Quasigroups and Related Systems **23** (2015), 243 − 256

Subquasigroups in the framework of fuzzy points

Young Bae Jun, Seok Zun Song and Ghulam Muhiuddin

Abstract. A relation between $(\in, \in\vee q)$-fuzzy subquasigroups and $(q, \in\vee q)$-fuzzy subquasigroups is provided, and conditions for an $(\in, \in\vee q)$-fuzzy subquasigroup to be a $(q, \in\vee q)$-fuzzy subquasigroup are considered. Conditions for the t-q-set (resp., the t-$\in\vee q$-set) to be a subquasigroup are provided. The notion of (ε, δ)-characteristic fuzzy sets is introduced. Given a subquasigroup S of a quasigroup $\mathcal{Q}$, conditions for the (ε, δ)-characteristic fuzzy set in $\mathcal{Q}$ to be an $(\in, \in\vee q)$-fuzzy subquasigroup, an $(\in, q)$-fuzzy subquasigroup, an $(\in, \in\wedge q)$-fuzzy subquasigroup, a (q, q)-fuzzy subquasigroup, a $(q, \in)$-fuzzy subquasigroup, a $(q, \in\vee q)$-fuzzy subquasigroup and a $(q, \in\wedge q)$-fuzzy subquasigroup are provided. Using the notions of (α, β)-fuzzy subquasigroup $\mu_S^{(\varepsilon,\delta)}$, conditions for the S to be a subquasigroup of $\mathcal{Q}$ are investigated where (α, β) is one of $(\in, \in\vee q)$, $(\in, \in\wedge q)$, $(\in, q)$, $(q, \in\vee q)$, $(q, \in\wedge q)$, $(q, \in)$ and (q, q).

1. Introduction

Quasigroups has useful applications in cryptography, physics and geometry etc. In mathematics, especially in abstract algebra, a quasigroup is an algebraic structure resembling a group in the sense that "division" is always possible. Quasigroups differ from groups mainly in that they need not be associative. The fuzzy subquasigroup of a quasigroup is studied by W. A. Dudek in the paper [3]. M. Akram and W. A. Dudek [1] introduced the notion of (α, β)-fuzzy subquasigroups where α, $\beta \in \{\in,\ q\,,\ \in\vee q\,,\ \in\wedge q\,\}$ and $\alpha \neq\ \in\wedge q$, and investigated some related properties. They characterized $(\in, \in\vee q\,)$-fuzzy subquasigroups by their level subquasigroups, and studied fuzzy subquasigroups with thresholds.

In this paper, we discuss a relation between $(\in, \in\vee q\,)$-fuzzy subquasigroups and $(q, \in\vee q\,)$-fuzzy subquasigroups, and provide conditions for an $(\in, \in\vee q\,)$-fuzzy subquasigroup to be a $(q, \in\vee q\,)$-fuzzy subquasigroup. We consider conditions for the t-q-set (resp., the t-$\in\vee q$-set) to be a subquasigroup. We introduce the notion of (ε, δ)-characteristic fuzzy sets in quasigroups. Given a subquasigroup S of a quasigroup $\mathcal{Q}$, we provide conditions for the (ε, δ)-characteristic fuzzy set in $\mathcal{Q}$ to be an $(\in, \in\vee q)$-fuzzy subquasigroup, an $(\in, q)$-fuzzy subquasigroup, an $(\in, \in\wedge q)$-fuzzy subquasigroup, a (q, q)-fuzzy subquasigroup, a $(q, \in)$-fuzzy subquasigroup, a $(q, \in\vee q)$-fuzzy subquasigroup and a $(q, \in\wedge q)$-fuzzy subquasigroup. Using the notions of (α, β)-fuzzy subquasigroup $\mu_S^{(\varepsilon,\delta)}$, we investigate conditions for the S

2010 Mathematics Subject Classification: 20N15, 94D05
Keywords: (fuzzy) subquasigroup, (α, β)-fuzzy subquasigroup, t-q-set, t-$\in\vee q$-set.

to be a subquasigroup of $\mathcal{Q}$ where (α, β) is one of $(\in, \in\vee q)$, $(\in, \in\wedge q)$, $(\in, q)$, $(q, \in\vee q)$, $(q, \in\wedge q)$, $(q, \in)$ and (q, q).

2. Preliminaries

A *quasigroup* $(Q, \cdot)$ is a set Q with a binary operation "$\cdot$" such that for each a and b in Q there exist unique elements x and y in Q such that $a \cdot x = b$ and $y \cdot a = b$. The unique solutions to these equations are denoted by $x = a\backslash b$ and $y = b/a$. The operations "$\backslash$" and "$/$" denote the defined binary opersations of left and right division (sometimes called *parastrophe*), respectively. This axiomatization of quasigroups requires existential quantification and hence first order logic. The second definition of a quasigroup is grounded in universal algebra, which prefers that algebraic structures be varieties, i.e., that structures be axiomatized solely by identities. An identity is an equation in which all variables are tacitly universally quantified, and the only operations are the primitive operations proper to the structure. Quasigroups can be axiomatized in this manner if left and right division are taken as primitive.

A quasigroup $\mathcal{Q} = (Q, \cdot, \backslash, /)$ is a type $(2, 2, 2)$ algebra satisfying the identities:

$$(x \cdot y)/y = x,\ x\backslash(x \cdot y) = y,\ (x/y) \cdot y = x,\ x \cdot (x\backslash y) = y$$

(cf. [2] or [4]). Hence if $(Q, \cdot)$ is a quasigroup according to the first definition, then $\mathcal{Q} = (Q, \cdot, \backslash, /)$ is an equivalent quasigroup in the universal algebra sense. We say also that $(Q, \cdot, \backslash, /)$ is an *equasigroup* (i.e. equationally definable quasigroup) [4] or a *primitive quasigroup* [2]. The equasigroup $\mathcal{Q} = (Q, \cdot, \backslash, /)$ corresponds to quasigroup $(Q, \cdot)$ where

$$x\backslash y = z \iff x \cdot z = y, \quad x/y = z \iff z \cdot y = x.$$

A nonempty subset S of a quaisgroup $\mathcal{Q} = (Q, \cdot, \backslash, /)$ is called a *subquasigroup* of $\mathcal{Q}$ if it is closed with respect to these three operations, i.e., $x * y \in S$ for all $x, y \in S$ and $* \in \{\cdot, \backslash, /\}$.

A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [5] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$. To say that $x_t \in \mu$ (resp. $x_t\ q\ \mu$), we mean $\mu(x) \geqslant t$ (resp. $\mu(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $x_t \in\vee q\ \mu$ (resp. $x_t \in\wedge q\ \mu$), we mean $x_t \in \mu$ or $x_t\, q\, \mu$ (resp. $x_t \in \mu$ and $x_t\, q\, \mu$). To say that $x_t\, \overline{\alpha}\, \mu$, we mean $x_t \alpha \mu$ does not hold, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

Definition 2.1. ([3, Definition 3.2]) A fuzzy set μ in a quasigroup $\mathcal{Q}$ is called a *fuzzy subquasigroup* of $\mathcal{Q}$ if it satisfies:

$$\mu(x * y) \geqslant \min\{\mu(x), \mu(y)\} \tag{1}$$

for all $x, y \in Q$ and $* \in \{\cdot, \backslash, /\}$.

We have the following characterization of a fuzzy subquasigroup.

Proposition 2.2. *Let $\mathcal{Q}$ be a quasigroup. A fuzzy set μ in $\mathcal{Q}$ is a fuzzy subquasigroup of $\mathcal{Q}$ if and only if the following assertion is valid.*

$$x_t \in \mu,\ y_s \in \mu \implies (x * y)_{\min\{t,s\}} \in \mu \tag{2}$$

for all $x, y \in Q$, $t, s \in (0, 1]$ and $* \in \{\cdot, \backslash, /\}$.

Proof. Straightforward. □

3. Subquasigroups in the framework of (α, β)-type fuzzy sets

In what follows, let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a quasigroup unless otherwise specified.

Definition 3.1. ([1, Definition 3.1]) A fuzzy set μ in Q is said to be an (α, β)-*fuzzy subquasigroup* of $\mathcal{Q}$, where $\alpha, \beta \in \{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha \neq \in\wedge q$, if it satisfies the following condition:

$$x_{t_1} \alpha\, \mu,\ y_{t_2} \alpha\, \mu \Rightarrow (x * y)_{\min\{t_1,t_2\}}\, \beta\, \mu. \tag{3}$$

for all $x, y \in Q$, $t_1, t_2 \in (0, 1]$ and $* \in \{\cdot, \backslash, /\}$.

Lemma 3.2. ([1, Theorem 3.13]) *A fuzzy set μ in $\mathcal{Q}$ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$ if and only if it satisfies:*

$$(\forall x, y \in Q)\, (\mu(x * y) \geqslant \min\{\mu(x), \mu(y), 0.5\}) \tag{4}$$

where $* \in \{\cdot, \backslash, /\}$.

We know that there are twelve different types of (α, β)-fuzzy subquasigroups in Q, that is, (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in\wedge q)$, $(\in, \in\vee q)$, $(q, \in)$, (q, q), $(q, \in\wedge q)$, $(q, \in\vee q)$, $(\in\vee q, \in)$, $(\in\vee q, q)$, $(\in\vee q, \in\wedge q)$, and $(\in\vee q, \in\vee q)$. Clearly, we have relations among these types which are described in the following diagrams.

$$\begin{tikzcd}
(\in,\in) \arrow[dr, Rightarrow] & (\in,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\in,q) \arrow[dl, Rightarrow] \\
& (\in,\in\vee q) & \\
& (\in\vee q,\in\vee q) \arrow[u, Rightarrow] & \\
(\in\vee q,\in) \arrow[ur, Rightarrow] & (\in\vee q,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[u, Rightarrow] & (\in\vee q,q) \arrow[ul, Rightarrow]
\end{tikzcd} \tag{5}$$

and

$$\begin{tikzcd}
(q,\in) \arrow[dr, Rightarrow] & (q,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (q,q) \arrow[dl, Rightarrow] \\
& (q,\in\vee q) &
\end{tikzcd} \tag{6}$$

If there exists $x \in Q$ such that $\mu(x) > 0.5$, then we have the following relation:

$$\begin{tikzcd}
(\in\wedge q,\in) \arrow[dr, Rightarrow] & (\in\wedge q,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\in\wedge q,\, q) \arrow[dl, Rightarrow] \\
& (\in\wedge q,\in\vee q) & \\
& (\in,\in\vee q) \arrow[u, Rightarrow] &
\end{tikzcd} \tag{7}$$

We provide a relation between $(\in, \in\vee q)$-fuzzy subquasigroups and $(q, \in\vee q)$-fuzzy subquasigroups.

Theorem 3.3. *Every $(q, \in\vee q)$-fuzzy subquasigroup is an $(\in, \in\vee q)$-fuzzy subquasigroup.*

Proof. Let μ be a $(q, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$. Let $* \in \{\cdot, \backslash, /\}$ and let $x, y \in Q$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geqslant t_1$ and $\mu(y) \geqslant t_2$. Suppose $(x * y)_{\min\{t_1,t_2\}} \overline{\in\vee q}\, \mu$. Then

$$\mu(x * y) < \min\{t_1, t_2\}, \tag{8}$$

$$\mu(x * y) + \min\{t_1, t_2\} \leqslant 1. \tag{9}$$

It follows that

$$\mu(x * y) < 0.5. \tag{10}$$

and from (8) and (10) that

$$\mu(x * y) < \min\{t_1, t_2, 0.5\}.$$

Thus

$$1 - \mu(x * y) > 1 - \min\{t_1, t_2, 0.5\} \geqslant \max\{1 - \mu(x), 1 - \mu(y), 0.5\},$$

and so there exists $\delta \in (0, 1]$ such that

$$1 - \mu(x * y) \geqslant \delta > \max\{1 - \mu(x), 1 - \mu(y), 0.5\}. \tag{11}$$

The right inequality in (11) induces $\mu(x) + \delta > 1$ and $\mu(y) + \delta > 1$, that is, $x_\delta \, q \, \mu$ and $y_\delta \, q \, \mu$. Since μ is a $(q, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$, it follows that $(x * y)_\delta = (x * y)_{\min\{\delta,\delta\}} \in\vee q \, \mu$. But, from the left inequality in (11), we get $\mu(x * y) + \delta \leqslant 1$, that is, $(x * y)_\delta \, \overline{q} \, \mu$, and $\mu(x * y) \leqslant 1 - \delta < 1 - 0.5 = 0.5 < \delta$, i.e., $(x*y)_\delta \, \overline{\in} \, \mu$. Hence $(x*y)_\delta \, \overline{\in\vee q} \, \mu$, a contradiction. Therefore $(x*y)_{\min\{t_1,t_2\}} \in\vee q \, \mu$, and thus μ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$. □

Regarding (α, β)-fuzzy subquasigroups, Theorem 3.3 and figure (6) induces the the following relations.

$$\begin{tikzcd}
(q, \in) & (q, \in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (q, q) \arrow[dl, Rightarrow] \\
 & (q, \in\vee q) \arrow[d, Rightarrow] & \\
 & (\in, \in\vee q) &
\arrow[from=1-1, to=2-2, Rightarrow]
\end{tikzcd} \tag{12}$$

The converse of Theorem 3.3 is not true in general (see [1, Example 3.6]).

We provide conditions for an $(\in, \in\vee q)$-fuzzy subquasigroup to be a $(q, \in\vee q)$-fuzzy subquasigroup.

Theorem 3.4. *If μ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$ in which $\mu(x) \leqslant 0.5$ for all $x \in Q$, then μ is a $(q, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$.*

Proof. Let μ be an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$ such that $\mu(x) \leqslant 0.5$ for all $x \in Q$. Let $* \in \{\cdot, \backslash, /\}$, $x, y \in Q$ and $t_1, t_2 \in (0, 0.5]$ be such that $x_{t_1} \, q \, \mu$ and $y_{t_2} \, q \, \mu$. Then $\mu(x) > 1 - t_1 \geqslant t_1$ and $\mu(y) > 1 - t_2 \geqslant t_2$, that is, $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Since μ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$, it follows that $(x * y)_{\min\{t_1,t_2\}} \in\vee q \, \mu$. Consequently, μ is a $(q, \in\vee q)$-fuzzy subquasigroup. □

The figure (5) and Theorem 3.4 induces the following corollary.

Corollary 3.5. *Let μ be an (α,β)-fuzzy subquasigroup of $\mathcal{Q}$ where (α,β) is any one of* $(\in,\in)$, $(\in,q)$, $(\in,\in\wedge q)$, $(\in\vee q,\in)$, $(\in\vee q,q)$, $(\in\vee q,\in\wedge q)$, *and* $(\in\vee q,\in\vee q)$. *If every fuzzy point has the value t in* $(0,0.5]$, *then μ is a $(q,\in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$.* □

For a fuzzy set μ in $\mathcal{Q}$ and $t\in(0,1]$, consider the q-set and $\in\vee q$-set with respect to t (briefly, t-q-set and t-$\in\vee q$-set, respectively) as follows:

$$\mathcal{Q}_q^t := \{x\in Q \mid x_t\, q\, \mu\} \text{ and } \mathcal{Q}_{\in\vee q}^t := \{x\in Q \mid x_t \in\vee q\, \mu\}.$$

Note that, $\mathcal{Q}_q^r \subseteq \mathcal{Q}_q^t$ and $\mathcal{Q}_{\in\vee q}^r \subseteq \mathcal{Q}_{\in\vee q}^t$ for all $t,r\in(0,1]$ with $t\geqslant r$. Obviously, $\mathcal{Q}_{\in\vee q}^t = U(\mu;t)\cup\mathcal{Q}_q^t$, where

$$U(\mu;t) := \{x\in Q \mid \mu(x)\geqslant t\}.$$

Theorem 3.6. *If μ is an $(\in,\in)$-fuzzy subquasigroup of $\mathcal{Q}$, then the t-q-set $\mathcal{Q}_q^t$ is a subquasigroup of $\mathcal{Q}$ for all $t\in(0,1]$ whenever it is nonempty.*

Proof. Let $*\in\{\cdot,\backslash,/\}$ and $x,y\in\mathcal{Q}_q^t$. Then $x_t\, q\, \mu$ and $y_t\, q\, \mu$, that is, $\mu(x)+t>1$ and $\mu(y)+t>1$. It follows that

$$\mu(x*y)+t \geqslant \min\{\mu(x),\mu(y)\}+t = \min\{\mu(x)+t,\mu(y)+t\}>1$$

and so that $(x*y)_t\, q\, \mu$. Hence $x*y\in\mathcal{Q}_q^t$, and therefore $\mathcal{Q}_q^t$ is a subquasigroup. □

Corollary 3.7. *If μ is an $(\in,\in\wedge q)$-fuzzy subquasigroup of $\mathcal{Q}$, then the t-q-set $\mathcal{Q}_q^t$ is a subquasigroup of $\mathcal{Q}$ for all $t\in(0,1]$ whenever it is nonempty.* □

Theorem 3.8. *If μ is a $(q,\in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$, then the t-q-set $\mathcal{Q}_q^t$ and the t-$\in\vee q$-set $\mathcal{Q}_{\in\vee q}^t$ are subquasigroups of $\mathcal{Q}$ for all $t\in(0.5,1]$ whenever it is nonempty.*

Proof. Let $*\in\{\cdot,\backslash,/\}$ and μ a $(q,\in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$. Let $x,y\in Q$ be such that $x\in\mathcal{Q}_q^t$ and $y\in\mathcal{Q}_q^t$ for all $t\in(0.5,1]$. Then $x_t\, q\, \mu$ and $y_t\, q\, \mu$, which imply that $(x*y)_t\in\vee q\, \mu$, i.e., $(x*y)_t\in\mu$ or $(x*y)_t\, q\, \mu$. If $(x*y)_t\, q\, \mu$, then $x*y\in\mathcal{Q}_q^t$. If $(x*y)_t\in\mu$, then $\mu(x*y)\geqslant t>1-t$ since $t>0.5$. Hence $(x*y)_t\, q\, \mu$, that is, $x*y\in\mathcal{Q}_q^t$. Therefore $\mathcal{Q}_q^t$ is a subquasigroup of $\mathcal{Q}$. Now, let $x,y\in\mathcal{Q}_{\in\vee q}^t$. Then $x_t\in\vee q\, \mu$ and $y_t\in\vee q\, \mu$. Hence we have the following four cases:

(i) $x_t\in\mu$ and $y_t\in\mu$,

(ii) $x_t\in\mu$ and $y_t\, q\, \mu$,

(iii) $x_t\, q\, \mu$ and $y_t\in\mu$,

(iv) $x_t\, q\, \mu$ and $y_t\, q\, \mu$.

For the first case, we have $\mu(x) + t \geqslant 2t > 1$ and $\mu(y) + t \geqslant 2t > 1$, that is, $x_t \, q \, \mu$ and $y_t \, q \, \mu$. It follows that $(x * y)_t \in \vee q \, \mu$ and so that $x * y \in \mathcal{Q}^t_{\in \vee q}$. In the case (ii), $x_t \in \mu$ implies $\mu(x) + t \geqslant 2t > 1$, that is, $x_t \, q \, \mu$. Hence $(x * y)_t \in \vee q \, \mu$ and so $x * y \in \mathcal{Q}^t_{\in \vee q}$. Similarly, the third case implies $x * y \in \mathcal{Q}^t_{\in \vee q}$. The last case implies $(x * y)_t \in \vee q \, \mu$ and so $x * y \in \mathcal{Q}^t_{\in \vee q}$. Consequently, t-$\in \vee q$-set $\mathcal{Q}^t_{\in \vee q}$ is a subquasigroup of $\mathcal{Q}$ for all $t \in (0.5, 1]$. □

Corollary 3.9. *If μ is any one of a $(q, \in)$-fuzzy subquasigroup, a $(q, \in \wedge q)$-fuzzy subquasigroup and a (q, q)-fuzzy subquasigroup of $\mathcal{Q}$, then the t-q-set $\mathcal{Q}^t_q$ and the t-$\in \vee q$-set $\mathcal{Q}^t_{\in \vee q}$ are subquasigroups of $\mathcal{Q}$ for all $t \in (0.5, 1]$ whenever it is nonempty.*

Proof. It follows from the figure (6) and Theorem 3.8. □

Lemma 3.10. ([1, Theorem 3.12]) *For a subquasigroup S of $\mathcal{Q}$, let μ be a fuzzy set in $\mathcal{Q}$ such that*

(1) $\mu(x) \geqslant 0.5$ *for all* $x \in S$,

(2) $\mu(x) = 0$ *for all* $x \in Q \setminus S$.

Then μ is a $(q, \in \vee q)$-fuzzy subquasigroup of $\mathcal{Q}$.

Using Theorem 3.8 and Lemma 3.10, we have the following result.

Theorem 3.11. *For a subquasigroup S of $\mathcal{Q}$, if μ is a fuzzy set in $\mathcal{Q}$ such that*

(1) $\mu(x) \geqslant 0.5$ *for all* $x \in S$,

(2) $\mu(x) = 0$ *for all* $x \in Q \setminus S$,

then the nonempty t-q-set $\mathcal{Q}^t_q$ and the t-$\in \vee q$-set $\mathcal{Q}^t_{\in \vee q}$ are subquasigroups of $\mathcal{Q}$ for all $t \in (0.5, 1]$. □

Theorem 3.12. *If μ is an $(\in, \in \vee q)$-fuzzy subquasigroup of $\mathcal{Q}$, then the nonempty t-q-set Q^t_q is a subquasigroup of $\mathcal{Q}$ for all $t \in (0.5, 1]$.*

Proof. Let $* \in \{\cdot, \backslash, /\}$. Assume that $Q^t_q \neq \emptyset$ for all $t \in (0.5, 1]$. Let $x, y \in Q^t_q$. Then $x_t \, q \, \mu$ and $y_t \, q \, \mu$, that is, $\mu(x) + t > 1$ and $\mu(y) + t > 1$. It follows from Lemma 3.2 that

$$\begin{aligned} \mu(x * y) + t &\geqslant \min\{\mu(x), \mu(y), 0.5\} + t \\ &= \min\{\mu(x) + t, \mu(y) + t, 0.5 + t\} \\ &> 1. \end{aligned}$$

So $(x * y)_t \, q \, \mu$. Hence $x * y \in Q^t_q$, and therefore Q^t_q is a subquasigroup. □

In what follows, let $\varepsilon, \delta \in [0,1]$ be such that $\varepsilon > \delta$ unless otherwise specified. For a nonempty subset S of $\mathcal{Q}$, define a fuzzy set $\mu_S^{(\varepsilon,\delta)}$ in $\mathcal{Q}$ as follows:

$$\mu_S^{(\varepsilon,\delta)}(x) := \begin{cases} \varepsilon & \text{if } x \in S, \\ \delta & \text{otherwise.} \end{cases}$$

We say that $\mu_S^{(\varepsilon,\delta)}$ is an (ε,δ)*-characteristic fuzzy set* in $\mathcal{Q}$ over S. In particular, the $(1,0)$-characteristic fuzzy set $\mu_S^{(1,0)}$ in $\mathcal{Q}$ over S is the characteristic function χ_S of S.

Theorem 3.13. *For any nonempty subset S of $\mathcal{Q}$, the following are equivalent:*

(1) *S is a subquasigroup of $\mathcal{Q}$.*

(2) *The (ε,δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a fuzzy subquasigroup of $\mathcal{Q}$.*

Proof. Assume that S is a subquasigroup of $\mathcal{Q}$ and let $x, y \in Q$. If $x, y \in S$, then $x * y \in S$ and so

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon = \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y)\right\}.$$

If $x \notin S$ or $y \notin S$, then $\mu_S^{(\varepsilon,\delta)}(x) = \delta$ or $\mu_S^{(\varepsilon,\delta)}(y) = \delta$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x * y) \geqslant \delta = \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y)\right\}.$$

Therefore $\mu_S^{(\varepsilon,\delta)}$ is a fuzzy subquasigroup of $\mathcal{Q}$.

Conversely, suppose that $\mu_S^{(\varepsilon,\delta)}$ is a fuzzy subquasigroup of $\mathcal{Q}$. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(y) = \varepsilon$. It follows that

$$\mu_S^{(\varepsilon,\delta)}(x * y) \geq \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y)\right\} = \varepsilon.$$

Thus $x * y \in S$, and therefore S is a subquasigroup of $\mathcal{Q}$. □

Theorem 3.14. *If S is a subquasigroup of $\mathcal{Q}$, then the (ε,δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\!\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$.*

Proof. Assume that S is a subquasigroup of $\mathcal{Q}$ and let $x, y \in Q$. If $x, y \in S$, then $x * y \in S$ and so

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geqslant \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), 0.5\right\}.$$

If $x \notin S$ or $y \notin S$, then $\mu_S^{(\varepsilon,\delta)}(x) = \delta$ or $\mu_S^{(\varepsilon,\delta)}(y) = \delta$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x * y) \geqslant \delta \geqslant \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), 0.5\right\}.$$

It follows from Lemma 3.2 that $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\!\vee q)$-fuzzy subquasigroup. □

In order to consider the converse of Theorem 3.14, we need additional conditions.

Theorem 3.15. *For any nonempty subset S of $\mathcal{Q}$, if $\delta < 0.5$ and the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$, then S is a subquasigroup of $\mathcal{Q}$.*

Proof. Assume that $\delta < 0.5$ and the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$. Using Lemma 3.2, we have

$$\begin{aligned}\mu_S^{(\varepsilon,\delta)}(x*y) &\geqslant \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), 0.5\right\}\\ &= \min\{\varepsilon, 0.5\}\\ &= \begin{cases} 0.5 & \text{if } \varepsilon \geqslant 0.5,\\ \varepsilon & \text{otherwise,}\end{cases}\end{aligned}$$

and so that $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$. Thus $x*y \in S$, and S is a subquasigroup of $\mathcal{Q}$. □

Corollary 3.16. *A nonempty subset S of $\mathcal{Q}$ is a subquasigroup of $\mathcal{Q}$ if and only if the characteristic function χ_S of S is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$.* □

Theorem 3.17. *If S is a subquasigroup of $\mathcal{Q}$, then the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy subquasigroup of $\mathcal{Q}$ whenever if any element t in $(0,1]$ satisfies $x_t \in \mu_S^{(\varepsilon,\delta)}$ for $x \in Q$ then $\delta < t$ and $1 - t < \varepsilon$.*

Proof. Let $* \in \{\cdot, \backslash, /\}$ and let $x, y \in Q$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1} \in \mu_S^{(\varepsilon,\delta)}$ and $y_{t_2} \in \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) \geqslant t_1 > \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) \geqslant t_2 > \delta$. It follows that $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x, y \in S$. Since S is a subquasigroup of $\mathcal{Q}$, we have $x*y \in S$. Hence $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$, and thus $\mu_S^{(\varepsilon,\delta)}(x*y) + \min\{t_1, t_2\} = \varepsilon + \min\{t_1, t_2\} > 1$ which shows that $(x*y)_{\min\{t_1,t_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Therefore $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy subquasigroup of $\mathcal{Q}$. □

Corollary 3.18. *If S is a subquasigroup of $\mathcal{Q}$, then the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$ whenever if any element t in $(0,1]$ satisfies $x_t \in \mu_S^{(\varepsilon,\delta)}$ for $x \in Q$ then $\delta < t$ and $1 - t < \varepsilon$.* □

Theorem 3.19. *Let S be a nonempty subset of $\mathcal{Q}$. If $\varepsilon + \delta \leqslant 1$ and the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy subquasigroup of $\mathcal{Q}$, then S is a subquasigroup of $\mathcal{Q}$.*

Proof. Let $* \in \{\cdot, \backslash, /\}$. Assume that $\varepsilon + \delta \leqslant 1$ and the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy subquasigroup of $\mathcal{Q}$. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}}\, q\, \mu_S^{(\varepsilon,\delta)}$, which implies that $\mu_S^{(\varepsilon,\delta)}(x*y)+\varepsilon>1$. Therefore $\mu_S^{(\varepsilon,\delta)}(x*y)>1-\varepsilon\geqslant\delta$, and thus $\mu_S^{(\varepsilon,\delta)}(x*y)=\varepsilon$, that is, $x*y\in S$. Consequently, S is a subquasigroup. □

Corollary 3.20. *Let S be a nonempty subset of $\mathcal{Q}$. If $\varepsilon+\delta\leqslant 1$ and the (ε,δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in,\in\wedge q)$-fuzzy subquasigroup of $\mathcal{Q}$, then S is a subquasigroup of $\mathcal{Q}$.* □

If we take $\varepsilon=1$ and $\delta=0$ in Theorems 3.17 and 3.19, then we have the following corollary.

Corollary 3.21. *A nonempty subset S of $\mathcal{Q}$ is a subquasigroup of $\mathcal{Q}$ if and only if the characteristic function χ_S of S is an $(\in,q)$-fuzzy subquasigroup of $\mathcal{Q}$.* □

Theorem 3.22. *If S is a subquasigroup of $\mathcal{Q}$, then the (ε,δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a (q,q)-fuzzy subquasigroup of $\mathcal{Q}$ whenever if any element t in $(0,1]$ satisfies $x_t\in\mu_S^{(\varepsilon,\delta)}$ for $x\in Q$ then $\delta\leqslant 1-t<\varepsilon$.*

Proof. Let $*\in\{\cdot,\backslash,/\}$. Let $x,y\in Q$ and $t_1,t_2\in(0,1]$ be such that $x_{t_1}\,q\,\mu_S^{(\varepsilon,\delta)}$ and $y_{t_2}\,q\,\mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x)+t_1>1$ and $\mu_S^{(\varepsilon,\delta)}(y)+t_2>1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x)>1-t_1\geqslant\delta$ and $\mu_S^{(\varepsilon,\delta)}(y)>1-t_2\geqslant\delta$. It follows that $\mu_S^{(\varepsilon,\delta)}(x)=\varepsilon=\mu_S^{(\varepsilon,\delta)}(y)$ and so that $x,y\in S$. Since S is a subquasigroup of $\mathcal{Q}$, we have $x*y\in S$ and so $\mu_S^{(\varepsilon,\delta)}(x*y)=\varepsilon$. Thus

$$\mu_S^{(\varepsilon,\delta)}(x*y)+\min\{t_1,t_2\}=\varepsilon+\min\{t_1,t_2\}>1,$$

that is, $(x*y)_{\min\{t_1,t_2\}}\,q\,\mu_S^{(\varepsilon,\delta)}$. This shows that $\mu_S^{(\varepsilon,\delta)}$ is a (q,q)-fuzzy subquasigroup. □

Corollary 3.23. *If S is a subquasigroup of $\mathcal{Q}$, then the (ε,δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q,\in\vee q)$-fuzzy subquasigroup of $\mathcal{Q}$ whenever if any element t in $(0,1]$ satisfies $x_t\in\mu_S^{(\varepsilon,\delta)}$ for $x\in Q$ then $\delta\leqslant 1-t<\varepsilon$.* □

Theorem 3.24. *Let S be a nonempty subset of $\mathcal{Q}$. Assume that $\varepsilon>\max\{\delta,0.5\}$ and $\varepsilon+\delta\leqslant 1$. If the (ε,δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a (q,q)-fuzzy subquasigroup of $\mathcal{Q}$, then S is a subquasigroup of $\mathcal{Q}$.*

Proof. Let $*\in\{\cdot,\backslash,/\}$ and let $x,y\in S$. Then $\mu_S^{(\varepsilon,\delta)}(x)=\varepsilon=\mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x)+\varepsilon=\varepsilon+\varepsilon>1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y)+\varepsilon=\varepsilon+\varepsilon>1,$$

that is, $x_\varepsilon\,q\,\mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon\,q\,\mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a (q,q)-fuzzy subquasigroup of $\mathcal{Q}$, it follows that $(x*y)_\varepsilon=(x*y)_{\min\{\varepsilon,\varepsilon\}}\,q\,\mu_S^{(\varepsilon,\delta)}$. Hence $\mu_S^{(\varepsilon,\delta)}(x*y)>1-\varepsilon\geqslant\delta$, and therefore $\mu_S^{(\varepsilon,\delta)}(x*y)=\varepsilon$. This proves that $x*y\in S$, and S is a subquasigroup. □

Corollary 3.25. *Let S be a nonempty subset of $\mathcal{Q}$. Assume that $\varepsilon > \max\{\delta, 0.5\}$ and $\varepsilon + \delta \leqslant 1$. If the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in \wedge\, q\,)$-fuzzy subquasigroup of $\mathcal{Q}$, then S is a subquasigroup.* □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.22 and 3.24, then we have the following corollary.

Corollary 3.26. *A nonempty subset S of $\mathcal{Q}$ is a subquasigroup of $\mathcal{Q}$ if and only if the characteristic function χ_S of S is a (q, q)-fuzzy subquasigroup of $\mathcal{Q}$.* □

Theorem 3.27. *For any nonempty subset S of $\mathcal{Q}$ and the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$, assume that if any element t in $(0, 1]$ satisfies $x_t \in \mu_S^{(\varepsilon,\delta)}$ for $x \in Q$ then $\delta \leqslant 1 - t$ and $t < \varepsilon$. If S is a subquasigroup of $\mathcal{Q}$, then $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subquasigroup of $\mathcal{Q}$.*

Proof. Let $* \in \{\cdot, \backslash, /\}$. Let $x, y \in Q$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1}\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_{t_2}\, q\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + t_1 > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + t_2 > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1 - t_1 \geqslant \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1 - t_2 \geqslant \delta$. Hence $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x, y \in S$. Since S is a subquasigroup of $\mathcal{Q}$, we have $x * y \in S$ and thus

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geqslant \min\{t_1, t_2\},$$

that is, $(x * y)_{\min\{t_1,t_2\}} \in \mu_S^{(\varepsilon,\delta)}$. This shows that $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subquasigroup of $\mathcal{Q}$. □

Corollary 3.28. *For any nonempty subset S of $\mathcal{Q}$ and the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$, assume that if any element t in $(0, 1]$ satisfies $x_t \in \mu_S^{(\varepsilon,\delta)}$ for $x \in Q$ then $\delta \leqslant 1 - t$ and $t < \varepsilon$. If S is a subquasigroup of $\mathcal{Q}$, then $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in \vee\, q\,)$-fuzzy subquasigroup of $\mathcal{Q}$.* □

Theorem 3.29. *Let S be a nonempty subset of $\mathcal{Q}$. Assume that $\varepsilon > \max\{\delta, 0.5\}$. If the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subquasigroup of $\mathcal{Q}$, then S is a subquasigroup of $\mathcal{Q}$.*

Proof. Let $* \in \{\cdot, \backslash, /\}$ and let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subquasigroup of $\mathcal{Q}$, it follows that $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in \mu_S^{(\varepsilon,\delta)}$ and so that $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$, that is, $x * y \in S$. Therefore S is a subquasigroup of $\mathcal{Q}$. □

Corollary 3.30. *Let S be a nonempty subset of $\mathcal{Q}$. Assume that $\varepsilon > \max\{\delta, 0.5\}$. If the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in \wedge\, q\,)$-fuzzy subquasigroup of $\mathcal{Q}$, then S is a subquasigroup of $\mathcal{Q}$.* □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.27 and 3.29, then we have the following corollary.

Corollary 3.31. *A nonempty subset S of Q is a subquasigroup of Q if and only if the characteristic function χ_S of S is a $(q, \in)$-fuzzy subquasigroup of Q.* □

Theorem 3.32. *If S is a subquasigroup of Q, then the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy subquasigroup of Q whenever if any element t in $(0, 1]$ satisfies $x_t \in \mu_S^{(\varepsilon,\delta)}$ for $x \in Q$ then $\delta < t$ and $1 - t < \varepsilon$.*

Proof. Let $x, y \in Q$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu_S^{(\varepsilon,\delta)}$ and $y_{t_2} \in \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) \geqslant t_1 > \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) \geqslant t_2 > \delta$, which imply that $x, y \in S$ and $\varepsilon \geqslant \min\{t_1, t_2\}$. Since S is a subquasigroup of Q, we have $x*y \in S$. Hence $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon \geqslant \min\{t_1, t_2\}$, i.e., $(x*y)_{\min\{t_1,t_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Now, $\mu_S^{(\varepsilon,\delta)}(x*y) + \min\{t_1, t_2\} = \varepsilon + \min\{t_1, t_2\} > 1$ and so $(x*y)_{\min\{t_1,t_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Therefore $(x*y)_{\min\{t_1,t_2\}} \in\wedge q\, \mu_S^{(\varepsilon,\delta)}$, and consequently $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy subquasigroup of Q. □

Corollary 3.33. *If S is a subquasigroup of Q, then the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is both an $(\in, \in\vee q)$-fuzzy subquasigroup and an $(\in, q)$-fuzzy subquasigroup of Q whenever if any element t in $(0, 1]$ satisfies $x_t \in \mu_S^{(\varepsilon,\delta)}$ for $x \in Q$ then $\delta < t$ and $1 - t < \varepsilon$.* □

Theorem 3.34. *Let S be a nonempty subset of Q. If $\varepsilon + \delta \leqslant 1$ and the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy subquasigroup of Q, then S is a subquasigroup of Q.*

Proof. Assume that $\varepsilon + \delta \leqslant 1$ and the (ε, δ)-characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy subquasigroup of Q. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in\wedge q\, \mu_S^{(\varepsilon,\delta)}$, that is, $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in \mu_S^{(\varepsilon,\delta)}$ and $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Hence $\mu_S^{(\varepsilon,\delta)}(x*y) \geqslant \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(x*y) + \varepsilon > 1$. If $\mu_S^{(\varepsilon,\delta)}(x*y) \geqslant \varepsilon$, then $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$ and thus $x*y \in S$. If $\mu_S^{(\varepsilon,\delta)}(x*y) + \varepsilon > 1$, then $\mu_S^{(\varepsilon,\delta)}(x*y) > 1 - \varepsilon \geqslant \delta$ and so $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$, which shows that $x*y \in S$. Therefore S is a subquasigroup of Q. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.32 and 3.34, then we have the following corollary.

Corollary 3.35. *A nonempty subset S of Q is a subquasigroup of Q if and only if the characteristic function χ_S of S is an $(\in, \in\wedge q)$-fuzzy subquasigroup of Q.* □

Theorem 3.36. *If S is a subquasigroup of Q, then the fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\wedge q)$-fuzzy subquasigroup of Q under the condition that if any element t in $(0, 1]$ satisfies $x_t \in \mu_S^{(\varepsilon,\delta)}$ for $x \in Q$ then $\delta \leqslant 1 - t$ and $t < \varepsilon$.*

Proof. Let $x, y \in Q$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1}\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_{t_2}\, q\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x)+t_1 > 1$ and $\mu_S^{(\varepsilon,\delta)}(y)+t_2 > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1-t_1 \geqslant \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1-t_2 \geqslant \delta$. Hence $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$ and $\varepsilon > \max\{1-t_1, 1-t_2\}$, and so $x, y \in S$. Since S is a subquasigroup of $\mathcal{Q}$, we have $x * y \in S$ and thus

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geqslant \min\{t_1, t_2\},$$

that is, $(x*y)_{\min\{t_1,t_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Now, $\mu_S^{(\varepsilon,\delta)}(x*y)+\min\{t_1,t_2\} = \varepsilon+\min\{t_1,t_2\} > 1$, and so $(x*y)_{\min\{t_1,t_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Hence $(x*y)_{\min\{t_1,t_2\}} \in\!\wedge\, q\, \mu_S^{(\varepsilon,\delta)}$, and $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\!\wedge\, q)$-fuzzy subquasigroup of $\mathcal{Q}$. □

Corollary 3.37. *If S is a subquasigroup of $\mathcal{Q}$, then the fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\!\vee\, q)$-fuzzy subquasigroup of $\mathcal{Q}$ under the condition that if any element t in $(0,1]$ satisfies $x_t \in \mu_S^{(\varepsilon,\delta)}$ for $x \in Q$ then $\delta \leqslant 1-t$ and $t < \varepsilon$.* □

Theorem 3.38. *Let S be a nonempty subset of $\mathcal{Q}$. Assume that $\varepsilon > \max\{\delta, 0.5\}$. If the fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\!\wedge\, q)$-fuzzy subquasigroup of $\mathcal{Q}$, then S is a subquasigroup of $\mathcal{Q}$.*

Proof. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\!\wedge\, q)$-fuzzy subquasigroup of $\mathcal{Q}$, it follows that $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in\!\wedge\, q\, \mu_S^{(\varepsilon,\delta)}$ and so that $\mu_S^{(\varepsilon,\delta)}(x*y) \geqslant \varepsilon$. Hence $x*y \in S$ and S is a subquasigroup of $\mathcal{Q}$. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.36 and 3.38, then we have the following corollary.

Corollary 3.39. *A nonempty subset S of $\mathcal{Q}$ is a subquasigroup of $\mathcal{Q}$ if and only if the characteristic function χ_S of S is a $(q, \in\!\wedge\, q)$-fuzzy subquasigroup of $\mathcal{Q}$.* □

Theorem 3.40. *Let S be a nonempty subset of $\mathcal{Q}$. Assume that if any element t in $(0,1]$ satisfies $x_t \in \mu_S^{(\varepsilon,\delta)}$ for $x \in Q$ then $\delta \leqslant 1-t$. If S is a subquasigroup of $\mathcal{Q}$, then the fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\!\vee\, q)$-fuzzy subquasigroup of $\mathcal{Q}$.*

Proof. Let $x, y \in Q$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1}\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_{t_2}\, q\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x)+t_1 > 1$ and $\mu_S^{(\varepsilon,\delta)}(y)+t_2 > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1-t_1 \geqslant \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1-t_2 \geqslant \delta$. Hence $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $\varepsilon > \max\{1-t_1, 1-t_2\}$ and $x, y \in S$. Since S is a subquasigroup of $\mathcal{Q}$, we have $x*y \in S$ and thus $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$ which implies that $\mu_S^{(\varepsilon,\delta)}(x*y)+\min\{t_1,t_2\} = \varepsilon+\min\{t_1,t_2\} > 1$, i.e., $(x*y)_{\min\{t_1,t_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. It follows that $(x*y)_{\min\{t_1,t_2\}} \in\!\vee\, q\, \mu_S^{(\varepsilon,\delta)}$. Therefore $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\!\vee\, q)$-fuzzy subquasigroup of $\mathcal{Q}$. □

Theorem 3.41. *Let S be a nonempty subset of $\mathcal{Q}$. Assume that $\varepsilon > \max\{\delta, 0.5\}$ and $\varepsilon + \delta \leqslant 1$. If the fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\vee\, q)$-fuzzy subquasigroup of $\mathcal{Q}$, then S is a subquasigroup of $\mathcal{Q}$.*

Proof. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\vee\, q)$-fuzzy subquasigroup of $\mathcal{Q}$, it follows that $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in\vee\, q\, \mu_S^{(\varepsilon,\delta)}$, that is, $\mu_S^{(\varepsilon,\delta)}(x * y) \geqslant \varepsilon$ or $\mu_S^{(\varepsilon,\delta)}(x*y) + \varepsilon > 1$. If $\mu_S^{(\varepsilon,\delta)}(x*y) \geqslant \varepsilon$, then $x*y \in S$. If $\mu_S^{(\varepsilon,\delta)}(x*y) + \varepsilon > 1$, then $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon \geqslant \delta$ and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Thus $x * y \in S$, and therefore S is a subquasigroup of $\mathcal{Q}$. □

Corollary 3.42. *Let S be a nonempty subset of $\mathcal{Q}$. Assume that $\varepsilon > \max\{\delta, 0.5\}$ and $\varepsilon + \delta \leqslant 1$. If the fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an (α, β)-fuzzy subquasigroup of $\mathcal{Q}$ for $(\alpha, \beta) \in \{(q, \in), (q, \in\wedge\, q), (q, q)\}$, then S is a subquasigroup of $\mathcal{Q}$.* □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.40 and 3.41, then we have the following corollary.

Corollary 3.43. *A nonempty subset S of $\mathcal{Q}$ is a subquasigroup of $\mathcal{Q}$ if and only if the characteristic function χ_S of S is a $(q, \in\vee\, q)$-fuzzy subquasigroup of $\mathcal{Q}$.* □

References

[1] **M. Akram and W.A. Dudek**, *Generalized fuzzy subquasigroups,* Quasigroups and Related Systems **16** (2008), 133 – 146.

[2] **V.D. Belousov**, *Foundations of the theory of quasigroups and loops,* Nauka, Moscow 1967.

[3] **W.A. Dudek**, *Fuzzy subquasigroups,* Quasigroups and Related Systems **5** (1998), 81 – 98.

[4] **H.O. Pflugfelder**, *Quasigroups and loops: introduction,* Sigma Series in Pure Math., vol. 7, Heldermann Verlag, Berlin 1990.

[5] **P.M. Pu and Y.M. Liu**, *Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence,* J. Math. Anal. Appl. **76** (1980), 571 – 599.

Received December 8, 2014

Y.B.Jun
Department of Mathematics Education, Gyeongsang National University, Jinju 660-701, Korea
e-mail: skywine@gmail.com

S.Z.Song
Department of Mathematics, Jeju National University, Jeju 690-756, Korea
e-mail: szsong@jejunu.ac.kr

G.Muhiuddin
Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia
e-mail: chishtygm@gmail.com