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Actions over monoids and hypergroups

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Abstract. We construct the hypergroups by actions over monoids. Particularly, some non-unital hypergroups are constructed. Here, hypergroups are obtained by orbit neighborhood collections that make a complete lattice.

1. Introduction and preliminaries

A generating technique of examples in a theory can be very useful, in particular if it is not given various fundamental examples in that theory. One of these theories is the theory of hypergroups which was introduced in 1934 by Marty [3].

A *hyperoperation* on a set H is a map $\cdot : H \times H \to P^*(H)$, where $P^*(H)$ is the set of all non-empty subsets of H. The set H with a hyperoperation $\cdot$ is called a *hypergroup* if for every $x, y, z \in H$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (*association law*), and $x \cdot H = H \cdot x = H$. For more information, see [1] and [4].

In the sense of category theory, an action over monoids is a monoid in the category T-**Act** of all T-acts for a monoid T. Let M be a monoid with no zero element. For any monoid T, a homomorphism of monoids:

$$\Phi : T \to H(M);\ t \mapsto \varphi_t : M \to M,$$

where $H(M)$ denotes the monoid of all endomorphisms of M is said to be an *action over monoids*. Note that $H(M)$ has a (unique) zero element which is a constant mapping equals 1. If T has a zero element 0, we impose the assumption that $T\backslash\{0\}$ is a monoid. So letting φ_0 be the zero element of $H(M)$, $\Phi : T \to H(M)$ is a homomorphism of semigroups. In this case, $m\varphi_0 = 1$ for every $m \in M$ and then Φ is called a *zero faithful action*.

In this paper a generating technique for constructing hypergroups is presented. Using neighborhood collections, we construct a class of hypergroups, and describe how an action over monoids can be applied to obtain a hypergroup. We consider hypergroup actions over monoids, which are those actions $\Phi : T \to H(M)$ over monoids for which $(M, \bullet)$ is a hypergroup. It is obtained a necessary and sufficient condition for a hypergroup action over monoids to be unital, that is, $1 \in x \bullet y$ for all $x, y \in M$.

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For a monoid M, on the monoid $H(M)$ of all endomorphisms of M we consider the operation $\star : M \to M$ defined by $\sigma \star \mu := \mu \circ \sigma$, for each $\mu, \sigma \in H(M)$. To denote the image of $x \in M$ under σ we will use the postfix notation. Also $Sub(M)$ denotes the set of all submonoids of M. Throughout M stands for a monoid with no zero element unless otherwise stated.

2. Actions over monoids

In this section first we give some instances of actions over monoids

Example 2.1. Each of the following is an action over monoids:

(i) For any commutative monoid M and $T = (\mathbb{N}, \cdot), \Phi : T \to H(M); m\varphi_k = m^k$.

(ii) For any submonoid T of $H(M), \Phi : T \to H(M); \Phi := id_T$ (*natural action*).

(iii) For any monoid T with zero, $\Phi : T \to H(M); \varphi_t := id_M$ for each $0 \neq t \in T$. If $t = 0, m\varphi_0 := 1$, for each $m \in M$. □

Let M be a monoid. By a *neighborhood collection* on M we mean the sequence $\mathcal{V} = \{V_x : x \in M\}$ indexed by M, such that for each $x \in M$, $V_x \subseteq M$ and $x \in V_x$. If $\mathcal{V}$ is a neighborhood collection, we define a hyperoperation, called *the hyperoperation induced by* $\mathcal{V}$ in the following way: for each $x, y \in M$, $x \bullet y = V_x V_y$, where $V_x V_y$ is the usual product of subsets V_x and V_y of M. It is clear that for every $x, y \in M$, $xy \in x \bullet y$. For every $a \in M$ and a non-empty subset X of M, we put $a \bullet X := \bigcup_{x \in X} a \bullet x$ and $X \bullet a := \bigcup_{x \in X} x \bullet a$. Clearly, $a \bullet M = V_a M$ and $M \bullet a = M V_a$ for each $a \in M$. Also we have:

Lemma 2.2. *If M is a group, then $(M, \bullet)$ is a hypergroup.*

Proof. Let M be a monoid and $\emptyset \neq A \subseteq M$. If there is an invertible element $a \in A$, then $AM = M = MA$. Moreover, if M is a group, then the operation on subsets of M is also associative. Therefore, if M is a group and $\bullet$ is a hyperoperation induced by any neighborhood collection, then M is a hypergroup. □

Definition 2.3. Let $\Phi : T \to H(M)$ be an action over monoids. Then the set $\{m\varphi_t : t \in T\}$ of all images of an element $m \in M$ under the mappings $\Phi(t)$ is usually called *the orbit* of m and it is denoted by $Orb_T(m)$. It is obvious that for each $m \in M, m \in Orb_T(m)$. Hence, $\mathcal{V}_T = \{V_m = Orb_T(m) : m \in M\}$ is the set of all orbits of elements from M which is called *the orbit neighborhood collection*.

From now on, $\bullet$ stands for the hyperoperation induced by orbit neighborhood collection. Also, for a submonoid S of $H(M)$, by $\mathcal{V}_S$ we mean the orbit neighborhood collection induced by the natural action from S to $H(M)$. In this case, $\mathcal{V}_S = \{Orb_S(m) : m \in M\}$, where $Orb_S(m) = \{m\sigma : \sigma \in S\}$ for each $m \in M$.

Definition 2.4. An action $\Phi : T \to H(M)$ over monoids is called *right* (*left*) *multiplicative* if for each $m \in M$ and $0 \neq t \in T$, there exists $x \in M$ such that $m\varphi_t = mx$ ($m\varphi_t = xm$).

The action over monoids in Example 2.1(i) is (left and right) multiplicative. If $\Phi : T \to H(M)$ is a (left) right multiplicative zero faithful action, then M is a group.

Proposition 2.5. *Let $\Phi : T \to H(M)$ be a (left) right multiplicative action over monoids. Then $(M, \bullet)$ is a hypergroup if and only if M is a group.*

Proof. Suppose $(M, \bullet)$ is a hypergroup. Let $m \in M$. By assumption, for each $t \in T$, $m\varphi_t M = mxM \subseteq mM$ for some $x \in M$. Thus

$$M = m \bullet M = Orb_T(m)M = \bigcup_{t \in T} m\varphi_t M \subseteq mM.$$

Then $M = mM$. Hence, M is a group. The converse follows from Lemma 2.2. □

Remark 2.6. An action $\Phi : T \to H(M)$ over monoids and the natural action $\Psi = id_{\Phi(T)} : \Phi(T) \to H(M)$ defined as in Example 2.1(ii) have the same orbits of elements from M.

Let M be a monoid and $\mathcal{V}, \mathcal{W}$ be two neighborhood collections on M. We say $\mathcal{V} \leq \mathcal{W}$ if for every $x \in M$, $V_x \subseteq W_x$. Clearly, $\leq$ is a partial order relation on the set of all neighborhood collections.

A neighborhood collection $\mathcal{V} = \{V_x : x \in M\}$ is called a *basis neighborhood collection* if for every $y \in V_x$, $V_y \subseteq V_x$. For instance, if $\Phi : T \to H(M)$ is an action over monoids, then the orbit neighborhood collection $\mathcal{V}_T$ is a basis neighborhood collection. Indeed, for any $x, y \in M$, $y \in Orb_T(x)$ implies that $y = x\varphi_t$ for some $t \in T$ and then $Orb_T(y) = \{x\varphi_{ts} : s \in T\} \subseteq Orb_T(x)$.

Lemma 2.7. *Let $\mathcal{V} = \{V_x : x \in M\}$ be a basis neighborhood collection and $S = \{\sigma \in H(M) : x\sigma \in V_x$ for all $x \in M\}$. The following statements hold:*

(i) *S is a submonoid of $H(M)$ and $\mathcal{V}_S \leq \mathcal{V}$.*

(ii) *For every action $\Phi : T \to H(M)$ over monoids satisfying $\mathcal{V}_T \leq \mathcal{V}$, we have $\mathcal{V}_T \leq \mathcal{V}_S$.*

Proof. (i) For every $x \in M$, $x\, id_M = x \in V_x$, so $id_M \in S$. Let $\sigma, \mu \in S$. Then $x\sigma \in V_x$ and $x\sigma\mu \in V_{x\sigma}$ for all $x \in M$, and so $x\sigma\mu = (x\sigma)\mu \in V_{x\sigma} \subseteq V_x$ because $\mathcal{V}$ is a basis. Therefore, $\sigma\mu \in S$. (ii) It follows from Remark 2.6. □

For a monoid M, let $\mathbf{ONC}(M)$ denote the set of all orbit neighborhood collections $\mathcal{V}_T$, for all monoids T such that there is an action over monoids T and M.

Theorem 2.8. *For a monoid M, $(\mathbf{ONC}(M), \leq)$ is a complete lattice.*

Proof. Let $\{T_i : i \in I\}$ be a non-empty family of monoids such that $\Phi_i : T_i \to H(M)$ is an action over monoids for all $i \in I$. For every $x \in M$, let $V_x = \bigcap_{i \in I} Orb_{T_i}(x)$. Also take $\mathcal{V} = \{V_x : x \in M\}$. It is easy to check that $\mathcal{V}$ is a basis neighborhood collection. Put $S := \{\sigma \in H(M) : x\sigma \in V_x$ for all $x \in M\}$. We claim that $\mathcal{V}_S = \bigwedge_{i \in I} \mathcal{V}_{T_i}$. By Lemma 2.7(i), $\mathcal{V}_S \leq \mathcal{V}$. So $\mathcal{V}_S \leq \mathcal{V}_{T_i}$ for all

$i \in I$. Suppose $\mathcal{V}_T \in \mathbf{ONC}(M)$ and $\mathcal{V}_T \leq \mathcal{V}_{T_i}$ for all $i \in I$. Let $x \in M$. We have $Orb_T(x) \subseteq Orb_{T_i}(x)$ for all $i \in I$. Then $Orb_T(x) \subseteq \bigcap_{i \in I} Orb_{T_i}(x) = V_x$. Since $\mathcal{V}$ is a basis neighborhood collection, $\mathcal{V}_T \leq \mathcal{V}_S$ by Lemma 2.7(ii), as desired. Note that $\mathcal{V}_{\{id_M\}}$ is the bottom element, and $\mathcal{V}_{H(M)}$ is the top element of $\mathbf{ONC}(M)$. $\square$

Remark 2.9. Let $\{T_i : i \in I\}$ be a non-empty family of submonoids of a monoid T and $\Phi : T \to H(M)$ be an action over monoids. Then, using Lemma 2.7 and Theorem 2.8, $\mathcal{V}_{\bigcap_{i \in I} T_i} \leq \bigwedge_{i \in I} \mathcal{V}_{T_i} \leq \mathcal{V}$, where $V_x = \bigcap_{i \in I} Orb_{T_i}(x)$ and $\mathcal{V} = \{V_x : x \in M\}$. But, $\mathcal{V}_{\bigcap_{i \in I} T_i}$ and $\mathcal{V}$ are not necessarily equal. For instance, let $M = (\mathbb{Z}_{100}, \cdot)$ and $T = (\mathbb{N}, \cdot)$. Consider the action $\Phi : T \to H(M)$ over monoids defined by $a\varphi_n := a^n$ for each $n \in \mathbb{N}$ and $a \in \mathbb{Z}_{100}$. Let $T_1 = \{2^k : k \in \mathbb{N} \cup \{0\}\}$ and $T_2 = \{3^k : k \in \mathbb{N} \cup \{0\}\}$. Then T_1 and T_2 are submonoids of T such that $T_1 \cap T_2 = \{1\}$. Let $a = 5 \in \mathbb{Z}_{100}$. We have $a^2 \neq a$, $a^3 \neq a$ and $a^2 = a^3$. So $a^2 \in Orb_{T_1}(a) \cap Orb_{T_2}(a)$, but $a^2 \notin Orb_{T_1 \cap T_2}(a)$. Therefore, $Orb_{T_1 \cap T_2}(a) \neq Orb_{T_1}(a) \cap Orb_{T_2}(a)$.

Question: The map $\psi : Sub(H(M)) \to \mathbf{ONC}(M)$ given by $T \mapsto \mathcal{V}_T$ is a poset homomorphism. Is ψ a lattice homomorphism? Generally: Let $\Phi : S \to H(M)$ be an action over monoids. When is the map $\phi : Sub(S) \to \mathbf{ONC}(M)$, given by $\phi(T) = \mathcal{V}_T$, a lattice homomorphism?

3. Non-unital hypergroup actions over monoids

In this section, we introduce and study the notion of hypergroup action over monoids and construct two kinds of non-unital hypergroup actions over monoids.

Definition 3.1. An action over monoids $\Phi : T \to H(M)$ is called a *hypergroup action over monoids* if $(M, \bullet)$ is a hypergroup, where the hyperoperation $\bullet$ is induced by orbit neighborhood collection.

In view of Lemma 2.2, any action $\Phi : T \to H(M)$ is a hypergroup action over monoids provided M is a group.

Proposition 3.2. *For every monoids T and M, $\Phi : T \to H(M)$ is a hypergroup action over monoids if and only if for every $m \in M$ there exist $s, t \in T$ such that $m\varphi_s$ is right invertible and $m\varphi_t$ is left invertible in M.*

Proof. clearly $\Phi : T \to H(M)$ is a hypergroup action over monoids if and only if for each $m \in M$, $Orb_T(m)M = M = MOrb_T(m)$. Then the assertion holds. $\square$

Example 3.3. Consider the monoid $T = \{0, 1\}$. For a zero faithful action $\Phi : T \to H(M)$ over monoids, $(M, \bullet)$ is a hypergroup by Proposition 3.2. To describe the hyperoperation $\bullet$ induced by $\mathcal{V}_T$, let $x, y \in M$. We have $x \bullet y = Orb_T(x)Orb_T(y) = \{1, x\}\{1, y\} = \{1, x, y, xy\}$. $\square$

Definition 3.4. Let M be a monoid without zero and $\odot$ be a hyperoperation on M. Then $(M, \odot)$ is called *unital* if for every $x, y \in M$, $1 \in x \odot y$. A hypergroup action $\Phi : T \to H(M)$ over monoids is called *unital* if $(M, \bullet)$ is unital, where $\bullet$ is the hyperoperation induced by $\mathcal{V}_T$.

Lemma 3.5. *A hypergroup action $\Phi : T \to H(M)$ over monoids is unital if and only if $1 \in Orb_T(x)$ for any $x \in M$.*

Proof. Let $\Phi : T \to H(M)$ be a hypergroup action over monoids. If Φ is non-unital, then there are $x, y \in M$ such that $1 \notin x \bullet y \supseteq Orb_T(xy)$ which is a contradiction. The converse follows from the fact that $x \bullet 1 = Orb_T(x)$ for each $x \in M$. □

By virtue of Proposition 3.2 and Lemma 3.5, the following is immediate:

Corollary 3.6. *Every zero faithful action is a unital hypergroup action over monoids.* □

Corollary 3.6 provides an easy construction of a unital hypergroup action over monoids. But, finding a non-unital hypergroup action over monoids is not so easy.

Let G be a non-trivial group. For any $* \notin G$, put $G^* := G \cup \{*\}$. Define $*a = a* = *$ for all $a \in G^*$. Then G^* is a monoid in which $*$ is a zero element, and every non-zero element is invertible. For a non-empty set X, let G_X denote the set of all mappings $f : X \to G^*$ satisfying $xf \neq *$ for some $x \in X$. For every $f, g \in G_X$ and $x \in X$, define $(x)fg := (xf)(xg)$. Under this multiplication, G_X is a monoid with no zero element. Also the identity of G_X is the map $\mathbb{I}_{G_X}$ given by $x\mathbb{I}_{G_X} = 1$, for each $x \in X$. Note that $H(G_X)$ has a zero element given by the endomorphism $\mathcal{O} : G_X \to G_X$ such that for any $f \in G_X, f\mathcal{O} = \mathbb{I}_{G_X}$.

Now, take a map $\alpha : X \to X$. Define $\tilde{\alpha} : G_X \to G_X$ by $f\tilde{\alpha} := \alpha f$ and put $T_X := \{\tilde{\alpha} \mid \alpha : X \to X \text{ is a map}\}$. Then we get the following:

Lemma 3.7.

(i) *$\tilde{\alpha} \in H(G_X)$, and T_X is a submonoid of $H(G_X)$ such that $\mathcal{O} \notin T_X$.*

(ii) *An $f \in G_X$ is invertible if and only if for every $x \in X$, $xf \neq *$. In this case, $xf^{-1} = (xf)^{-1}$ for all $x \in X$.*

Proof. (i) For every $g, h \in G_X$, $(gh)\tilde{\alpha} = \alpha(gh) = (\alpha g)(\alpha h) = (g\tilde{\alpha})(h\tilde{\alpha})$. So $\tilde{\alpha}$ is an endomorphism of G_X. To prove T_X is a submonoid of $H(G_X)$, let $\alpha, \beta : X \to X$ be two maps. For every $f \in G_X$ we have $f\tilde{\alpha}\tilde{\beta} = (\alpha f)\tilde{\beta} = \beta(\alpha f) = (\beta\alpha)f = f\widetilde{\beta\alpha}$. Then $\tilde{\alpha}\tilde{\beta} = \widetilde{\beta\alpha} \in T_X$. Also $id_{G_X} = \widetilde{id_X} \in T_X$. Finally, if $\mathcal{O} \in T_X$, then there is a mapping $\alpha : X \to X$ such that $\tilde{\alpha} = \mathcal{O}$. Take an $f \in G_X$ satisfying $xf \neq 1$ for each $x \in X$. Then $\alpha f = f\tilde{\alpha} = f\mathcal{O} = \mathbb{I}_{G_X}$ which is a contradiction.

(ii) Note that $f \in G_X$ is invertible if and only if xf is invertible for all $x \in X$. Since every $a \neq *$ in G^* is invertible, the assertion holds. □

In light of Lemma 3.7(i), we have the natural action $\Phi_X : T_X \to H(G_X)$ over monoids. Now the following result is obtained.

Theorem 3.8. *The natural action $\Phi_X : T_X \to H(G_X)$ is a non-unital hypergroup action over monoids.*

Proof. First we show that for every $f \in G_X$ there exists an endomorphism $\sigma : G_X \to G_X$ such that $\sigma \in T_X$ and $f\sigma$ is invertible. To this end, let $f \in G_X$, $Y = \{x \in X : xf \neq *\} \neq \emptyset$ and $\alpha : X \to X$ be a mapping such that $X\alpha = Y$. Considering $\sigma = \tilde{\alpha} \in T_X$, we have $xf\sigma = xf\tilde{\alpha} = x\alpha f \neq *$ for each $x \in X$. Then it follows from Lemma 3.7(ii) that $f\sigma$ is invertible. Now, using Proposition 3.2, Φ_X is a hypergroup action over monoids. To complete the proof, using Lemma 3.5, it suffices to find an $f \in G_X$ such that $\mathbb{I}_{G_X} \notin Orb_{T_X}(f)$. Take any $a \in G$ such that $a \neq 1$, and the constant map f corresponding to a such that $xf = a$ for all $x \in X$. For every map $\alpha : X \to X$ and $x \in X$, $xf\tilde{\alpha} = x\alpha f = a$. Thus $f\tilde{\alpha} \neq \mathbb{I}_{G_X}$, and hence $\mathbb{I}_{G_X} \notin Orb_{T_X}(f)$. □

Let M be a monoid and G be a non-trivial group. Then $M \times G$ is a monoid without zero under the usual componentwise binary operation. Define $\xi : M \times G \to M \times G$ by $(m, g)\xi = (1, g)$ for every $m \in M, g \in G$. Clearly, ξ is an endomorphism of $M \times G$ such that for every $x \in M \times G$, $x\xi$ is invertible, and $\xi^2 = \xi$. Now we have the following:

Proposition 3.9. *If T is a submonoid of $H(M \times G)$ without zero that contains ξ, then the natural action $\Phi : T \to H(M \times G)$ is a hypergroup action over monoids. In particular, $\Phi_\xi : T_\xi \to H(M \times G)$ is a non-unital hypergroup action over monoids, where $T_\xi = \{id, \xi\}$.*

Proof. We have $\xi \in T$ and $x\xi$ is invertible for all $x \in M \times G$. It follows from Proposition 3.2 that $\Phi : T \to H(M \times G)$ is a hypergroup action over monoids. Consider T_ξ, and let $x = (1, g) \in M \times G$ such that $g \neq 1$. Then we get $(1, 1) \notin Orb_{T_\xi}(x) = \{x\}$. Using Lemma 3.5, Φ_ξ is a non-unital hypergroup action over monoids. □

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