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An investigation on fuzzy hyperideals of ordered semihypergroups

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Abstract. We introduce the notions of fuzzy hyperideals, fuzzy bi-hyperideals and fuzzy quasi-hyperideals of an ordered semihypergroup and show that every fuzzy quasi-hyperideal is a fuzzy bi-hyperideal and in a regular ordered semihypergroup, fuzzy quasi-hyperideals and fuzzy bi-hyperideals coincide. Moreover, we show that in an ordered semihypergroup every fuzzy quasi-hyperideal is an intersection of a fuzzy right hyperideal and a fuzzy left hyperideal.

1. Introduction

The concept of algebraic hyperstructures was introduced in 1934 by Marty [13]. The concept of a semihypergroup is a generalization of the concept of a semigroup. Semihypergroups are studied by many authors, for example, Bonansinga and Corsini [1], Davvaz [4, 5], De Salvo et al. [6], Freni [7], Hila et al. [9], Leoreanu [14], and many others. In [8], Heidari and Davvaz studied a semihypergroup $(H, \circ)$ with a binary relation $\leqslant$, where $\leqslant$ is a partial order so that the monotony condition is satisfied. This structure is called an ordered semihypergroup. The study of fuzzy algebras was started in [15] by Rosenfeld. In [10], the relationships between some types of fuzzy ideals in ordered semigroups were investigated. In [11], some equivalent definitions of fuzzy ideals of ordered semigroups were given. In [3], Davvaz introduced the concept of a fuzzy right (resp. left, two-sided) hyperideal of a semihypergroup and proved some results in this respect. Now, in this paper we study the notions of fuzzy hyperideals of ordered semihypergroups.

The paper is structured as follows. After an introduction, in Section 2 we present some basic notions and examples on ordered semihypergroups. In Section 3, we introduce the notions of fuzzy hyperideals, fuzzy bi-hyperideals and fuzzy quasi-hyperideals of an ordered semihypergroup and we give some results in this respect. In particular, we show that every fuzzy quasi-hyperideal is a fuzzy bi-hyperideal and in a regular ordered semihypergroup, fuzzy quasi-hyperideals and fuzzy bi-hyperideals coincide. Moreover, we show that in an ordered semihypergroup every fuzzy quasi-hyperideal is an intersection of a fuzzy right hyperideal and a fuzzy left hyperideal.

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2. Preliminaries

A *hypergroupoid* consists of a non-empty set H and a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ called a *hyperoperation*, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H. We denote by $a \circ b$ the image of the pair (a, b) in $H \times H$.

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if it satisfies the associative property, namely,

$$(a \circ b) \circ c = a \circ (b \circ c).$$

For any non-empty subsets A, B of H, we denote

$$A \circ B := \bigcup_{a \in A, b \in B} a \circ b.$$

Instead of $\{a\} \circ A$ and $B \circ \{a\}$, we write $a \circ A$ and $B \circ a$, respectively.

Definition 2.1. Let H be a non-empty set and $\leqslant$ be an ordered relation on H. The triplet $(H, \circ, \leqslant)$ is called an *ordered semihypergroup* if the following conditions are satisfied.

(1) $(H, \circ)$ is a semihypergroup,

(2) $(H, \leqslant)$ is a partially order set,

(3) for every $a, b, c \in H$, $a \leqslant b$ implies $a \circ c \leqslant b \circ c$ and $c \circ a \leqslant c \circ b$, where $a \circ c \leqslant b \circ c$ means that for every $x \in a \circ c$ there exists $y \in b \circ c$ such that $x \leqslant y$.

A non-empty subset A of an ordered semihypergroup $(H, \circ, \leqslant)$ is called a *subsemihypergroup* of H if $(A, \circ, \leqslant)$ is an ordered semihypergroup.

We note that for every $a, b, c, d, e, f \in H$ with $a \circ b \leqslant c \circ d$ and $e \leqslant f$, we obtain $a \circ b \circ e \leqslant c \circ d \circ f$.

For $K \subseteq H$, we denote

$$(K] := \{a \in H \mid a \leqslant k \text{ for some } k \in K\}.$$

Definition 2.2. A non-empty subset A of an ordered semihypergroup $(H, \circ, \leqslant)$ is called a *right* (resp. *left*) *hyperideal* of H if

(1) $A \circ H \subseteq A$ (resp. $H \circ A \subseteq A$),

(2) for every $a \in H$, $b \in A$ and $a \leqslant b$ implies $a \in A$.

If A is both right hyperideal and left hyperideal of H, then A is called a *hyperideal (or two-side hyperideal)* of H.

Definition 2.3. A subsemihypergroup A of an ordered semihypergroup $(H, \circ, \leqslant)$ is called a *bi-hyperideal* of H if

(1) $A \circ H \circ A \subseteq A$,

(2) for every $a \in H$, $b \in A$ and $a \leqslant b$ implies $a \in A$.

Definition 2.4. A non-empty subset Q of an ordered semihypergroup $(H, \circ, \leqslant)$ is called a *quasi-hyperideal* of H if

(1) $(Q \circ H] \cap (H \circ Q] \subseteq Q$,

(2) for every $a \in H$, $b \in Q$ and $a \leqslant b$ implies $a \in Q$.

Example 2.5. The set $H = \{a, b, c, d, e\}$ and the hyperoperation defined by the table

$\circ$	a	b	c	d	e
a	a	$\{a,b,d\}$	a	$\{a,b,d\}$	$\{a,b,d\}$
b	a	b	a	$\{a,b,d\}$	$\{a,b,d\}$
c	a	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	$\{a,b,c,d,e\}$
d	a	$\{a,b,d\}$	a	$\{a,b,d\}$	$\{a,b,d\}$
e	a	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	$\{a,b,c,d,e\}$

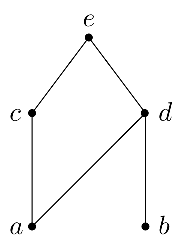
is a semihypergroup (cf. [2]).

We define order relation $\leqslant$ as follows:

$$\leqslant := \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,c),(a,d),(a,e),(b,d),(b,e),(c,e),(d,e)\}.$$

We give the covering relation $\prec$ and the figure of H:

$$\prec = \{(a,c),(a,d),(b,d),(c,e),(d,e)\}.$$



Now, $(H, \circ, \leqslant)$ is an ordered semihypergroup, $\{a, b, d\}$ is a hyperideal and $\{a\}, \{a, c\}$ are left hyperideals and also bi-hyperideals of $(H, \circ, \leqslant)$. □

Now, we use the ordered semigroup defined in Example 3.3 in [16] to construct a semihypergroup in a similarly way of Example 3.10 in [2] and give an example of quasi-hyperideals of an ordered semihypergroup.

Example 2.6. Let $H = \{a, b, c, d, e\}$. Define the hyperoperation $\circ$ on H by the following table.

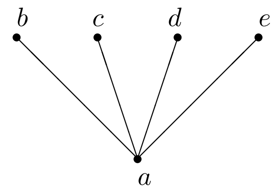
$\circ$	a	b	c	d	e
a	a	a	a	a	a
b	a	$\{a,b\}$	a	$\{a,d\}$	a
c	a	$\{a,e\}$	$\{a,c\}$	$\{a,c\}$	$\{a,e\}$
d	a	$\{a,b\}$	$\{a,d\}$	$\{a,d\}$	$\{a,b\}$
e	a	$\{a,e\}$	a	$\{a,c\}$	a

Suppose that the order relation $\leqslant$ as follows:

$$\leqslant := \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(a,d),(a,e)\}.$$

We give the covering relation $\prec$ and the figure of H:

$$\prec = \{(a,b),(a,c),(a,d),(a,e)\}.$$



Now, $(H,\circ,\leqslant)$ is an ordered semihypergroup. It is easy to show that all proper quasi-hyperideals of H are $\{a\}$, $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{a,e\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{a,b,e\}$ and $\{a,c,e\}$. $\square$

Let $(H,\circ,\leqslant)$ be an ordered semihypergroup and $a \in H$. We denote

$$A_a := \{(b,c) \in H \times H \mid a \leqslant b \circ c\}.$$

A *fuzzy subset* μ of a semihypergroup $(H,\circ)$ is a function $\mu : H \to [0,1]$. Let $t \in [0,1]$ and μ be a fuzzy subset of H. The set $\mu_t = \{a \in H \mid \mu(a) \geqslant t\}$ is called a *level subset* of μ. For fuzzy subsets μ and ν of H, we define the fuzzy subset $\mu \circ \nu$ of H by letting $a \in H$,

$$(\mu \circ \nu)(a) := \begin{cases} \sup\limits_{(b,c)\in A_a} \{\min\{\mu(b),\nu(c)\}\}, & \text{if } A_a \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

At the end of this section, we recall the notions of a fuzzy subsemihypergroup of a semihypergroup introduced by Davvaz in [5] as the following. A fuzzy subset μ of a semihypergroup is called a *fuzzy subsemihypergroup* if $\inf\limits_{x\in a\circ b} \mu(x) \geqslant \min\{\mu(a),\mu(b)\}$ for all $a, b \in H$.

3. Fuzzy hyperideals of ordered semihypergroups

In this section, we define the concepts of a fuzzy right (left) hyperideal, a fuzzy bi-hyperideal and a fuzzy quasi-hyperideal and give relationships between them.

Definition 3.1. Let $(H, \circ, \leqslant)$ be an ordered semihypergroup. A fuzzy subset $\mu : H \to [0, 1]$ is called a *fuzzy right* (resp. *left*) *hyperideal* of H if

(1) $\mu(a) \leqslant \inf\limits_{c \in a \circ b}\{\mu(c)\}$ (resp. $\mu(b) \leqslant \inf\limits_{c \in a \circ b}\{\mu(c)\}$ for every $a, b \in H$,

(2) for every $a, b \in H, a \leqslant b$ implies $\mu(b) \leqslant \mu(a)$.

If μ is both fuzzy right hyperideal and fuzzy left hyperideal of H, then μ is called a *fuzzy hyperideal* (or *fuzzy two-sided hyperideal*) of H.

Example 3.2. Consider the ordered semihypergroup $(H, \circ, \leqslant)$ defined in Example 2.5. We define two fuzzy subset μ and λ of H as follows:

$$\mu(x) = \begin{cases} 0.7 & \text{if} \quad x = a, b, d \\ 0.3 & \text{if} \quad x = c, e. \end{cases} \qquad \lambda(x) = \begin{cases} 0.9 & \text{if} \quad x = a \\ 0.8 & \text{if} \quad x = c \\ 0.5 & \text{if} \quad x = b, d, e \end{cases}$$

Then, μ is a fuzzy hyperideal and λ is a fuzzy left hyperideal of H. □

Definition 3.3. Let $(H, \circ, \leqslant)$ be an ordered semihypergroup. A fuzzy subsemihypergroup $\mu : H \to [0, 1]$ is called a *fuzzy bi-hyperideal* of H if the following assertions are satisfied:

(1) $\min\{\mu(b), \mu(d)\} \leqslant \inf\limits_{a \in b \circ c \circ d}\{\mu(a)\}$ for every $b, c, d \in H$,

(2) for every $a, b \in H, a \leqslant b$ implies $\mu(b) \leqslant \mu(a)$.

Definition 3.4. Let $(H, \circ, \leqslant)$ be an ordered semihypergroup. A fuzzy subset $\mu : H \to [0, 1]$ is called a *fuzzy quasi-hyperideals* of H if the following assertions are satisfied:

(1) $(\mu \circ 1) \cap (1 \circ \mu) \subseteq \mu$,

(2) for every $a, b \in H, a \leqslant b$ implies $\mu(b) \leqslant \mu(a)$,

where $1 : H \to [0, 1]$ is a constant function defined by $1(a) = 1$ for all $a \in H$.

Lemma 3.5. *Let $(H, \circ, \leqslant)$ be an ordered semihypergroup and μ be a fuzzy subset of H. Then, μ is a fuzzy right* (*resp. left*) *hyperideal of H if and only if for every $t \in [0, 1]$, the non-empty level subset μ_t is a right* (*resp. left*) *hyperideal of H.*

Proof. Assume that μ is a fuzzy right hyperideal of H. Let $t \in [0,1]$ with $\mu_t \neq \emptyset$. Let $a \in \mu_t \circ H$. We have $a \in b \circ h$ for some $b \in \mu_t, h \in H$. By assumption, $t \leqslant \mu(b) \leqslant \inf\limits_{a \in b \circ h}\{\mu(a)\}$, we have $\mu(a) \geqslant t$. This implies $\mu_t \circ H \subseteq \mu_t$. Let $x \in \mu_t, y \in H$ with $y \leqslant x$. Since $t \leqslant \mu(x) \leqslant \mu(y)$, we obtain $y \in \mu_t$. Therefore, μ_t is a right hyperideal of H.

Conversely, we assume that for every $t \in [0,1]$, μ_t is a right hyperideal of H. We show that $\mu(a) \leqslant \inf\limits_{c \in a \circ b}\{\mu(c)\}$ for all $a, b \in H$. We put $t_0 = \mu(a)$. By assumption μ_{t_0} is a right hyperideal of H. Since $a \in \mu_{t_0}$, $a \circ b \subseteq \mu_{t_0}$. Then, for every $c \in a \circ b$, we obtain $t_0 \leqslant \mu(c)$ and hence, $\mu(a) = t_0 \leqslant \inf\limits_{c \in a \circ b}\{\mu(c)\}$. Let $a, b \in H$ with $a \leqslant b$. Since $a \leqslant b, b \in \mu_{\mu(b)}$ and $\mu_{\mu(b)}$ is a right hyperideal of H, we get $a \in \mu_{\mu(b)}$. So, $\mu(b) \leqslant \mu(a)$. Therefore, μ is a fuzzy right hyperideal of H. □

Corollary 3.6. *Let $(H, \circ, \leqslant)$ be an ordered semihypergroup and χ_I be the characteristic function of I. Then, I is a left (resp. right) hyperideal of H if and only if χ_I is a fuzzy left (resp. right) hyperideal of H.* □

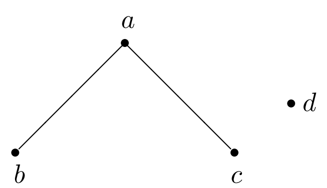
Example 3.7. Let $H = \{a, b, c, d\}$. We consider the ordered semihypergroup $(H, \circ, \leqslant)$, where the hyperoperation $\circ$ and the order relation $\leqslant$ on H are defined as follows:

$\circ$	a	b	c	d
a	a	$\{a,b\}$	$\{a,c\}$	a
b	a	$\{a,b\}$	$\{a,c\}$	a
c	a	$\{a,b\}$	$\{a,c\}$	a
d	a	$\{a,b\}$	$\{a,c\}$	a

$$\leqslant := \{(a,a), (b,b), (c,c), (d,d), (b,a), (c,a)\}.$$

We give the covering relation $\prec$ and the figure of H:

$$\prec = \{(b,a), (c,a)\}.$$



Now, we define the fuzzy subset μ of H as follows:

$$\mu(x) = \begin{cases} 0.8 & \text{if} \quad x = a, b, c, \\ 0.3 & \text{if} \quad x = d. \end{cases}$$

Then, μ is a fuzzy hyperideal of H. □

Lemma 3.8. *Let $(H, \circ, \leqslant)$ be an ordered semihypergroup and μ be a fuzzy subset of H. Then, μ is a fuzzy bi-hyperideal of H if and only if for every $t \in [0,1]$, the non-empty level subset μ_t is a bi-hyperideal of H.*

Proof. Assume that μ is a fuzzy bi-hyperideal of H. Let $t \in [0,1]$ with $\mu_t \neq \emptyset$. Let $a \in \mu_t \circ H \circ \mu_t$. We have $a \in b \circ h \circ c$ for some $b, c \in \mu_t, h \in H$. Since $t \leqslant \min\{\mu(b), \mu(c)\} \leqslant \inf\limits_{a \in b \circ h \circ c}\{\mu(a)\}$, we have $\mu(a) \geqslant t$. This implies $\mu_t \circ H \circ \mu_t \subseteq \mu_t$. Let $x \in \mu_t, y \in H$ with $y \leqslant x$. Since $t \leqslant \mu(x) \leqslant \mu(y)$, we obtain $y \in \mu_t$. Therefore, μ_t is a bi-hyperideal of H.

Conversely, we assume that for every $t \in [0,1]$, μ_t is a bi-hyperideal of H. We show that $\min\{\mu(b), \mu(c)\} \leqslant \inf\limits_{a \in b \circ h \circ c}\{\mu(a)\}$ for all $b, c, h \in H$. We choose $t_0 = \min\{\mu(b), \mu(c)\}$. By assumption μ_{t_0} is a bi-hyperideal of H. Since $b, c \in \mu_{t_0}$, $b \circ h \circ c \subseteq \mu_{t_0}$. Then, for every $a \in b \circ h \circ c$, we have $t_0 \leqslant \mu(a)$ and so $\min\{\mu(b), \mu(c)\} = t_0 \leqslant \inf\limits_{a \in b \circ h \circ c}\{\mu(a)\}$. Let $a, b \in H$ with $a \leqslant b$. Since $a \leqslant b$, $b \in \mu_{\mu(b)}$ and $\mu_{\mu(b)}$ is a bi-hyperideal of H, we get $a \in \mu_{\mu(b)}$. So, $\mu(b) \leqslant \mu(a)$. Therefore, μ is a bi-hyperideal of H. □

Corollary 3.9. *Let $(H, \circ, \leqslant)$ be an ordered semihypergroup and χ_I be the characteristic function of I. Then, I is a bi-hyperideal of H if and only if χ_I is a fuzzy bi-hyperideal of H.* □

Lemma 3.10. *Let $(H, \circ, \leqslant)$ be an ordered semihypergroup and μ be a fuzzy subset of H. Then, μ is a fuzzy quasi-hyperideal of H if and only if for every $t \in [0,1]$, the non-empty level subset μ_t is a quasi-hyperideal of H.*

Proof. Assume that μ is a fuzzy quasi-hyperideal of H. Let $t \in [0,1]$ with $\mu_t \neq \emptyset$. We show that $(\mu_t \circ H] \cap (H \circ \mu_t] \subseteq \mu_t$. Let $a \in (\mu_t \circ H] \cap (H \circ \mu_t]$. Then, $a \in (\mu_t \circ H]$ and $a \in (H \circ \mu_t]$, i.e., $a \leqslant b \circ h$ and $a \leqslant k \circ c$ for some $b, c \in \mu_t, h, k \in H$, i.e., $(b,h), (k,c) \in A_a$. This implies $(\mu \circ 1)(a) = \sup\limits_{(x,y) \in A_a}\{\min\{\mu(x), 1(y)\}\} \geqslant t$ and $(1 \circ \mu)(a) = \sup\limits_{(x,y) \in A_a}\{\min\{1(x), \mu(y)\}\} \geqslant t$. By assumption, we obtain $\mu(a) \geqslant \min\{(\mu \circ 1)(a), (1 \circ \mu)(a)\} \geqslant t$. Therefore, $(\mu_t \circ H] \cap (H \circ \mu_t] \subseteq \mu_t$. Let $x \in \mu_t, y \in H$ with $y \leqslant x$. Since $t \leqslant \mu(x) \leqslant \mu(y)$, we obtain $y \in \mu_t$. Therefore, μ_t is a quasi-hyperideal of H.

Conversely, we assume that for every $t \in [0,1], \mu_t$ is a quasi-hyperideal of H. We show that $(\mu \circ 1) \cap (1 \circ \mu) \subseteq \mu$. Let $a \in H$. If $A_a = \emptyset$, then it is clear that $\min\{(\mu \circ 1)(a), (1 \circ \mu)(a)\} \leqslant \mu(a)$. If $A_a \neq \emptyset$, then there exist $x, y \in H$ such that $a \leqslant x \circ y$. Let $t_0 = \min\{\mu(x), \mu(y)\}$. Since μ_{t_0} is a quasi-hyperideal, $a \leqslant x \circ y$ and $x, y \in \mu_{t_0}$, we have $a \in (\mu_{t_0} \circ H] \cap (H \circ \mu_{t_0}] \subseteq \mu_{t_0}$. Then, $\mu(a) \geqslant t_0$. This means $\mu(a) \geqslant \min\{\mu(x), \mu(y)\}$ for all $(x,y) \in A_a$. Now, we have:

$$\begin{aligned} ((\mu \circ 1) \cap (1 \circ \mu))(a) &= \min\{(\mu \circ 1)(a), (1 \circ \mu)(a)\} \\ &= \min\{\sup_{(x,y) \in A_a}\{\min\{\mu(x), 1(y)\}\}, \sup_{(x,y) \in A_a}\{\min\{1(x), \mu(y)\}\}\} \end{aligned}$$

$$
\begin{aligned}
&= \min\{\sup_{(x,y)\in A_a}\{\mu(x)\}, \sup_{(x,y)\in A_a}\{\mu(y)\}\} \\
&= \sup_{(x,y)\in A_a}\{\min\{\mu(x), \mu(y)\}\} \\
&\leqslant \mu(a).
\end{aligned}
$$

Thus, $(\mu \circ 1) \cap (1 \circ \mu) \subseteq \mu$. Let $a, b \in H$ with $a \leqslant b$. Since $a \leqslant b, b \in \mu_{\mu(b)}$ and $\mu_{\mu(b)}$ is a quasi-hyperideal of H, we get $a \in \mu_{\mu(b)}$. So, $\mu(b) \leqslant \mu(a)$. Therefore, μ is a qausi-hyperideal of H. □

Corollary 3.11. *Let $(H, \circ, \leqslant)$ be an ordered semihypergroup and χ_I be the characteristic function of I. Then, I is a qausi-hyperideal of H if and only if χ_I is a fuzzy quasi-hyperideal of H.* □

Example 3.12. Consider the ordered semihypergroup $(H, \circ, \leqslant)$ defined in Example 2.5. Define fuzzy subsets μ and ν of H by letting $x \in H$,

$$
\mu(x) = \begin{cases} 0.7 & \text{if} \quad x = a, \\ 0.5 & \text{if} \quad x = b, d, \\ 0.3 & \text{if} \quad x = c, e, \end{cases} \qquad \nu(x) = \begin{cases} 0.8 & \text{if} \quad x = a, \\ 0.6 & \text{if} \quad x = c, \\ 0.2 & \text{if} \quad x = b, d, e. \end{cases}
$$

By Lemma 3.5, μ is a fuzzy left hyperideal of H, since every non-empty level subset of μ is a left hyperideal of H. Similarly, by Lemma 3.8, ν is a fuzzy bi-hyperideal of H. □

Example 3.13. Consider the ordered semihypergroup $(H, \circ, \leqslant)$ defined in Example 2.6. Define a fuzzy subset $\rho : H \to [0, 1]$ by letting $x \in H$,

$$
\rho(x) = \begin{cases} 0.9 & \text{if} \quad x = a \\ 0.8 & \text{if} \quad x = c \\ 0.5 & \text{if} \quad x = d \\ 0 & \text{if} \quad x = b, e. \end{cases}
$$

By Lemma 3.10, ρ is a fuzzy quasi-hyperideal of H, since every non-empty level subset of ρ is a quasi-hyperideal of H. □

Theorem 3.14. *Let $(H, \circ, \leqslant)$ be an ordered semihypergroup. Then, every fuzzy right (resp. left) hyperideal of H is a fuzzy quasi-hyperideal of H.*

Proof. Let μ be a fuzzy right hyperideal of H and $a \in H$. We have

$$((\mu \circ 1) \cap (1 \circ \mu))(a) = \min\{(\mu \circ 1)(a), (1 \circ \mu)(a)\}.$$

If $A_a = \emptyset$, then it is clear that $\min\{(\mu \circ 1)(a), (1 \circ \mu)(a)\} \subseteq \mu$.

Let $A_a \neq \emptyset$. Let $(x, y) \in A_a$. We have $a \leqslant x \circ y$. This means $a \leqslant z$ for some $z \in x \circ y$. Since μ is a fuzzy right hyperideal of H, $\mu(a) \geqslant \mu(z) \geqslant \inf_{z \in x \circ y}\{\mu(z)\} \geqslant$

$\mu(x)$. It follows

$$\begin{aligned}\mu(a) &\geqslant \sup_{a\leqslant x\circ y}\{\mu(x)\}\\ &= \sup_{(x,y)\in A_a}\{\min\{\mu(x),1(y)\}\} = (\mu\circ 1)(a)\\ &\geqslant \min\{(\mu\circ 1)(a),(1\circ\mu)(a)\}\\ &= ((\mu\circ 1)\cap(1\circ\mu))(a).\end{aligned}$$

Therefore, μ is a fuzzy quasi-hyperideal of H. □

Theorem 3.15. *Let $(H,\circ,\leqslant)$ be an ordered semihypergroup. Then, every fuzzy quasi-hyperideal of H is a fuzzy bi-hyperideal of H.*

Proof. Let μ be a fuzzy quasi-hyperideal of H and $x,y,z\in H$. We show that $\min\{\mu(x),\mu(z)\}\leqslant \inf\limits_{a\in x\circ y\circ z}\{\mu(a)\}$. Since $a\in x\circ y\circ z\leqslant x\circ(y\circ z)$, we obtain

$$(\mu\circ 1)(a) = \sup_{(u,v)\in A_a}\{\min\{\mu(u),1(v)\}\} \geqslant \min\{\mu(x),1(w)\} = \mu(x),\quad \forall w\in y\circ z.$$

Since $a\in x\circ y\circ z\leqslant (x\circ y)\circ z$, we obtain

$$(1\circ\mu)(a) = \sup_{(u,v)\in A_a}\{\min\{1(u),\mu(v)\}\} \geqslant \min\{1(t),\mu(z)\} = \mu(z),\quad \forall t\in x\circ y.$$

By assumption, we have

$$\mu(a)\geqslant ((\mu\circ 1)\cap(1\circ\mu))(a) = \min\{(\mu\circ 1)(a),(1\circ\mu)(a)\}.$$

Hence,

$$\inf_{a\in x\circ y\circ z}\{\mu(a)\}\geqslant \min\{(\mu\circ 1)(a),(1\circ\mu)(a)\}\geqslant \min\{\mu(x),\mu(z)\}.$$

Therefore, μ is a bi-hyperideal of H. □

An ordered semihypergroup $(H,\circ\leqslant)$ is called *regular*, if for every $a\in H$, there exists $x\in H$ such that $a\leqslant a\circ x\circ a$.

Theorem 3.16. *Let $(H,\circ,\leqslant)$ be a regular ordered semihypergroup and μ is a fuzzy subset of H. Then, μ is a fuzzy quasi-hyperideal if and only if μ is a fuzzy bi-hyperideal.*

Proof. Let μ is a fuzzy bi-hyperideal. We show that μ is a fuzzy quasi-hyperideal of H, i.e., $(\mu\circ 1)\cap(1\circ\mu)\subseteq\mu$.

Let $a\in H$. If $A_a=\emptyset$, then it is clear that $((\mu\circ 1)\cap(1\circ\mu))(a)\leqslant\mu(a)$.

If $A_a\neq\emptyset$, then

$$(\mu\circ 1)(a) = \sup_{(x,y)\in A_a}\{\min\{\mu(x),1(y)\} \text{ and } (1\circ\mu)(a) = \sup_{(u,v)\in A_a}\{\min\{1(u),\mu(v)\}.$$

If $(\mu \circ 1)(a) \leqslant \mu(a)$, then $\mu(a) \geqslant ((\mu \circ 1) \cap (1 \circ \mu))(a)$.

If $(\mu \circ 1)(a) > \mu(a)$, then there exists $(x, y) \in A_a$ such that $\min\{\mu(x), 1(y)\} = \mu(x) > \mu(a)$. We claim that $(1 \circ \mu)(a) \leqslant \mu(a)$. Let $(u, v) \in A_a$. Since H is regular, there exists $w \in H$ such that $a \leqslant a \circ w \circ a$. It turns out $a \leqslant x \circ y \circ w \circ u \circ v$, i.e., there exists $b \in x \circ y \circ w \circ u \circ v$ such that $a \leqslant b$. Since μ is a fuzzy bi-hyperideal of H,

$$\mu(a) \geqslant \mu(b) \geqslant \inf_{c \in x \circ y \circ w \circ u \circ v} \{\mu(c)\} \geqslant \min\{\mu(x), \mu(v)\}.$$

If $\min\{\mu(x), \mu(v)\} = \mu(x)$, then $\mu(a) \geqslant \mu(x)$. This gives a contradiction. Then, $\min\{\mu(x), \mu(v)\} = \mu(v)$ and so $\mu(a) \geqslant \mu(v) = \min\{1(u), \mu(v)\}$ for all $(u, v) \in A_a$. Hence, $\mu(a) \geqslant \sup_{(u,v) \in A_a} \min\{1(u), \mu(v)\} = (1 \circ \mu)(a)$. Now, the claim was proved. Therefore, $\min\{(\mu \circ 1)(a), (1 \circ \mu)(a)\} \leqslant \mu(a)$, i.e., $(\mu \circ 1) \cap (1 \circ \mu) \subseteq \mu$. □

Lemma 3.17. *Let $(H, \circ, \leqslant)$ be an ordered semihypergroup and μ a fuzzy subset of H such that $\mu(a) \geqslant \mu(b)$ for every $a, b \in H$ with $a \leqslant b$. Then,*

(1) *$\mu \cup (1 \circ \mu)$ is a fuzzy left hyperideal of H,*

(2) *$\mu \cup (\mu \circ 1)$ is a fuzzy right hyperideal of H.*

Proof. (1) Let $a, b \in H$ and $c \in a \circ b$. We have

$$\begin{aligned} (\mu \cup (1 \circ \mu))(c) &= \max\{\mu(c), (1 \circ \mu)(c)\} \\ &\geqslant (1 \circ \mu)(c) = \sup_{(x,y) \in A_c} \{\min\{1(x), \mu(y)\}\} = \sup_{(x,y) \in A_c} \{\mu(y)\} \\ &\geqslant \mu(b), \quad (\text{ since } c \in a \circ b \text{ and then } (a, b) \in A_c). \end{aligned}$$

Next, we show that $(1 \circ \mu)(c) \geqslant (1 \circ \mu)(b)$. Let $A_b \neq \emptyset$ and $(r, s) \in A_b$. Since $(a, b) \in A_c$, we have

$$\begin{aligned} (r, s) \in A_b &\Rightarrow b \leqslant r \circ s \\ &\Rightarrow a \circ b \leqslant (a \circ r) \circ s \\ &\Rightarrow c \leqslant (a \circ r) \circ s \\ &\Rightarrow c \leqslant t \circ s, \qquad \text{for some } t \in a \circ r. \end{aligned}$$

We have $(1 \circ \mu)(c) \geqslant \min\{1(t), \mu(s)\} = \mu(s) = \min\{1(r), \mu(s)\}$. Thus, $(1 \circ \mu)(c) \geqslant \sup_{(r,s) \in A_b} \{\min\{1(r), \mu(s)\}\} = (1 \circ \mu)(b)$ and then $(\mu \cup (1 \circ \mu)(c) \geqslant (1 \circ \mu)(b)$. This implies $\inf_{c \in a \circ b} \{(\mu \cup (1 \circ \mu))(c)\} \geqslant (\mu \cup (1 \circ \mu))(b)$. Next, we show that for any $a, b \in H$ and $a \leqslant b$ implies $(\mu \cup (1 \circ \mu))(a) \geqslant (\mu \cup (1 \circ \mu))(b)$. Since $A_a \supseteq A_b$, we have $(1 \circ \mu)(a) \geqslant (1 \circ \mu)(b)$. Then, $\max\{\mu(a), (1 \circ \mu)(a)\} \geqslant \max\{\mu(b), (1 \circ \mu)(b)\}$. This means $(\mu \cup (1 \circ \mu)(a) \geqslant (\mu \cup (1 \circ \mu)(b)$. Altogether, $\mu \cup (1 \circ \mu)$ is a fuzzy left hyperideal of H.

(2) It can be proved similarly. □

Similarly to Corollary 1 in [10], we have the following lemma.

Lemma 3.18. *If H is an ordered semihypergroup, then the set of all fuzzy subsets of H is a distributive lattice.* □

Now, we show that every fuzzy quasi-hyperideal is exactly an intersection of a fuzzy right hyperideal and a fuzzy left hyperideal and vice versa.

Theorem 3.19. *Let $(H, \circ, \leqslant)$ be an ordered semihypergroup and μ a fuzzy subset of H. Then, μ is a fuzzy quasi-hyperideal of H if and only if there exist a fuzzy right hyperideal ν and a fuzzy left hyperideal ρ of H such that $\mu = \nu \cap \rho$.*

Proof. By Lemma 3.17 and Lemma 3.18, we have $\mu = (\mu \cup (1 \circ \mu)) \cap (\mu \cup (\mu \circ 1))$. Conversely, let ν be a fuzzy right hyperideal and ρ be a fuzzy left hyperideal of H such that $\mu = \nu \cap \rho$. We show that μ is a fuzzy quasi-hyperideal of H. Let $a \in H$. If $A_a = \emptyset$, then it is clear that $((\mu \circ 1) \cap (1 \circ \mu)) \subseteq \mu$. Let $A_a \neq \emptyset$ and $(x, y) \in A_a$. We have $a \leqslant x \circ y$. Then, there exists $b \in x \circ y$ such that $a \leqslant b$. Since ν is a fuzzy right hyperideal of H and $\mu = \nu \cap \rho$, we have $\nu(a) \geqslant \nu(b) \geqslant \inf\limits_{c \in x \circ y} \{\nu(c)\} \geqslant \nu(x) \geqslant \mu(x)$. Now, we have $\nu(a) \geqslant \mu(x)$ for all $(x, y) \in A_a$. Hence,

$$(\mu \circ 1)(a) = \sup_{(x,y) \in A_a} \{\min\{\mu(x), 1(y)\}\} = \sup_{(x,y) \in A_a} \{\min\{\mu(x)\}\} \leqslant \nu(a).$$

Similarly, we can show that $(1 \circ \mu)(a) \leqslant \rho(a)$. Thus,

$$\begin{aligned} ((\mu \circ 1) \cap (1 \circ \mu))(a) &= \min\{(\mu \circ 1)(a), (1 \circ \mu)(a)\} \\ &\leqslant \min\{\nu(a), \rho(a)\} \\ &= (\nu \cap \rho)(a) = \mu(a). \end{aligned}$$

Therefore, μ is a fuzzy quasi-hyperideal of H. □

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References

[1] **P. Bonansinga and P. Corsini**, *On semihypergroup and hypergroup homomorphisms*, Boll. Un. Mat. Ital. B (6) **1(2)** (1982), 717 – 727.

[2] **P. Corsini, M. Shabir and T. Mahmood**, *Semisimple semihypergroups in terms of hyperideals and fuzzy hyperideals*, Iran. J. Fuzzy Syst. **8** (2011), 95 – 111.

[3] **B. Davvaz**, *Fuzzy hyperideals in semihypergroups*, Italian J. Pure and Appl. Math. **8** (2000), 67 – 74.

[4] **B. Davvaz**, *Some results on congruences in semihypergroups*, Bull. Malays. Math. Sci. So.(2) **23** (2000), 53 – 58.

[5] **B. Davvaz**, *Characterizations of sub-semihypergroups by various triangular norms*, Czech. Math. J. **55(4)** (2005), 923 – 932.

[6] **M. De Salvo, D. Freni and G. Lo Faro**, *Fully simple semihypergroups*, J. Algebra **399** (2014), 358 − 377.

[7] **D. Freni**, *Minimal order semihypergroups of type U on the right, II*, J. Algebra **340** (2011), 77 − 89.

[8] **D. Heidari and B. Davvaz**, On ordered hyperstructures, U.P.B. Sci. Bull. Series A **73** (2011), 85 − 96.

[9] **K. Hila, B. Davvaz and K. Naka**, *On quasi-hyperideals in semihypergroups*, Commun. Algebra **39** (2011), 4183 − 4194.

[10] **N. Kehayopulu and M. Tsingelis**, *Fuzzy ideals in ordered semigroups*, Quasigroups and Related Systems **15** (2007), 279 − 289.

[11] **N. Kehayopulu and M. Tsingelis**, *Fuzzy right, left, quasi-ideals, bi-ideals in ordered semigroups*, Lobachevskii J. Math. **30(1)** (2009), 17 − 22.

[12] **A. Khan, N. H. Sarmin, B. Davvaz and F.M. Khan**, *New types of fuzzy bi-ideals in ordered semigroups*, Neural Comput. Appl. **21** (2012), 295 − 305.

[13] **F. Marty**, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandinaves, Stockholm (1934), 45 − 49.

[14] **V. Leoreanu**, *About the simplifiable cyclic semihypergroups*, Italian J. Pure Appl. Math. **7** (2000), 69 − 76.

[15] **A. Rosenfeld**, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512 − 517.

[16] **J. Tang and X. Xie**, *On fuzzy quasi-ideals of ordered semigroups*, J. Math. Research Appl. **32** (2012), 589 − 598.

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