = 10$, which is a contradiction with Corollary 2.16(i). If $m_6(G) \in \{8t, 32t\}$, then we get a contradiction with $M(G) = 20$.

(ii). If $v = 2$, then $|\mathrm{G}_3| = 9$. So, G_3 is a 3-elementary abelian group. Set $C := C_G(\langle x\rangle)$. If C is abelian, then we can see that $C = C_3 \times C_3 \times C_2 \times C_2$ and hence, $m_6(C) = 8 \cdot 3 = 24$, which is a contradiction with $(*)$. Thus C is not abelian and hence, $C \cong C_6 \times \mathbb{S}_3$, where $\mathbb{S}_3$ denotes the symmetric group of degree 3. Therefore, $m_6(C) = m_6(C_6) \cdot |\mathbb{S}_3| + m_3(C_6) \cdot m_2(\mathbb{S}_3) + m_2(C_6) \cdot m_3(\mathbb{S}_3) = 20$ and hence, C is normal in G. This forces $\langle x\rangle = Z(C)$ to be normal in G. Thus $[G : N_G(\langle x\rangle)] = 1$ and hence, (1) guarantees $|G||72$. If $|G| = 72$, then $\pi_e(G) \subseteq \{1, 2, 3, 4, 6\}$. Thus by Lemma 2.13, $9|m_2 + m_4 + m_6 = m_2 + m_4 + 20$ and $8|m_3 + m_6 = m_3 + 20$. So, there exist the natural numbers s, t such that $s, t \geqslant 3$, $m_2 + m_4 + 20 = 9t$ and $m_3 + 20 = 8s$. Therefore, $1 + m_2 + m_3 + m_4 + m_6 = 72$ forces $8s + 9t = 91$. Thus considering the different values of s and t shows that $s = 8$ and $t = 3$. So, $m_3 = 64 - 20 = 44$. But $n_3(G) = 3u + 1|8$ and hence $n_3(G) \leq 4$. This shows that $44 = m_3(G) \leqslant n_3(G).(|\mathrm{G}_3| - 1) \leqslant 4 \cdot 8 = 32$, which is a contradiction. Thus $|G| = 36$ and hence, $G = C \cong C_6 \times \mathbb{S}_3$.

(3). If $k = 10$, then Lemma 2.7 shows that $|G||2^{\alpha_1} \cdot 5^{\alpha_2}$, where for $i \in \{1, 2\}$, $\alpha_i > 0$. Let $x \in G$ such that $o(x) = 10$. Then $|C_G(\langle x\rangle)| = 2^u \cdot 5^v$. According to $(*)$, we can see that $u \leqslant 2$ and $v = 1$. Since $[G : N_G(\langle x\rangle)]||G|$, Lemma 2.8 shows that $[G : N_G(\langle x\rangle)] \in \{1, 2, 4, 5\}$. Note that if $[G : N_G(\langle x\rangle)] \in \{4, 5\}$, then there exists $y \in G$ such that $[G : N_G(\langle y\rangle)] \in \{1, 2\}$. So, without loss of generality, we can assume that $[G : N_G(\langle x\rangle)]|2$ and hence, (1) shows that $|G||2^5 \cdot 5^2$. Since $[N_G(\langle x\rangle) : C_G(\langle x\rangle)]|2$, we deduce that $\mathrm{G}_5 \leqslant C_G(\langle x\rangle)$. If $\mathrm{G}_5 \ntrianglelefteq G$, then $|G : N_G(\mathrm{G}_5)| \geqslant 6$. Thus Corollary 2.16$(ii)$ shows that $m_{10}(G) = n_5(G) \cdot \phi(10) \cdot t \geqslant 6 \cdot 4 \cdot t = 24t$, where $t = m_2(C_G(\mathrm{G}_5))$. Obviously, this is a contradiction with $(*)$. If $\mathrm{G}_5 \trianglelefteq G$, then Corollary 2.16$(ii)$ shows that $m_{10}(G) = 4t = 20$, where $t = m_2(C_G(\mathrm{G}_5)) = 5$. Since $C_G(\mathrm{G}_5) = \mathrm{G}_5 \times \mathrm{G}_2(C_G(\mathrm{G}_5))$, we deduce that $|\mathrm{G}_2(C_G(\mathrm{G}_5))| - 1 = t = 5$, which is impossible.

(4). If $k = 12$, then by applying the argument in Case (2), Subcase (i) of the proof of Theorem 2.21, we get a contradiction.

(5). If $k = 22$, then the main theorem leads us to get a contradiction and if $k = 44$, then the main theorem shows that $G \cong C_{44} \rtimes (C_u \times C_l)$, where $u|10$ and $l|2$.

(6). If $k \in \{25, 50\}$, then the main theorem completes the proof. □

References

[1] **A.A. Buturlakin**, *Spectra of finite linear and unitary groups*, Algebra and Logic **47(2)** (2008), 91 – 99.

[2] **G.Y. Chen and W. Shi**, *Finite groups with* 30 *elements of maximal order*, Appl. Categorical Structures **16(1)** (2008), 239 – 247.

[3] **Y. Cheng**, *Finite groups based on the numbers of elements of maximal order*, (Chinese), Chin. Ann. Math. **14** (1993), 561 – 567.

[4] **X.L. Du and Y.Y. Jiang**, *On finite groups with exact* 4p *elements of maximal order are solvable*, (Chinese), Chin. Ann. Math., **25A(5)** (2004), 607 – 612..

[5] **G. Frobenius**, *Verallgemeinerung des sylowschen satze*, Berliner sitz (1895), 981 – 993.

[6] **The GAP group**, *GAP- Groups, Algorithms, and Programming*, Version 4.3, Aachen, Andrews, 2002, (http://www.gap-system.org).

[7] **D. Gorenstein**, *Finite groups*, New York, 1968.

[8] **M. Herzog**, *On fnite simple groups of order divisible by three primes only*, J. Algebra **10** (1968), 383 – 388.

[9] **Y. Jiang**, *A theorem of finite groups with* 18p *elements having maximal order*, Algebra Coll., **15** (2008), 317 – 329.

[10] **Q. Jiang and C. Shao**, *Finite groups with* 24 *elements of maximal order*, Frontiers Math. in China **5** (2010), 665 – 678.

[11] **G.A. Miller**, *Addition to a theorem due to Frobenius*, Bull. Amer. Math. Soc. **11** (1904), 6 – 7.

[12] **H. Xu, G. Chen and Y. Yan**, *A new characterization of simple K_3-groups by their orders and large degrees of their irreducible character*, Commun. Algebra **42** (2014), 5374 – 5380.

[13] **L. Zhang and X. Liu**, *Characterization of the projective general linear groups $PGL_2(q)$ by their orders and patterns*, Intern. J. Algebra Computation **19** (2009), 873 – 889.

Received December 1, 2015

Department of Pure Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P.O. Box 115, Shahrekord, Iran
E-mails: asadian.bahare@stu.sku.ac.ir, ahanjideh.neda@sci.sku.ac.ir