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Some properties of a graph associated to a lattice

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Abstract. Some properties of the graph $\Gamma_S(L)$, where L is a lattice and S is a $\wedge$-closed subset of L, are obtained. Moreover, the graph structure of $\Gamma_S(L)$ under graph operations union, join, lexicographic product and tensor product are determined. The graph associated to quotient lattice is also studied.

1. Introduction

Making connection between various algebraic structures and graph theory by assigning graphs to an algebraic structure and investigating the properties of one from the another is an exciting research methods in the last decade. Barati et al. [2] associated a simple graph $\Gamma_S(R)$ to a multiplicatively closed subset S of a commutative ring R with all elements of R as vertices, and two distinct vertices x, y are adjacent if and only if $x + y \in S$. Afkhami et al. [1] introduced the same graph structure on a lattice. They considered a lattice L and defined a graph $\Gamma_S(L)$ with all elements of L as vertices and two distinct vertices $x, y \in L$ are adjacent if and only if $x \vee y \in S$ where S is a subset of L which is closed under $\wedge$ operation.

Throughout this paper L means a finite bounded lattice. Let x, y be two distinct elements of L, whenever $x < y$ and there is no element z in L such that $x < z < y$, we say that y *covers* x. In bounded lattice L an element $p \in L$ is said to be an *atom* if it covers 0, also an element $m \in L$ is a *coatom* of L if 1 covers it. We denote the set of all coatoms of L by $Coatom(L)$ and the set of atoms of L by $Atom(L)$. The set of all lower bounds of a subset A of L is denoted by A^ℓ and the set of all upper bounds of A is denoted by A^u i.e.,

$$A^\ell = \{x \in L : x \leqslant a \ \ for\ all\ a \in A\},$$

$$A^u = \{x \in L : a \leqslant x \ \ for\ all\ a \in A\},$$

$\{x\}^\ell$ and $\{x\}^u$ (or simply x^ℓ and x^u) are also denoted by $(x]$ and $[x)$ respectively.

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Let L and L' be lattices. A mapping $\theta : L \longrightarrow L'$ is called a *homomorphism* if for all $a, b \in L$, $\theta(a \vee b) = \theta(a) \vee \theta(b)$ and $\theta(a \wedge b) = \theta(a) \wedge \theta(b)$. If the map θ is also bijective, we call θ to be an *isomorphism*.

A mapping $\theta : L \longrightarrow L'$ is called an *anti-homomorphism* if $\theta(a \vee b) = \theta(a) \wedge \theta(b)$ and $\theta(a \wedge b) = \theta(a) \vee \theta(b)$ for all $a, b \in L$. A bijective anti-homomorphism is called an *anti-isomorphism*. An equivalence relation R on a lattice L is called a *congruence* if $a_1 R b_1$ and $a_2 R b_2$ imply $(a_1 \wedge a_2) R (b_1 \wedge b_2)$ and $(a_1 \vee a_2) R (b_1 \vee b_2)$. The set of all such relations is denoted by $Con(L)$ or L/R. It is well-known that the set of all congruence relations, under inclusion constitutes a complete lattice. The (ordinal) sum $P + Q$ of P and Q can be defined on the (disjoint) union $P \cup Q$ ordered as follows: for the elements $x, y \in P \cup Q$, define $x \leqslant y$ if one of the following conditions holds:

i) $x, y \in P$ and $x \leqslant_P y$,

ii) $x, y \in Q$ and $x \leqslant_Q y$,

iii) $x \in P$ and $y \in Q$.

For an order set P with unit 1_P, and an order set Q with zero, 0_Q, the *glued sum*, $P \dot{+} Q$, is obtained from $P + Q$ by identifying 1_P and 0_Q[5, p. 8]. We refer to [4, 5] for a complete description of these notions.

Let G be an undirected graph with the vertex set $V(G)$. The notation $ab \in E$ means that vertices a and b are adjacent in G. The degree of a vertex v is denoted by $deg(v)$ and the notations P_n, C_n, S_n and K_n are used for the path, cycle, star and complete graphs with n vertices, respectively. Recall that a subgraph H of a graph G is a graph whose the set of vertices and the set of edges are both subsets of G. A *vertex-induced subgraph* of graph G is one that consists of some of the vertices of G and all of the edges that connect them in G. An edge-induced subgraph of graph G consists of some of the edges of G and the vertices that are at their endpoints. The *complement* of G is a graph denoted by $\overline{G}$ with the same vertex set as G and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. The complement of the complete graph K_n is called the *null graph* on n vertices, see [3] for more details.

We now recall some graph operations [6]. Suppose G and H are graphs with disjoint vertex sets. The *disjoint union* $G + H$ is a graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. The *join* $G \oplus H$ defined as $\overline{\bar{G} + \bar{H}}$. The *tensor product* (or *direct product*) $G \times H$ of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ in such a way that vertices (g, h) and (g', h') are adjacent if and only if $gg' \in E(G)$ and $hh' \in E(H)$.

2. Main results

The aim of this section is to compute $\Gamma_S(L)$, for some special lattice L and a subset S of L. We start by an example:

Example 2.1. Let L be a chain with n elements and S be any nonempty subset of L. Then $deg_{\Gamma_S(L)}(x) = |x^l| + |S \cap x^u| - 2$, $x \in S$, and for any $x \in S^c$, $deg(x) = |x^u \cap S|$. In some special cases, we have:

- If $S = y^u$ for some $y \in L$, then $deg(x) = |L| - 1$, for all $x \in S$ and $deg(x) = |S|$, for every $x \in S^c$.
- If $S = y^l$ for some $y \in L$, then $deg(x) = |S| - 1$, for all $x \in S$ and $deg(x) = 0$, for every $x \in S^c$.

Proposition 2.2. *We have:*

(i) $\Gamma_S(L)$ *is a cycle if and only if* $|L| = 3$ *and* $\Gamma_S(L)$ *is complete. On the other word* $\Gamma_S(L) \neq C_n$ *for all subset* S *of* L*, unless* $n = 3$.

(ii) $\Gamma_S(L)$ *is a tree if and only if it is a star.*

Proof. (i). Since the cycle is two regular, if $\Gamma_S(L)$ is a cycle, then $deg(1) = deg(0) = 2$. Hence by [1, Lemma 2.2], $1 \in S$ and $deg(1) = |L| - 1 = 2$ i.e., $|L| = 3$. On the other hand, if $0 \in S$, then $deg(0) = |S| - 1 = 2$ and $|S| = 3$ i.e., $S = L$, and if $0 \notin S$, then $deg(0) = |S| = 2$ i.e., $S = L \setminus \{0\}$ [1, Lemma 2.2]. So, $\Gamma_S(L)$ is a complete graph [1, Proposition 2.4].

(ii). If $\Gamma_S(L)$ is a tree, then it is connected, so, $1 \in S$ [1, Theorem 2.3]. Thus $deg(1) = |L| - 1$ [1, Lemma 2.2]. Since $\Gamma_S(L)$ is a tree, it has no other edge, so, $|Coatom(L)| = 1$ and by [1, Lemma 2.2], $S = \{1\}$ or $S = \{0, 1\}$. The result follows from [1, Theorem 2.5]. □

Lemma 2.3. *Let* L *be a bounded lattice. Then*

(1) $\Gamma_S(L)$ *is null graph if and only if* $S = \{0\}$ *or* $S = \emptyset$.

(2) $\Gamma_S(L) = P_2 + \overline{K}_{|L|-2}$ *if and only if* $S = \{p\}$ *or* $S = \{0, p\}$ *that* $p \in Atom(L)$*, in fact in this case,* $deg(p) = deg(0) = 1$ *and* $deg(x) = 0$*, for every* $x \neq 0, p$.

(3) $\Gamma_S(L) = P_3 + \overline{K}_{|L|-3}$ *if and only if* $S = \{p_1, p_2\}$ *or* $S = \{0, p_1, p_2\}$ *for some* $p_1, p_2 \in Atom(L)$*, in this case,* $deg(p_1) = deg(p_2) = 1$*,* $deg(0) = 2$ *and for every* $x \neq 0, p_1, p_2$*,* $deg(x) = 0$.

(4) $\Gamma_S(L) = C_3 + \overline{K}_{|L|-3}$ *if and only if* $S = \{0, p, x\}$ *such that* $x^\ell = \{0, p\}$ *and* $p \in Atom(L)$.

(5) $\Gamma_S(L) = S_\alpha + \overline{K}_{|L|-\alpha}$ *(where* $\alpha = |S| - 1$ *or* $\alpha = |S^l|$*) if and only if* $S \subseteq \{0\} \cup AtomL$ *or* $S = \{x\}$*, for some nonzero element of lattice* L.

Proof. The proof is straightforward and so it is omitted. □

Remark 2.4. Suppose that S is a $\wedge$-closed subset of a lattice L and $a, b, x \in L$, we know that $a \vee (a \vee b) = b \vee (a \vee b) = (a \wedge b) \vee (a \vee b) = (a \wedge x) \vee (a \vee b) = (b \wedge x) \vee (a \vee b) = a \vee b$. So if in a graph $\Gamma_S(L)$ a, b are adjacent i.e., $a \vee b \in S$, then $a \vee (a \vee b) \in S$, $b \vee (a \vee b) \in S$, $(a \wedge b) \vee (a \vee b) \in S$ and $(a \wedge x) \vee (a \vee b) \in S$. Hence, summarizing, we have:

If $n \geqslant 3$, then $\Gamma_S(L) \neq P_n + \overline{K}_{|L|-(n)}$ for all $\wedge$-closed subsets S of L; and if $n \geqslant 4$, then $\Gamma_S(L) \neq C_n + \overline{K}_{|L|-n}$ for all subset S of L.

Remark 2.5. If S is a sublattice of L, then the subgraph $\Gamma_S(L)$ on S is complete. Since for all $a, b \in S$ we have $a \vee b \in S$, every two elements of subset S in $\Gamma_S(L)$ are adjacent.

Remark 2.6. It is easy to show that $\Gamma_{S'}(L)$ is a subgraph of $\Gamma_S(L)$, when S, S' are two $\wedge$-closed subsets of L and $S' \subseteq S$. But in general $\Gamma_{S'}(L)$ is neither edge-induced nor vertex-induced subgraph of $\Gamma_S(L)$. For example, let L be the modular lattice M_3 containing $0, 1$ and three incomparable elements a, b, c. Define $S = \{0, b, c, 1\}$ and $S' = \{0, b\}$. Then it is clear to see that $\Gamma_{S'}(L)$ is not edge-induced and vertex-induced subgraph of $\Gamma_S(L)$.

Theorem 2.7. *A $\wedge$-closed subset S of L is an ideal if and only if*

$$\Gamma_S(L) = K_{|S|} + \overline{K}_{|S^c|}.$$

Proof. Suppose $\Gamma_S(L) = K_{|S|} + \overline{K}_{|S^c|}$. Then by definition of $\Gamma_S(L)$, we have $a \vee b \in S$ if and only if $a, b \in S$ which implies that S is an ideal.

Conversely, if S is an ideal of L. Then, S is closed under taking join of elements, consequently all vertices of S are adjacent in graph $\Gamma_S(L)$. Moreover, since S is a lower set, for all $a, b \in S^c$, $a \vee b \notin S$. In fact, if in contrary $a \vee b \in S$ then $a \wedge (a \vee b) = a \in S$ which is a contraction. So, all vertices of S^c aren't adjacent in $\Gamma_S(L)$. Moreover, since S is a lower set, it follows that all $a \in S$ and $b \in S^c$ aren't adjacent in $\Gamma_S(L)$. Therefore, $\Gamma_S(L) = K_{|S|} + \overline{K}_{|S^c|}$. □

Clearly we have:

Lemma 2.8. *Let $\alpha : L \longrightarrow L'$ be a lattice isomorphism and S be a $\wedge$-closed subset of L. Then*

$$\Gamma_S(L) \cong \Gamma_{\alpha(S)}(L').$$

Theorem 2.9. *A $\wedge$-closed subset S of L is a prime filter if and only if*

$$\Gamma_S(L) = K_{|S|} \oplus \overline{K}_{|S^c|}.$$

Proof. Assume that S is a prime filter. Then for any $x, y \in S$, we have $x \vee y \in S$, i.e., $xy \in E(\Gamma_S(L))$. Since S is an upper subset of L, $x \vee y \in S$ for each $x \in S$ and $y \in S^c$. This means that x and y are adjacent. In addition, since S is a prime filter, S^c is an ideal. Hence for any $x, y \in S^c$, $x \vee y \in S^c$ and so $x \vee y \notin S$. This implies that x, y aren't adjacent in $\Gamma_S(L)$. On the other hand, if $\Gamma_S(L) = K_{|S|} \oplus \overline{K}_{|S^c|}$, then obviously for any $x \in S$ and $y \in L$, $x \vee y \in S$ and if $x \vee y \in S$, then $x \in S$ or $y \in S$. This completes the proof. □

A *semiregular graph* is a graph in which the set of degree of vertices includes only two elements. The following corollary immediately follows from Theorem 2.9.

Corollary 2.10. *If S is a prime filter of L, then $\Gamma_S(L)$ is a semiregular graph.*

Proof. Suppose S is a prime filter of L. Then by Theorem 2.9, we conclude that $deg(x) = |L| - 1$, for all $x \in S$ and $deg(y) = |S|$, for all $y \in S^c$, and the proof is completed. □

Proposition 2.11. *Assume that $\alpha : L \longrightarrow L'$ is a lattice isomorphism and S is a prime ideal or a filter of L, then*

$$\overline{\Gamma_S(L)} \cong \Gamma_{\alpha(S)^c}(L').$$

Proof. It is easy to show that if S is a prime ideal or a filter of L, then $\alpha(S)^c$ is a $\wedge$-closed subset of L'. The details are left to the readers. □

Corollary 2.12. *If S is a filter or a prime ideal of L, then $\overline{\Gamma_S(L)} = \Gamma_{S^c}(L)$.*

Proof. The proof by Proposition 2.11 and $\alpha = IdL$ (the identity map) is done. □

The *disjunction graph* $G \vee H$ of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ in such a way that vertices (g, h) and (g', h') are adjacent if and only if $gg' \in E(G)$ or $hh' \in E(H)$.

Theorem 2.13. *Let L, L' be two lattices and $L \times L'$ be its direct product. If S and T are $\wedge$-closed subset of L, L', respectively, Then*

(1) $\Gamma_{S\times T}(L \times L') = \Gamma_S(L) \times \Gamma_T(L')$,

(2) $\Gamma_S(L) + \Gamma_T(L') = \Gamma_{S\cup T}(L + L')$,

(3) *Let $S_0 = S \times L'$ and $T_0 = L \times T$. If S or T is a lower set, then we have* $\Gamma_{S_0\cup T_0}(L \times L') = \Gamma_S(L) \vee \Gamma_T(L')$.

Proof. (1). At first, we notice that $S\times T$ is a $\wedge$-closed subset of $L\times L'$. Two distinct vertices (a, b) and (c, d) of $\Gamma_{S\times T}(L \times L')$ are adjacent if and only if $(a, b) \vee (c, d) = (a \vee c, b \vee d) \in S \times T$, which is equivalent to $a \vee c \in S$ and $b \vee d \in T$. This means that a, c are adjacent in $\Gamma_S(L)$ and b, d are adjacent in $\Gamma_T(L')$. Therefore, (a, b) and (c, d) are adjacent in $\Gamma_S(L) \times \Gamma_T(L')$.

(2). If a, b are adjacent in $\Gamma_{S\cup T}(L + L')$, then $a \vee b \in S \cup T$. So, $a \vee b \in S$ or $a \vee b \in T$, i.e., a, b are adjacent in $\Gamma_S(L)$ or a, b are adjacent in $\Gamma_T(L')$ which implies that a, b are adjacent in $\Gamma_S(L) + \Gamma_T(L')$. On the other hand, if a, b are adjacent in $\Gamma_S(L) + \Gamma_T(L')$, then a, b are adjacent in $\Gamma_S(L)$ or a, b are adjacent in $\Gamma_T(L')$. So, $a \vee b \in S$ or $a \vee b \in T$, i.e., $a \vee b \in S \cup T$. Hence a, b are adjacent in $\Gamma_{S\cup T}(L + L')$,

(3). Since S or T is a lower set, $S_0 \cup T_0$ is a $\wedge$-closed subset of $L \times L'$. Two distinct vertices (a, b) and (c, d) are adjacent in $\Gamma_{S_0\cup T_0}(L \times L')$ if and only if $(a \vee c, b \vee d) \in (S \times L') \cup (L \times T)$ if and only if $a \vee c \in S$ or $b \vee d \in T$ and this means that a, c are adjacent in $\Gamma_S(L)$ or b, d are adjacent in $\Gamma_T(L')$. The later is equivalent to (a, b) and (c, d) are adjacent in $\Gamma_S(L) \vee \Gamma_T(L')$. □

Recall that the *lexicographic product* of two graph G and H, denoted by $G[H]$, is defined as $V(G[H]) = V(G) \times V(H)$ where two vertices $(a,b),(c,d)$ of $G[H]$ are adjacent whenever $ac \in E(G)$, or $a = c$ and $bd \in E(H)$ [6, p. 43].

If P and Q are two partially ordered sets, then $P\times Q$, by ordering $(a,b) \leqslant (c,d)$ if $a <_P c$, or $a = c$ and $b \leqslant_Q d$ will be a partially ordered set again. We use the notation $P \oslash Q$ to denote $(P \times Q, \leqslant)$. Notice that if P and Q are totally ordered sets, then $P \oslash Q$ is a totally ordered set too. One can check at once that if L and L' are two lattices and L' is bounded, then $L \oslash L'$ is a lattice [5, p. 260] with join and meet operations as follows:

$$(a,b) \wedge (c,d) = \begin{cases} (a, b \wedge d) & \text{if } a = c, \\ (a,b)(\text{or } (c,d)) & \text{if } a < c \text{ (or } c < a), \\ (a \wedge c, 1) & \text{if } a \parallel c, \end{cases}$$

$$(a,b) \vee (c,d) = \begin{cases} (a, b \vee d) & \text{if } a = c, \\ (c,d)(\text{or } (a,b)) & \text{if } a < c \text{ (or } c < a), \\ (a \vee c, 0) & \text{if } a \parallel c. \end{cases}$$

Theorem 2.14. *Let L, L' be two totally ordered lattices and L' be bounded. If S and T are subsets of L, L', respectively, then*

$$\Gamma_{S\times T}(L \oslash L') = \Gamma_S(L)[\Gamma_T(L')].$$

Proof. Since L, L' are totally ordered, $L \oslash L'$ is totally ordered and so $S \times T$ is a $\wedge$-closed subset of $L \oslash L'$ and $\Gamma_{S\times T}(L \oslash L')$ is well defined. We now assume that (a,b) and (c,d) are two distinct vertices of $\Gamma_{S\times T}(L \oslash L')$. These two vertices are adjacent if and only if $(a,b) \vee (c,d) \in S \oslash T$ if and only if $(a,b) \in S \times T$ or $(c,d) \in S \times T$ if and only if $(a > c \textit{ or } a = c, b > d)$ or $(a < c \textit{ or } a = c, b < d)$, equivalently $a \vee c \in S$ or $(a = c, b \vee d \in T)$. This is equivalent to $ac \in E(\Gamma_S(L))$ or $a = c$, $bd \in E(\Gamma_T(L'))$. So, (a,b) and (c,d) are adjacent in $\Gamma_S(L)[\Gamma_T(L')]$. □

Proposition 2.15. *Let L and L' be lattices and L' be bounded. Suppose that T is a $\wedge$-closed subset of L' and S is a lower set of L. We also assume that $S_0 = S \times L'$ and $T_0 = L \times T$, then $\Gamma_S(L)[\Gamma_T(L')]$ is a subgraph of $\Gamma_{S_0\cup T_0}(L \oslash L')$.*

Proof. At first, since T is a $\wedge$-closed subset of L' and S is a lower set of L, $S_0 \cup T_0$ is a $\wedge$-closed subset of $L\oslash L'$, so $\Gamma_{S_0\cup T_0}(L\oslash L')$ can be defined. On the other hand, $V(\Gamma_S(L)[\Gamma_T(L')]) = V(\Gamma_{S_0\cup T_0}(L\oslash L')) = L\times L'$. Also, if two distinct vertices (a,b) and (c,d) are adjacent in $\Gamma_S(L)[\Gamma_T(L')]$, by definition of lexicographic product of graphs, one of the following two cases are occurred:

1. a and c are adjacent in graph $\Gamma_S(L)$,
2. $a = c$ and b and d are adjacent in graph $\Gamma_T(L')$.

Thus we have $a\vee c \in S$ or ($a = c$ and $b\vee d \in T$). Hence, according to join operation in a lattice $L \oslash L'$, we conclude that $(a,b) \vee (c,d) \in S_0 \cup T_0$, so two vertices (a,b) and (c,d) are adjacent in $\Gamma_{S_0\cup T_0}(L \oslash L')$. This completes the proof. □

Corollary 2.16. *Let L, L' be two totally ordered lattices and L' be bounded. If S and T are ($\wedge$-closed) subsets of L, L', respectively, then $\Gamma_{S\times T}(L\oslash L')$ is a subgraph of $\Gamma_{S_0\cup T_0}(L\oslash L')$.*

The *Cartesian product* of two graph G and H is a graph, denoted by $G\Box H$, whose vertex set is $V(G)\times V(H)$ and two vertices (a,b) and (c,d) are adjacent if $a=c$ and $bd\in E(G)$, or $ac\in E(H)$ and $b=d$ [6, p. 35].

Proposition 2.17. *Let L and L' be lattices and T, S are $\wedge$-closed subsets of L, L', respectively. We also assume that $S_0=S\times L'$ and $T_0=L\times T$. Then $\Gamma_S(L)\Box\Gamma_T(L')$ is a subgraph of $\Gamma_{S_0\cup T_0}(L\times L')$.*

Proof. Assume that (a,b) and (c,d) are two distinct vertices of $\Gamma_S(L)\Box\Gamma_T(L')$. These two vertices are adjacent if and only if $(a=c, bd\in E(\Gamma_T(L')))$ or $(ac\in E(\Gamma_S(L)), b=d)$, if and only if $(a=c, b\vee d\in T)$ or $(a\vee c\in S, b=d)$, equivalently $(a,b)\vee(c,d)=(a\vee c, b\vee d)\in S_0\cup T_0$. So, (a,b) and (c,d) are adjacent in $\Gamma_{S_0\cup T_0}(L\times L')$. □

The *strong product* of two graph G and H is the graph denoted as $G\boxtimes H$, whose vertex set is $V(G)\times V(H)$ and $E(G\boxtimes H)=E(G\Box H)\cup E(G\times H)$ [6, p. 36].

Corollary 2.18. *Let L and L' be lattices and T, S are $\wedge$-closed subsets of L, L' respectively. We also assume that $S_0=S\times L'$ and $T_0=L\times T$. Then $\Gamma_S(L)\boxtimes\Gamma_T(L')$ is a subgraph of $\Gamma_{S\times T}(L\times L')\cup\Gamma_{S_0\cup T_0}(L\times L')$.*

Proof. The result follows from definition of $G\boxtimes H$, part (1) of Theorem 2.13 and previous preposition. □

Suppose Π is a partition of the vertices of a graph G. The *quotient graph* G/Π is a graph with vertex set Π, and for which distinct classes $C_1, C_2\in\Pi$ are adjacent if some vertex in C_1 is adjacent to a vertex of C_2 [6, p. 159]. In the following, we let $\varphi: L\longrightarrow K$ be an onto lattice homomorphism and α be the congruence relation of L defined by $x\equiv_\alpha y$ if and only if $\varphi(x)=\varphi(y)$. Therefore, $L/\alpha\cong K$. In other words, a homomorphic image of L is isomorphic to some quotient lattice of L. Obviously, if S is a $\wedge$-closed subset of L, then S_1, the set of all equivalence classes of α on S, is a $\wedge$-closed subset of L/α. So, we can define graph $\Gamma_{S_1}(L/\alpha)$. We have the following description for the graph associated to L/α.

Theorem 2.19. *Suppose that $\varphi: L\longrightarrow K$ is an onto lattice homomorphism and α is corresponding congruence relation with it. If S is an ideal of L and S_1 is the set of all equivalence classes of α on S, then*

$$\Gamma_{S_1}(L/\alpha)=\Gamma_S(L)/\alpha.$$

Proof. Consider $\alpha=\{[x]_\alpha : x\in L\}$ to be a partition for the vertex set of $\Gamma_S(L)$. So, the vertices of $\Gamma_S(L)/\alpha$ and $\Gamma_{S_1}(L/\alpha)$ are equal. On the other hand, according to definition of a quotient graph, if two distinct vertices $[x]$ and $[y]$ are adjacent in $\Gamma_S(L)/\alpha$, there exists $a \in [x]$ and $b \in [y]$ which are adjacent in $\Gamma_S(L)$ i.e. $a \vee b \in S$. So, $[a] \vee [b] = [a \vee b] \in S_1$. Thus $[a], [b]$ are adjacent in $\Gamma_{S_1}(L/\alpha)$, which is equivalent to $[x]$ and $[y]$ are adjacent in $\Gamma_{S_1}(L/\alpha)$.

Moreover, if $[x]$ and $[y]$ are adjacent in $\Gamma_{S_1}(L/\alpha)$, then $[x \vee y] = [x] \vee [y] \in S_1$. So, there exists a $s \in S$ such that $x \vee y \equiv_\alpha s$. According to the properties of congruence relations we have:

$$x = x \wedge (x \vee y) \equiv_\alpha x \wedge s, \qquad y = y \wedge (x \vee y) \equiv_\alpha y \wedge s.$$

So, $s \wedge x \in [x]$ and $s \wedge y \in [y]$. Since S is an ideal, $s \wedge x, s \wedge y \in S$ and $(s \wedge x) \vee (s \wedge y) \in S$. Thus $s \wedge x$ and $s \wedge y$ are adjacent in $\Gamma_S(L)$. This follows that $[x]$ and $[y]$ are adjacent in $\Gamma_S(L)/\alpha$ and the proof is complete. □

Corollary 2.20. *Suppose that $\varphi : L \longrightarrow K$ is an onto lattice anti-homomorphism and α is corresponding congruence relation with it. If S is a filter of L and S_1 is the set of all equivalence classes of α on S and $(L', \vee', \wedge')$ is dual of a lattice L, then*

$$\Gamma_{S_1}(L/\alpha) = \Gamma_S(L')/\alpha.$$

Proof. At first the vertex set of $\Gamma_S(L')/\alpha$ and $\Gamma_{S_1}(L/\alpha)$ are equal. On the other hand, if two distinct vertices $[x]$ and $[y]$ are adjacent in $\Gamma_S(L')/\alpha$, there exists $a \in [x]$ and $b \in [y]$ which are adjacent in $\Gamma_S(L')$ i.e. $a \vee' b \in S$. So, by definition of S_1, $[a \vee' b] \in S_1$ i.e., $[x] \vee [y] = [a] \vee [b] = [a \wedge b] = [a \vee' b] \in S_1$, so $[x]$ and $[y]$ are adjacent in $\Gamma_{S_1}(L/\alpha)$. Moreover, if $[x]$ and $[y]$ are adjacent in $\Gamma_{S_1}(L/\alpha)$, then $[x \wedge y] = [x] \vee [y] \in S_1$. So, there exist some $s \in S$ such that $x \wedge y \equiv_\alpha s$. By the properties of congruence relations, we have:

$$x = x \vee (x \wedge y) \equiv_\alpha x \vee s, \qquad y = y \vee (x \wedge y) \equiv_\alpha y \vee s.$$

So, $s \vee x \in [x]$ and $s \vee y \in [y]$. Since S is a filter, $s \vee x, s \vee y \in S$ and $(s \vee x) \wedge (s \vee y) \in S$. Thus $(s \vee x) \vee' (s \vee y) \in S$, i.e., $s \vee x$ and $s \vee y$ are adjacent in $\Gamma_S(L')$. This follows that $[x]$ and $[y]$ are adjacent in $\Gamma_S(L')/\alpha$. □

From now on L is a distributive lattice and S is a filter of L. We state here an important result of Stone [5, Theorem 115] as follows:

Theorem 2.21. *Let L be a distributive lattice, let I be an ideal, let D be a filter of L, and let $I \cap D = \emptyset$. Then there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \emptyset$.*

For a filter S of L and arbitrary element $x \in S^c$, by Stone theorem, there exists a prime ideal P_x such that $P_x \cap S = \emptyset$ and $(x] \subseteq P_x$. This means that S^c is a union of some prime ideals. Hence $S^c = \bigcup_{x \in S^c} P_x$. Set $I = \bigcap_{x \in S^c} P_x$ and define a congruence relation θ_0 on L as follows;

$$\theta_0 = \bigwedge \{\theta \in Con(L) \ : \ I^2 \subseteq \theta\}.$$

We consider $\tilde{S} = \{[x]_{\theta_0} \ : \ x \in S\}$.

Example 2.22. Suppose $L = \{0, x_1, x_2, x_3, x_4, x_5, \ldots\}$. Define an order $\leqslant$ on L as follows: for each $i \geqslant 1$, $0 \leqslant x_i$. Moreover, $x_1, x_2 \leqslant x_3$ and for $i, j \geq 3$ that $i \leqslant j$, $x_i \leqslant x_j$. Define $S = [x_5)$. So, $I = (x_3]$ and by definition of θ_0, we have $L/\theta_0 = \{I\} \cup \{\{x\} : x \notin I\}$.

Lemma 2.23. *$\tilde{S} = \{[x]_{\theta_0} : x \in S\}$ is a filter of L/θ_0.*

Proof. S is a $\wedge$-closed subset of L and in L/θ_0, we have $[a \wedge b] = [a] \wedge [b]$. So, $\tilde{S}$ is a $\wedge$-closed subset of L/θ_0. It is now enough to show that if $[a] \wedge [b] \in \tilde{S}$ then $[a], [b] \in \tilde{S}$. To do this, suppose that $[a \wedge b] = [a] \wedge [b] \in \tilde{S}$. Hence, there exist some element $s \in S$ such that $a \wedge b \equiv_{\theta_0} s$. According to properties of congruence relations, we have $a = a \vee (a \wedge b) \equiv_{\theta_0} a \vee s$, $b = b \vee (a \wedge b) \equiv_{\theta_0} b \vee s$. This means that $[a] = [a \vee s], [b] = [b \vee s]$. Since S is a filter of L, $b \vee s, a \vee s \in S$ which implies that $[a], [b] \in \tilde{S}$. □

Theorem 2.24. *$\Gamma_{\tilde{S}}(L/\theta_0)$ is connected.*

Proof. By [1, Theorem 2.3] the graph $\Gamma_S(L)$ is connected if and only if $1 \in S$. Now the result follows from Lemma 2.23. □

Theorem 2.25. *If $\Gamma_S(L)$ is complete, then $\Gamma_{\tilde{S}}(L/\theta_0)$ is complete.*

Proof. Suppose that $\Gamma_S(L)$ is complete. Thus $S = L$ or $S = L \setminus \{0\}$ [1, Theorem 4.2] and we have the following two cases:

- If $S = L$, then $I = \emptyset$, so $\theta_0 = \bigwedge\{\theta : I^2 \subseteq \theta\} = L \times L$. Thus $\tilde{S} = \{[x]_{\theta_0} : x \in S\}$ $= L/\theta_0$ and therefore $\Gamma_{\tilde{S}}(L/\theta_0)$ is complete.
- If $S = L \setminus \{0\}$ then $I = \{0\}$. So, $\theta_0 = \bigwedge\{\theta : I^2 \subseteq \theta\} = \triangle$. Hence, $\tilde{S} = (L/\theta_0) \setminus \{[0]\}$ and so $\Gamma_{\tilde{S}}(L/\theta_0)$ is complete. □

Notice that the converse of previous theorem is not true in general. Suppose $L = \{0, x_1, x_2, x_3, x_4, x_5, x_6, \ldots\}$. Define an order $\leqslant$ on L as follows: for each $i \geqslant 1$, $0 \leqslant x_i$. Moreover, $x_1 \leqslant x_2, x_3$ and $x_2, x_3 \leqslant x_4$ and for $i, j \geqslant 4$ that $i \leqslant j$, $x_i \leqslant x_j$. Define $S = [x_5)$, so $I = (x_4]$ and $\theta_0 = \{I\} \cup \{\{x\} : x \notin I\}$. Therefore, $\{I\}$ is zero element of a lattice L/θ_0 and so $\tilde{S} = (L/\theta_0) \setminus \{[0]\}$. By [1, Proposition 2.4] the graph $\Gamma_{\tilde{S}}(L/\theta_0)$ is complete. But by Theorem 2.9, $\Gamma_S(L)$ is not complete.

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