Quasigroups and Related Systems **23** (2015), 325 − 332

Relations between n-ary and binary comodules

Biljana Zekovich

Abstract. We construct a binary algebra $R = C^{\otimes(n-1)}/I$ for an n-ary algebra C and prove that M is an n-ary left C-module if and only if M is a binary left R-module. In the dual case, for an n-ary coalgebra C, we construct a binary coalgebra:

$$C^{\square(n-1)} = \bigcap_{j=1}^{n-2} \operatorname{Ker}\left[\Delta \otimes 1_C^{\otimes(n-2)} - 1_C^{\otimes j} \otimes \Delta \otimes 1_C^{\otimes(n-2-j)}\right] \subset C^{\otimes(n-1)}$$

and prove that M is an n-ary right C-comodule if and only if M is a binary right $C^{\square(n-1)}$-comodule. In the end, we prove that for n-ary finite generated coalgebra C over a field k, $C^{\square(n-1)}$ is the binary coalgebra, on the other hand, C^* is an n-ary algebra, for which, we construct the binary algebra $R = (C^*)^{\otimes(n-1)}/I$. If C is a finite-dimensional n-ary coalgebra over a field k, then C^* is a n-ary algebra and $(\mathrm{C}^{\square(n-1)})^* \cong (C^*)^{\otimes(n-1)}/I$. Dually, if C is an n-ary finite generated algebra over a field k, then $R = C^{\otimes(n-1)}/I$ is a binary algebra and C^* is an n-ary coalgebra. Moreover, $(C^*)^{\square(n-1)} \cong \left(C^{\otimes(n-1)}/I\right)^*$.

1. Introduction

Let k be a ground commutative associative ring with a unit, C and M modules over k. In what follows, $\otimes$ is a tensor product over k. All homomorphisms are k-linear maps. In [3], the concept of *n-ary algebra* (C, m) is defined, where

$$m : C \otimes \cdots \otimes C \to C$$

is n-ary multiplication, which is associative. It means that the following diagram is commutative:

$$\begin{array}{ccc} C^{\otimes(2n-1)} & \xrightarrow{\ m\otimes 1_C^{\otimes(n-1)}\ } & C^{\otimes n} \\ {\scriptstyle 1_C^{\otimes i}\otimes m\otimes 1_C^{\otimes(n-i-1)}}\downarrow & & \downarrow{\scriptstyle m} \\ C^{\otimes n} & \xrightarrow{\ \ m\ \ } & C \end{array}$$

i.e.,

$$m \circ (m \otimes 1_C^{\otimes(n-1)}) = m \circ (1_C^{\otimes i} \otimes m \otimes 1_C^{\otimes(n-i-1)}).$$

2010 Mathematics Subject Classification: 20N15
Keywords: n-ary (co)algebra, n-ary (co)module, isomorphism of binary (co)algebras

The concept of *n-ary coalgebra* (C, Δ) is defined in [4], where

$$\Delta : C \to C \otimes \cdots \otimes C$$

is n-ary comultiplication, which is coassociative, that is the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{\ \Delta\ } & C^{\otimes n} \\ {\scriptstyle \Delta}\downarrow & & \downarrow {\scriptstyle 1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-1)}} \\ C^{\otimes n} & \xrightarrow{\ \Delta \otimes 1_C^{\otimes(n-1)}\ } & C^{\otimes(n-1)} \end{array}$$

i.e.,

$$(\Delta \otimes 1_C^{\otimes(n-1)}) \circ \Delta = (1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-1)}) \circ \Delta.$$

Similary, the concept of *n-ary bialgebra* (C, m, Δ) is introduced, where m is an associative n-ary multiplication and Δ is a coassociative n-ary comultiplication and Δ is a homomorphism of n-ary algebras. An example of n-ary algebra is given in [6]. We do not suppose the existence of an unit and a counit.

In the paper [3], the notion of homomorphism of n-ary algebras

$$(C, m_C) \to (C', m_{C'})$$

is defined as a morphism $f : C \to C'$, such that the following diagram is commutative

$$\begin{array}{ccc} C^{\otimes n} & \xrightarrow{\ f^{\otimes n}\ } & (C')^{\otimes n} \\ {\scriptstyle m_C}\downarrow & & \downarrow {\scriptstyle m_{C'}} \\ C & \xrightarrow{\ f\ } & C' \end{array}$$

i.e.,

$$f \circ m_C = m_{C'} \circ f^{\otimes n}.$$

Let C be an n-ary coalgebra and a finitely generated projective k-module. Denote by C^* the k-module $\mathrm{Hom}(C, k)$. Then C^* is an n-ary algebra with multiplication $l_1 * \cdots * l_n$, where for $c \in C$

$$(l_1 * \cdots * l_n)(c) = \sum_{(c)} l_1(c_{(1)}) \cdots l_n(c_{(n)}) \tag{1}$$

if

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(n)} \in C^{\otimes n}.$$

Conversely, let C be an n-ary algebra and a finitely generated projective k-module. Define an n-ary comultiplication in $C^* = \mathrm{Hom}(C, k)$ by the rule:

$$(\Delta l)(x_1 \otimes \cdots \otimes x_n) = l(x_1 \cdots x_n) \tag{2}$$

where $x_1, \ldots, x_n \in C$. Hence we use the isomorphism of k-modules:

$$(C \otimes \cdots \otimes C)^* = C^* \otimes \cdots \otimes C^*$$

(cf. [2]), because C is a finitely generated projective k-module. Then, C^* is an n-ary coalgebra. If C is an n-ary (co)algebra, then $(C^*)^* \cong C$ (cf. [3] and [4]).

In [5] are defined the concepts of a right (left) n-ary (co)modules in the following way: k-module M is called a *right n-ary C-comodule*, where C is an n-ary coalgebra, if there is a map $\rho : M \to M \otimes C^{\otimes(n-1)}$, such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\quad\rho\quad} & M \otimes C^{\otimes(n-1)} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle 1_M \otimes 1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-2)}} \\ M \otimes C^{\otimes(n-1)} & \xrightarrow[\rho \otimes 1_C^{\otimes(n-1)}]{} & M \otimes C^{\otimes 2(n-1)} \end{array}$$

i.e.,

$$(1_M \otimes 1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-2)}) \circ \rho = (\rho \otimes 1_C^{\otimes(n-1)}) \circ \rho.$$

k-module M is called a *left n-ary C-module*, where C is an n-ary algebra, if there is a map $\gamma : C^{\otimes(n-1)} \otimes M \to M$, such that the following diagram is commutative:

$$\begin{array}{ccc} C^{\otimes(n-1)} \otimes M & \xrightarrow{\quad\gamma\quad} & M \\ \uparrow{\scriptstyle 1_C^{\otimes(n-1)} \otimes \gamma} & & \uparrow{\scriptstyle\gamma} \\ C^{\otimes 2(n-1)} \otimes M & \xrightarrow[1_C^{\otimes i} \otimes m \otimes 1_C^{\otimes(n-i-2)} \otimes 1_M]{} & C^{\otimes(n-1)} \otimes M \end{array}$$

i.e.,

$$\gamma \circ (1_C^{\otimes(n-1)} \otimes \gamma) = \gamma \circ (1_C^{\otimes i} \otimes m \otimes 1_C^{\otimes(n-i-2)} \otimes 1_M).$$

Now, we define the concept of an n-ary ideal: a submodule I of the module C is called an *n-ary ideal*, if

$$C^{\otimes i} \otimes I \otimes C^{\otimes(n-i-1)} \subseteq I,$$

where $0 \leqslant i \leqslant n-1$, C is an n-ary algebra.

2. Relations between n-ary and binary modules

Let C be an n-ary algebra over commutative ring k. There is not necessarily a unit in C, but the multiplication is associative, i.e.,

$$(c_1 \cdots c_n)c_{n+1} \cdots c_{2n-1} = c_1 \cdots c_j(c_{j+1} \cdots c_{j+n})c_{j+n+1} \cdots c_{2n-1} \tag{3}$$

for all $j = 0, \dots, n-1$ and $c_1, \dots, c_{2n-1} \in C$. Consider the submodule I in the tensor-degree $C^{\otimes(n-1)}$ (see [4]), which is generated by all differences:

$$(c_1 \cdots c_n) \otimes c_{n+1} \otimes \cdots \otimes c_{2n-2} - c_1 \otimes \cdots \otimes c_j \otimes (c_{j+1} \cdots c_{j+n}) \otimes c_{j+n+1} \otimes \cdots \otimes c_{2n-2}$$

for $c_1, \dots, c_{2n-2} \in C$ and $j = 0, \dots, n-2$. Then, I is an n-ary ideal in the n-ary algebra $C^{\otimes(n-1)}$. Denote by R the factor-module $C^{\otimes(n-1)}/I$.

Theorem 2.1. *R is an associative binary k-algebra with respect to multiplication*

$$(c_1 \otimes \cdots \otimes c_{n-1} + I)(c_n \otimes \cdots \otimes c_{2n-2} + I) = (c_1 \cdots c_n) \otimes c_{n+1} \otimes \cdots \otimes c_{2n-2} + I \quad (4)$$

Proof. Let us check that the multiplication (4) is correctly defined. It is sufficient to show that:

$$\begin{aligned}
[(c_1 \cdots c_n) \otimes c_{n+1} \otimes \cdots \otimes c_{2n-2} + I\, [c_{2n-1} \otimes \cdots \otimes c_{3n-3} + I] \\
= [c_1 \otimes \cdots \otimes c_j \otimes (c_{j+1} \cdots c_{j+n}) \otimes c_{j+n+1} \otimes \cdots \otimes c_{2n-2} + I] \\
\cdot [c_{2n-1} \otimes \cdots \otimes c_{3n-3} + I]
\end{aligned}$$

for all $c_1, \dots, c_{3n-3} \in C$.

Similar equality holds after multiplication by $c_{2n-1} \otimes \cdots \otimes c_{3n-3} + I$, on the left. By (4), we have:

$$\begin{aligned}
[(c_1 \cdots c_n) \otimes c_{n+1} \otimes \cdots \otimes c_{2n-2} + I]\, [c_{2n-1} \otimes \cdots \otimes c_{3n-3} + I] \\
= [(c_1 \cdots c_n) c_{n+1} \cdots c_{2n-1}] \otimes c_{2n} \otimes \cdots \otimes c_{3n-3} + I
\end{aligned}$$

On the other hand:

$$\begin{aligned}
[c_1 \otimes \cdots \otimes c_j \otimes (c_{j+1} \cdots c_{j+n}) \otimes c_{j+n+1}! \otimes \cdots \otimes c_{2n-2} + I]\, [c_{2n-1} \otimes \cdots \otimes c_{3n-3} + I] \\
= [c_1 \cdots c_j (c_{j+1} \cdots c_{j+n}) c_{j+n+1} \cdots c_{2n-2} c_{2n-1}] \otimes c_{2n} \otimes \cdots \otimes c_{3n-3} + I.
\end{aligned}$$

By the associativity(3), the previous products are equal. The condition:

$$\begin{aligned}
&[c_{2n-1} \otimes \cdots \otimes c_{3n-3} + I]\, [(c_1 \cdots c_n) \otimes c_{n+1} \otimes \cdots \otimes c_{2n-2} + I] \\
&= [c_{2n-1} \otimes \cdots \otimes c_{3n-3} + I]\, [c_1 \otimes \cdots \otimes c_j \otimes (c_{j+1} \cdots c_{j+n}) \otimes c_{j+n+1} \otimes \cdots \otimes c_{2n-2} + I]
\end{aligned}$$

is checked in a similar way. Consequently, the multiplication in R is well defined.

Let us show that it is associative. We have:

$$\begin{aligned}
&[(c_1 \otimes \cdots \otimes c_{n-1} + I)(c_n \otimes \cdots \otimes c_{2n-2} + I)]\, (c_{2n-1} \otimes \cdots \otimes c_{3n-3} + I) \\
&= [(c_1 \cdots c_n) \otimes c_{n+1} \otimes \cdots \otimes c_{2n-2} + I]\, (c_{2n-1} \otimes \cdots \otimes c_{3n-3} + I) \\
&= [(c_1 \cdots c_n) c_{n+1} \cdots c_{2n-1}] \otimes c_{2n} \otimes \cdots \otimes c_{3n-3} + I.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(c_1 \otimes \cdots \otimes c_{n-1} + I)\, [(c_n \otimes \cdots \otimes c_{2n-2} + I)\, (c_{2n-1} \otimes \cdots \otimes c_{3n-3} + I)] \\
&= (c_1 \otimes \cdots \otimes c_{n-1} + I)\, [(c_n \dots c_{2n-1}) \otimes c_{2n} \cdots \otimes c_{3n-3} + I] \\
&= [c_1 \cdots c_{n-1}\, (c_n \dots c_{2n-1})] \otimes c_{2n} \otimes \cdots \otimes c_{3n-3} + I
\end{aligned}$$

By (3), we obtain that the multiplication in R is associative. □

Theorem 2.2. *M is an n-ary left C-module if and only if M is a binary left R-module.*

Proof. Suppose that M is an n-ary left C-module. If $c_1, \dots, c_{n-1} \in C$ and $m \in M$, then we put:

$$(c_1 \otimes \cdots \otimes c_{n-1} + I)m = (c_1 \otimes \cdots \otimes c_{n-1})m.$$

The definition of the ideal I and the n-ary C-module implies that $I \cdot m = 0$. So, M is a left R-module.

Conversely, if M is a left R-module, then for $c_1, \dots, c_{n-1} \in C$ and $m \in M$, we put

$$(c_1 \otimes \cdots \otimes c_{n-1})m = (c_1 \otimes \cdots \otimes c_{n-1} + I)m.$$

We see that M is an n-ary left C-module. □

What is proved here is an equivalence of categories between the category of n-ary left modules over C and the category of left modules over R.

3. Dual situation

Let C be an n-ary coalgebra over a field k. Denote by $C^{\square(n-1)}$ the set:

$$\bigcap_{j=1}^{n-2} \mathrm{Ker}[\Delta \otimes 1_C^{\otimes(n-2)} - 1_C^{\otimes j} \otimes \Delta \otimes 1_C^{\otimes(n-2-j)}] \subset C^{\otimes(n-1)}.$$

In the other words, $C^{\square(n-1)}$ contains all elements

$$f = \sum c_1 \otimes \cdots \otimes c_{n-1} \in C^{\otimes(n-1)},$$

such that

$$\sum \Delta c_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} = \sum c_1 \otimes \cdots \otimes c_j \otimes \Delta c_{j+1} \otimes c_{j+2} \otimes \cdots \otimes c_{n-1}$$

for all $j = 0, \dots, n-2$.

Theorem 3.1. *The n-ary comultiplication in C induces a comultiplication:*

$$\Delta' : C^{\square(n-1)} \to C^{\square(n-1)} \otimes C^{\square(n-1)}$$

i.e., $C^{\square(n-1)}$ is a binary coalgebra.

Proof. Define the map

$$\Delta' : C^{\otimes(n-1)} \to C^{\otimes(n-1)} \otimes C^{\otimes(n-1)}$$

by the following rule:

$$\Delta'(c_1 \otimes \cdots \otimes c_{n-1}) = \Delta c_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} \in\in C^{\otimes n} \otimes C^{\otimes(n-2)} = C^{\otimes(n-1)} \otimes C^{\otimes(n-1)}.$$

It is necessary show that

$$\Delta'(C^{\square(n-1)}) \subseteq C^{\square(n-1)} \otimes C^{\square(n-1)}.$$

Let $f \in C^{\square(n-1)}$. Then, for $j = 1, \ldots, n-2$:

$$\begin{aligned} &\left\{\left[\Delta \otimes 1_C^{\otimes(n-2)} - 1_C^{\otimes j} \otimes \Delta \otimes 1_C^{\otimes(n-2-j)}\right] \otimes 1_C^{\otimes(n-1)}\right\} \Delta'(f) \\ &= \left\{\left[\Delta \otimes 1_C^{\otimes(n-2)} - 1_C^{\otimes j} \otimes \Delta \otimes 1_C^{\otimes(n-2-j)}\right] \otimes 1_C^{\otimes(n-1)}\right\} (\Delta \otimes 1_C^{\otimes(n-2)}) f \\ &= \left\{\left[(\Delta \otimes 1_C^{\otimes(n-1)} - 1_C^{\otimes j} \otimes \Delta \otimes 1_C^{\otimes(n-1-j)}) \otimes 1_C^{\otimes(n-2)}\right] (\Delta \otimes 1_C^{\otimes(n-2)})\right\} f = 0 \end{aligned}$$

by the coassociativity. Analogously, for $j = 1, \ldots, n-2$:

$$\left\{1_C^{\otimes(n-1)} \otimes \left[\Delta \otimes 1_C^{\otimes(n-2)} - 1_C^{\otimes j} \otimes \Delta \otimes 1_C^{\otimes(n-2-j)}\right]\right\} \Delta'(f) = 0,$$

see [2]. □

Theorem 3.2. *k-module M is an n-ary right C-comodule if and only if M is a binary right $C^{\square(n-1)}$-comodule.*

Proof. If M is a binary right $C^{\square(n-1)}$-comodule, then M is an n-ary right C-comodule, because $C^{\square(n-1)} \subset C^{\otimes(n-1)}$.

Conversely, let M be an n-ary right C-comodule and $\rho: M \to M \otimes C^{\otimes(n-1)}$. It is necessary show that

$$\rho(M) \subseteq M \otimes C^{\square(n-1)},$$

i.e.,

$$(\Delta \otimes 1_C^{\otimes(n-2)} - 1_C^{\otimes j} \otimes \Delta \otimes 1_C^{\otimes(n-2-j)})\rho = 0.$$

This follows from the definition of an n-ary C-comodule. □

What is proved here is an equivalence of categories between the category of n-ary right comodules over C and the category of right comodules over $C^{\square(n-1)}$.

4. Isomorphisms of binary (co)algebras

In this part, as in previous, we shall suppose that k is a field.

Theorem 4.1. *Let C be an n-ary finite dimensional coalgebra over the field k. Then $C^{\square(n-1)}$ is a binary coalgebra. Moreover, C^* is an n-ary algebra, for which we construct the binary algebra $R = (C^*)^{\otimes(n-1)}/I$. Then there exists an isomorphism of binary algebras:*

$$(C^{\square(n-1)})^* \cong (C^*)^{\otimes(n-1)}/I.$$

Proof. By definition:

$$C^{\square(n-1)} = \bigcap_{j=1}^{n-2} \mathrm{Ker}[\Delta \otimes 1_C^{\otimes(n-2)} - 1_C^{\otimes j} \otimes \Delta \otimes 1_C^{\otimes(n-2-j)}].$$

In other words, we obtain the exact sequence of the vector spaces:

$$0 \to C^{\square(n-1)} \to C^{\otimes(n-1)} \xrightarrow{\varphi} \bigoplus_{j=1}^{n-2} C^{\otimes(2n-2)},$$

where

$$\varphi(x) = \left(\Delta \otimes 1_C^{\otimes(n-2)} - 1_C \otimes \Delta \otimes 1_C^{\otimes(n-3)}\right)(x) + \cdots \\ + \left(\Delta \otimes 1_C^{\otimes(n-2)} - 1_C^{\otimes(n-2)} \otimes \Delta\right)(x).$$

Moving to the dual finite dimensional spaces, we obtain the exact sequence:

$$0 \leftarrow (C^{\square(n-1)})^* \leftarrow (C^{\otimes(n-1)})^* \xleftarrow{\varphi^*} \bigoplus_{j=1}^{n-2} (C^{\otimes(2n-2)})^* \tag{5}$$

Since C has finite dimension:

$$(C^{\otimes(n-1)})^* = (C^*)^{\otimes(n-1)}$$
$$(C^{\otimes(2n-2)})^* = (C^*)^{\otimes(2n-2)}.$$

Moreover, for $l_1, \ldots, l_{2n-2}$ from j-th summand $(C^*)^{\otimes(2n-2)}$, we have:

$$\varphi^*(l_1 \otimes \cdots \otimes l_{2n-2}) = (l_1 * \cdots * l_n) \otimes l_{n+1} \otimes \cdots \otimes l_{2n-2} \\ - l_1 \otimes \cdots \otimes l_j \otimes (l_{j+1} * \cdots * l_{j+n}) \otimes l_{j+n+1} \otimes \cdots \otimes l_{2n-2} \tag{6}$$

In that way, by the exactness of the sequence (5), we obtain that:

$$(\mathrm{C}^{\square(n-1)})^* \cong (C^*)^{\otimes(n-1)}/I,$$

where I is the subspace generated by all elements of the form (6). We need to show that the constructed isomorphism

$$(C^*)^{\otimes(n-1)}/I \to (\mathrm{C}^{\square(n-1)})^*$$

is an isomorphism of binary algebras. Let

$$l_1, \ldots, l_{2n-2} \in C^* \quad \text{and} \quad f = \sum c_1 \otimes \cdots \otimes c_{n-1} \in C^{\square(n-1)}.$$

Then,

$$[(l_1 \otimes \cdots \otimes l_{n-1} + I)(l_n \otimes \cdots \otimes l_{2n-2} + I)]\,(f) \\ = [(l_1 * \cdots * l_n) \otimes l_{n+1} \otimes \cdots \otimes l_{2n-2} + I]\,(f) \\ = \mu(l_1 \otimes \cdots \otimes l_{2n-2})(\Delta \otimes 1_C^{\otimes(n-2)})(f)$$

But, for $u, v \in (\mathrm{C}^{\square(n-1)})^*$ and $\mathrm{f} \in \mathrm{C}^{\square(n-1)}$:

$$(u * v)(f) = \mu(u \otimes v)\Delta'(f) == \mu(u \otimes v)(\Delta \otimes 1^{\otimes(n-2)})f$$

Let

$$u = l_1 \otimes \cdots \otimes l_{n-1} + I, \quad v = l_n \otimes \cdots \otimes l_{2n-2} + I.$$

Then,

$$\mu(u \otimes v)(\Delta \otimes 1^{\otimes(n-2)}) = \mu(l_1 \otimes \cdots \otimes l_{2n-2})(\Delta \otimes 1^{\otimes(n-2)})$$

i.e., the map

$$R \to (\mathrm{C}^{\square(n-1)})^*$$

is a homomorphism of binary algebras. □

Analogically, we prove:

Theorem 4.2. *Let C be an n-ary finite dimensional algebra over a field. Then, $R = C^{\otimes(n-1)}/I$ is a binary algebra, and C^* is an n-ary coalgebra. Moreover,*

$$(C^*)^{\square(n-1)} \cong R^*.$$

References

[1] **N. Burbaki**, *Algebra*, Moscow, 1962.

[2] **N. Burbaki**, *Commutative algebra*, Mir, Moscow, 1971.

[3] **B. Zekovich**, *On n-ary bialgebras. I*, (Russian), Chebyshevskii Sb. **4** (2003), N3, 65 – 73.

[4] **B. Zekovich**, *On n-ary bialgebras. II,* (Russian), Chebyshevskii Sb. **4** (2003), N3, 73 – 80.

[5] **B. Zekovich**, *n-ary comodules over n-ary (co)algebras*, Algebra and Discrete Math. (2008), N4, 80 – 89.

[6] **B. Zekovich**, *Example of n-ary bialgebra*, Days of Difraction 2012, 250 – 253.

Received June 28, 2014

Faculty of Natural Science
Department of Mathematics
University of Montenegro
Podgorica, Montenegro
E-mail: biljanaz@t-com.me