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On two-sided bases of ternary semigroups

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Abstract. We introduce the concept of two-sided bases of a ternary semigroup, and study the structure of ternary semigroups containing two-sided bases.

1. Introduction

The notion of a ternary semigroup which is a natural generalization of a ternary group was defined as follows: a *ternary semigroup* is a non-empty set T together with a ternary operation, written as $(a, b, c) \mapsto [abc]$, satisfying the *associative law*

$$[[abc]uv] = [a[bcu]v] = [ab[cuv]]$$

for all $a, b, c, u, v \in T$.

A non-empty subset A of a ternary semigroup T is called

- a *left ideal* of T if $[TTA] \subseteq A$;
- a *right ideal* of T if $[ATT] \subseteq A$;
- a *middle ideal* of T if $[TAT] \subseteq A$.

If A is both a left and a right ideal of T then A is called a *two-sided ideal* of T. Finally, A is called an *ideal* of T if it is a left, a right and a middle ideal of T (see [6], [9]). Note that the union of two two-sided ideals of T is a two-sided ideal of T, and the intersection of two two-sided ideals of T, if it is non-empty, is a two-sided ideal of T.

It is known that, for a non-empty subset A of a ternary semigroup T,

$$A_t = A \cup [TTA] \cup [ATT] \cup [T[TAT]T]$$

is the two-sided ideal of T containing A (see [7], [9]). If $A = \{a\}$ we write A_t as $(a)_t$, called the *principal two-sided ideal* of T generated by a.

We introduce the *quasi-ordering* on a ternary semigroup T as follows:

$$a \leqslant_t b \text{ if and only if } (a)_t \subseteq (b)_t.$$

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Tamura [10] introduced one-sided bases including left bases and right bases of a semigroup. Fabrici [4] introduced two-sided bases of a semigroup and studied the structure of a semigroup containing two-sided bases. In the line of Fabrici, the results were extended to ordered semigroups by the second author and Summaprab [1]. The purpose of this paper is to introduce two-sided bases of a ternary semigroup and study the structure of a ternary semigroup containing two-sided bases.

2. Two-sided bases of a ternary semigroup

As in [4], we define two-sided bases of a ternary semigroup as follows.

Definition 2.1. A subset A of a ternary semigroup T is called a *two-sided* base of T if it satisfies the following two conditions:

(i) $A_t = T$;

(ii) there exists no a proper subset B of A such that $B_t = T$.

Example 2.2. Consider the multiplication over the complex numbers, the set $T = \{-i, 0, i\}$ is a ternary semigroup [3]. We have $\{i\}$ and $\{-i\}$ are the two-sided bases of T.

Example 2.3. Under the usual multiplication of integers, the set $\mathbb{Z}^-$ of all negative integers is a ternary semigroup. We have $\{-1\}$ is a two-sided base of $\mathbb{Z}^-$.

Example 2.4. Let $T = \mathbb{Z}^- \times \mathbb{Z}^- = \{(a, b) \mid a, b \in \mathbb{Z}^-\}$. Then (cf. [5]) T is a ternary semigroup under the ternary operation which is defined by

$$[(a, b)(c, d)(e, f)] = (a, f).$$

Then, for all $(a, b) \in T$, $\{(a, b)\}$ is a two-sided base of T.

Example 2.5. Let T be a non-empty set such that $0 \in T$ and the cardinality $|T| > 3$. Then T with the ternary operation defined by

$$[xyz] = \begin{cases} x & \text{if } x = y = z; \\ 0 & \text{otherwise,} \end{cases}$$

is a ternary semigroup [8]. We have $T \setminus \{0\}$ is a two-sided base of T.

Example 2.6. Consider a ternary semigroup

$$T = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

under the matrix multiplication [2], we have

$$A = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a two-sided base of T.

Example 2.7. Let $T = \{0, 1, 2, 3, 4, 5\}$. Define the ternary operation on T by

$$[abc] = (a * b) * c \text{ for all } a, b, c \in T$$

where the binary operation $*$ is defined by

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Then T is a ternary semigroup [8] and $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$ are two-sided bases of T.

We now give some elementary results:

Lemma 2.8. *Let A be a two-sided base of a ternary semigroup T. If $a, b \in A$ and $a \in [TTb] \cup [bTT] \cup [T[TbT]T]$, then $a = b$.*

Proof. Let $a, b \in A$ be such that $a \in [TTb] \cup [bTT] \cup [T[TbT]T]$. Suppose that $a \neq b$. We set $B = A \setminus \{a\}$; then $b \in B$. By

$$(a)_t \subseteq [TTb] \cup [bTT] \cup [T[TbT]T] \subseteq (b)_t \subseteq B_t,$$

it follows that $A_t \subseteq B_t$, and so $T = B_t$. This is a contradiction. Hence $a = b$. □

Theorem 2.9. *A non-empty subset A of a ternary semigroup T is a two-sided base of T if and only if A satisfies the following conditions:*

(1) *for any $x \in T$ there exists $a \in A$ such that $x \leqslant_t a$;*

(2) *for any $a, b \in A$, if $a \neq b$, then a and b are incomparable.*

Proof. Assume that A is a two-sided base of ternary semigroup T, and let $x \in T$. Thus $x \in A_t$. Then there exists $a \in A$ such that $x \in (a)_t$, and hence $x \leqslant_t a$. This shows that (1) hold. Let $a, b \in A$ be such that $a \neq b$ and $a \leqslant_t b$. Then $(a)_t \subseteq (b)_t$. Since $a \neq b$, we have $a \in (b)_t \setminus \{b\}$. By Lemma 2.8, $a = b$. This is a contradiction. Thus (2) follows.

Conversely, assume that the conditions (1) and (2) hold. By (1), for any $x \in T$, there is $a \in A$ such that $(x)_t \subseteq (a)_t \subseteq A_t$. Thus $T = A_t$. Suppose that there exists a proper subset B of A such that $T = B_t$. Let $a \in A \setminus B$. Then

$$a \in A_t = T = B_t.$$

By (1), there exists $b \in B \subseteq A$ such that $a \leqslant_t b$. This contradicts to (2). Hence A is a two-sided base of T. □

3. Main results

Throughout this section, the symbol $\subset$ stands for proper inclusion for sets.

Theorem 3.1. *Let A be a two-sided base of a ternary semigroup T such that $(a)_t = (b)_t$ for some $a \in A$ and $b \in T$. If $a \neq b$, then T contains at least two two-sided bases.*

Proof. Let $a \neq b$ be such that $(b)_t = (a)_t$, it follows that

$$b \in [TTa] \cup [aTT] \cup [T[TaT]T].$$

By Lemma 2.8, $b \notin A$. Hence $b \in T \setminus A$. We set $B = (A \setminus \{a\}) \cup \{b\}$. Thus $A \neq B$. We will show that B is a two-sided base of T. Let $x \in T$. Since A is a two-sided base of T, there exists $c \in A$ such that $x \leqslant_t c$. If $c \neq a$, then $c \in B$. If $c = a$, then $(c)_t = (a)_t = (b)_t$; hence $x \leqslant_t c \leqslant_t b \in B$. Therefore B satisfies the condition (1) of Theorem 2.9. Let $x, y \in B$ be such that $x \neq y$. If $x \neq b$ and $y \neq b$, then $x, y \in A$, that is, neither $x \leqslant_t y$ nor $y \leqslant_t x$. There are two cases to consider: $x = b$ or $y = b$. If $x = b$, then $y \in A$. Suppose that $x \leqslant_t y$. Then $a \leqslant_t b = x \leqslant_t y$ and $a, y \in A$. This is a contradiction. Suppose that $y \leqslant_t x$. Then $y \leqslant_t x = b \leqslant_t a$ and $a, y \in A$. This is a contradiction. Thus neither $x \leqslant_t y$ nor $y \leqslant_t x$. The case $y = b$ can be probed in the same manner. Therefore, B satisfies the condition (2) of Theorem 2.9. □

By Theorem 3.1, we have the following.

Corollary 3.2. *Let A be a two-sided base of a ternary semigroup T, and let $a \in A$. If $(a)_t = (x)_t$ for some $x \in T$ and $x \neq a$, then x is an element of a two-sided base of T which is different from A.*

Theorem 3.3. *Any two two-sided bases of a ternary semigroup T have the same cardinality.*

Proof. Let A and B be two-sided bases of a ternary semigroup T. Let $a \in A$. Since B is a two-sided base of T, we have $a \leqslant_t b$ for some $b \in B$. For $a \in A$, we choose and fix $b \in B$ such that $a \leqslant_t b$ and define a mapping $f : A \to B$ by $f(a) = b$ for all $a \in A$.

If $a_1, a_2 \in A$ such that $f(a_1) = f(a_2) = b$. We have $a_1 \leqslant_t b$ and $a_2 \leqslant_t b$. Since A is a two-sided base of T, we have $b \leqslant_t a'$ for some a' in A. Thus $a_1 \leqslant_t a', a_2 \leqslant_t a'$ and $a_1, a_2, a' \in A$. By Theorem 2.9, we have $a_1 = a' = a_2$. Hence f is one to one. Now, let $b \in B$. Then there exists $a \in A$ such that $b \leqslant_t a$. Similarly, there exists $b' \in B$ such that $a \leqslant_t b'$. Then $b \leqslant_t b'$. By Theorem 2.9, we have $b = b'$. Thus $a \leqslant_t b' = b$. Let $f(a) = c$ for some $c \in B$. Then $a \leqslant_t c$. Since $c, b \in T$ and A is a two-sided base of T, there exist $a', a'' \in A$ such that $c \leqslant_t a'$ and $b \leqslant_t a''$. Then $a \leqslant_t a'$ and $a \leqslant_t a''$. By Theorem 2.9, we have $a = a' = a''$. Then $b \leqslant_t a'' = a \leqslant_t c$. Thus $b = c$ by Theorem 2.9. Hence f is onto. □

A two-sided base of a ternary semigroup need not to be a ternary subsemigroup, in general. Consider Example 2.2 we have $\{i\}$ is a two-sided base of T, but it is not a ternary subsemigroup of T.

Theorem 3.4. *Let A be a two-sided base of a ternary semigroup T. Then A is a ternary subsemigroup of T if and only if it has only one element.*

Proof. Let $a, b \in A$, where A is a ternary subsemigroup of T. Then $[aab] \in A$. Since

$$[aab] \in [TTb] \cup [bTT] \cup [T[TbT]T]$$

and

$$[aab] \in [TTa] \cup [aTT] \cup [T[TaT]T],$$

it follows by Lemma 2.8 that $[aab] = a = b$. Then $A = \{a\}$.

The converse statement is obvious. □

Theorem 3.5. *Let $\mathcal{A}$ be the union of all two-sided bases of a ternary semigroup T. If $M = T \setminus \mathcal{A}$ is non-empty, then it is a two-sided ideal of T.*

Proof. Let $x, y \in T$ and $a \in M$. Suppose that $[xya] \notin M$ or $[axy] \notin M$. Then $[xya] \in \mathcal{A}$ or $[axy] \in \mathcal{A}$. Thus, there exists a two-sided base B of T such that $[xya] \in B$ or $[axy] \in B$. Hence, there is $b \in B$ such that $[xya] = b$ or $[axy] = b$. It implies $b \in (a)_t$. Then $(b)_t \subseteq (a)_t$. Thus $b \leqslant_t a$. If $(b)_t = (a)_t$, then $a \in \mathcal{A}$. This contradicts to $a \in M$. Hence $(b)_t \neq (a)_t$. Since B is a two-sided base of T, there exists $c \in B$ such that $a \leqslant_t c$. If $b = c$, then $(a)_t \subseteq (c)_t = (b)_t \subseteq (a)_t$; hence $(a)_t = (b)_t$. This is a contradiction. Thus $b \neq c$. We have $b \leqslant_t a \leqslant_t c$, $b \neq c$ and $b, c \in B$. This contradicts to Theorem 2.9. Therefore, $[xya], [axy] \in M$. □

Theorem 3.6. *Let $\mathcal{A}$ be the union of all two-sided bases of a ternary semigroup T such that $\emptyset \neq \mathcal{A} \subset T$. Let M^* be a maximal two-sided ideal of T containing all proper two-sided ideals of T. The following statements are equivalent:*

(1) *$T \setminus \mathcal{A}$ is a maximal two-sided ideal of T;*

(2) *$\mathcal{A} \subseteq (a)_t$ for every $a \in \mathcal{A}$;*

(3) *$T \setminus \mathcal{A} = M^*$;*

(4) *every two-sided base of T has only one element.*

Proof. (1) $\Leftrightarrow$ (2). Assume that $T \setminus \mathcal{A}$ is a maximal two-sided ideal of T. Suppose that $\mathcal{A} \nsubseteq (a)_t$. Since $\mathcal{A} \nsubseteq (a)_t$, there exists $x \in \mathcal{A}$ such that $x \notin (a)_t$. Thus $x \notin T \setminus \mathcal{A}$. Then $(T \setminus \mathcal{A}) \cup (a)_t \neq T$, and thus $(T \setminus \mathcal{A}) \cup (a)_t$ is a proper two-sided ideal of T such that $(T \setminus \mathcal{A}) \subset (T \setminus \mathcal{A}) \cup (a)_t$. This contradicts to the maximality of $T \setminus \mathcal{A}$.

Conversely, assume that $\mathcal{A} \subseteq (a)_t$ for every element $a \in \mathcal{A}$. By Theorem 3.5, $T \setminus \mathcal{A}$ is a proper two-sided ideal of T. Suppose that M is a two-sided ideal of T such that $T \setminus \mathcal{A} \subset M \subset T$. Then $M \cap \mathcal{A}$ is non-empty. Let $c \in M \cap \mathcal{A}$. We have $(c)_t \subseteq M$, and so

$$T = (T \setminus \mathcal{A}) \cup \mathcal{A} \subseteq (T \setminus \mathcal{A}) \cup (c)_t \subseteq M.$$

This is a contradiction. Hence $T \setminus \mathcal{A}$ is a maximal two-sided ideal of T.

(3) $\Leftrightarrow$ (4). Assume that $T \setminus \mathcal{A} = M^*$. Then $T \setminus \mathcal{A}$ is a maximal two-sided ideal of T. Let $a \in \mathcal{A}$. Using (1) $\Leftrightarrow$ (2), $\mathcal{A} \subseteq (a)_t$. Then $T = \mathcal{A}_t \subseteq (a)_t$. This implies $T = (a)_t$. Hence, for any $a \in \mathcal{A}$, $\{a\}$ is a two-sided base of T. Let B be a two-sided base of T, and let $a, b \in B$. Then $B \subseteq \mathcal{A}$, that is, $a, b \in \mathcal{A}$. Hence $b \in T = (a)_t$. By Lemma 2.8, $a = b$ (i.e., B has only one element).

Conversely, assume that every two-sided base of T has only one element. Then $T = (a)_t$ for all $a \in \mathcal{A}$. Suppose that there is a proper two-sided ideal M of T such that M is not contained in $T \setminus \mathcal{A}$. Then there exists $x \in \mathcal{A} \cap M$. Since $x \in M$, $T = (x)_t \subseteq M$, and so $T = M$. This is a contradiction.

(1) $\Leftrightarrow$ (3). Assume that $T \setminus \mathcal{A}$ is a maximal two-sided ideal of T. Let M be a two-sided ideal of T such that M is not contained in $T \setminus \mathcal{A}$. Hence, there exists $x \in M \cap \mathcal{A}$. Using (1) $\Leftrightarrow$ (2), $\mathcal{A} \subseteq (x)_t \subseteq M$. Thus $M = \mathcal{A} \cup X$ for some $X \subseteq T \setminus \mathcal{A}$. For any $y \in T$, there exists $c \in \mathcal{A}$ such that $y \leqslant_t c$. Then $y \in (y)_t \subseteq (c)_t \subseteq M$. This implies that $M = T$. Thus $T \setminus \mathcal{A} = M^*$.

The converse is obvious. □

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