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Eventually regular perfect semigroups

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Abstract. A congruence ρ on a semigroup S is called *perfect* if $(a\rho)(b\rho) = (ab)\rho$ for all $a, b \in S$, as sets, and S is said to be *perfect* if each of its congruences is perfect. We show that all eventually regular perfect semigroups are necessarily regular. Finally, we apply our result to perfect group-bound semigroups.

1. Introduction and preliminaries

The concept of a perfect semigroup was introduced by Vagner [12]. Groups are very well-known examples of perfect semigroups. Another examples of such semigroups are congruence-free semigroups S with the property $S = S^2$ (i.e., S is *globally idempotent*; note that perfect semigroups have this property). Perfect semigroups were studied first by Fortunatov (see e.g. [4, 5]) and then by Hamilton and Tamura [8], Hamilton [7], and by Goberstein [6]. In [1] the authors gave an example of a cancellative simple perfect semigroup without idempotents.

It is known that *any commutative perfect semigroup is inverse*, and that *all finite perfect semigroups are regular*; recall that a semigroup S is *regular* if S coincides with the set $\mathrm{Reg}(S)$ of its *regular* elements, where

$$\mathrm{Reg}(S) = \{s \in S : s \in sSs\}.$$

We extend the last result for eventually regular semigroups (a semigroup S is *eventually regular* if every element of S has a regular power, that is, for all $a \in S$ there is a positive integer $n = n(a)$ such that $a^n \in \mathrm{Reg}(S)$ [3]). Moreover, we apply this result to perfect group-bound semigroups (Corollary 2.2, below). Before we start our study, we recall some definitions and facts. For undefined terms, we refer the reader to the books [2, 9, 10].

Denote the set of all *idempotents* of a semigroup S by E_S, that is,

$$E_S = \{e \in S : e^2 = e\}.$$

If A is an *ideal* of a semigroup S, i.e., $AS \cup SA \subseteq A$, then the relation

$$\rho_A = (A \times A) \cup 1_S,$$

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where 1_S is the identity relation on S, is a congruence on S (the so-called *Rees congruence* on S). It is obvious that A is an idempotent ρ_A-class of S. Finally, we shall write S/A instead of S/ρ_A.

A generalization of the concept of regularity will also prove convenient. Define a semigroup S to be *idempotent-surjective* if and only if whenever ρ is a congruence on S and $a\rho$ is an idempotent of S/ρ, then $a\rho$ contains some idempotent of S. Edwards showed that eventually regular semigroups are idempotent-surjective [3].

Let S be a semigroup and let $a \in S$. Denote by S^1 the semigroup obtained from S by adjoining an identity if necessary. Then S^1aS^1 is the least ideal of S containing a. Denote it by $J(a)$. We shall say that the elements a, b of S are $\mathcal{J}$-*related* if $J(a) = J(b)$. Also, an equivalence $\mathcal{J}$-class containing a will be denoted by J_a. We can define a partial order on $S/\mathcal{J}$ by the rule:

$$J_a \leq J_b \iff J(a) \subseteq J(b)$$

for all $a, b \in S$ (a similar notation may be used for the Green's relations $\mathcal{L}$ and $\mathcal{R}$, cf. Section 2.1 of [10]).

We say that a semigroup S without zero is *simple* if and only if it has no proper ideals, that is, if and only if $SaS = S$ for every a of S. Further, a semigroup S with zero is called 0-*simple* if S is *not null* (i.e., $S^2 \neq \{0\}$) and S contains exactly two ideals (namely: $\{0\}$ and S). Clearly, S is 0-simple if and only if $S^2 \neq \{0\}$ and $S/\mathcal{J} = \{\{0\}, S \setminus \{0\}\}$.

By a 0-*minimal* ideal of a semigroup S we shall mean an ideal of S that is a minimal element in the set of all non-zero ideals of S.

The following result of Clifford is well-known.

Lemma 1.1. [2] *Any* 0-*minimal ideal of a semigroup is either null, or it is a* 0-*simple semigroup.* □

Let a be an element of a semigroup S. Suppose first that J_a is minimal among the $\mathcal{J}$-classes of S. Then $J(a) = J_a$ is the least ideal of S. On the other hand, if J_a is not minimal in $S/\mathcal{J}$, then the set

$$I(a) = \{b \in J(a) : J_b \leq J_a \ \&\ J_b \neq J_a\}$$

is an ideal of S such that $J(a) = I(a) \cup J_a$ (and this union is disjoint), and if B is a proper ideal of $J(a)$ and $I(a) \subseteq B$, then $I(a) = B$. This implies that $J(a)/I(a)$ is a 0-minimal ideal of $S/I(a)$, i.e., $J(a)/I(a)$ is either null, or it is a 0-simple semigroup (Lemma 1.1). For convenience, we shall write $J(a)/\emptyset = J(a)$. The semigroups $J(a)/I(a)$ ($a \in S$) are the so-called *principal factors* of S. Remark that we can think of the principal factor $J(a)/I(a)$ as consisting of the $\mathcal{J}$-class $J_a = J(a) \setminus I(a)$ with zero adjoined (if $I(a) \neq \emptyset$). Clearly, $J(a)/I(a)$ is null if and only if the product of any two elements of J_a always falls into a lower $\mathcal{J}$-class. In particular, if J_a is a subsemigroup of S, then the principal factor $J(a)/I(a)$ is not null. Finally, $J(a)/I(a)$ is simple if and only if $I(a)$ is empty.

Recall that among idempotents in an arbitrary semigroup there is a *natural partial order relation* defined by the rule that

$$e \leqslant f \Leftrightarrow e = ef = fe.$$

We say that an idempotent $e \neq 0$ of a semigroup S is *primitive* if it is minimal (with respect to the natural partial order) within the set of non-zero idempotents of S. Also, a (0)-simple semigroup is called *completely* (0)*-simple* if it is (0)-simple and contains a primitive idempotent. Notice that in the both cases each non-zero idempotent of S is primitive. For some equivalent definitions of these notions, we refer the reader to the book [10] (cf. Section 3.2). Munn showed that a (0)-simple semigroup S is completely (0)-simple if and only if it is *group-bound* (a semigroup S is called *group-bound* if every element of S has a power which belongs to some subgroup of S). Obviously, group-bound semigroups are eventually regular.

A semigroup is called (*completely*) *semisimple* if each of its principal factors is either (completely) 0-simple or (completely) simple. Recall that a semigroup is semisimple if and only if all its ideals are globally idempotent (see e.g. [2]).

Observe that every idempotent congruence class of a perfect semigroup S is globally idempotent. In particular, all ideals of S are globally idempotent, that is, S is semisimple.

Recall that an idempotent commutative semigroup is *semilattice*. Clearly, the least semilattice congruence η on an arbitrary semigroup S exists (note that $\mathcal{J} \subseteq \eta$). This relation induces the greatest semilattice decomposition of S, say $[Y; S_\alpha]$ $(\alpha \in Y)$, where $Y \cong S/\eta$, each S_α is an η-class and $S = \bigcup\{S_\alpha : \alpha \in Y\}$. To indicate this fact we shall always write $S = [Y; S_\alpha]$ $(\alpha \in Y)$ or briefly $S = [Y; S_\alpha]$. Notice that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$, where $\alpha\beta$ is the product of α and β in the semilattice Y.

We say that a semigroup S is *intra-regular* if for every $a \in S$, $a\,\mathcal{J}\,a^2$ [2]. It is easy to see that if S is intra-regular, then $\mathcal{J}$ is a semilattice congruence on S, so we have the following well-known result [2].

Lemma 1.2. *A semigroup S is intra-regular if and only if $\eta = \mathcal{J}$, where every $\mathcal{J}$-class is a simple semigroup.* □

We say that a $\mathcal{J}$-class J of a semigroup is *regular* if consists entirely of regular elements.

The following result, which is contained in the paper of Jones et al. [11], is due to Cirič.

Lemma 1.3. *Let a $\mathcal{J}$-class J of an eventually regular semigroup contains an idempotent. Then J is regular. Equivalently,* 0*-simple eventually regular semigroups are regular.* □

We recall now some known results concerning perfect semigroups in general. For beginning, from the First and Second Isomorphism Theorems we obtain the following result [5].

Lemma 1.4. *Every homomorphic image of a perfect semigroup is a perfect semigroup.* □

An ideal A of a semigroup S is called *completely prime* if $ab \in A$ implies that $a \in A$ or $b \in A$.

The following fact [5] follows from the definition of a Rees congruence.

Lemma 1.5. *Every non-zero ideal of a perfect semigroup is completely prime.* □

It is not difficult to see that every chain is perfect. Also, if the elements a, b of a semilattice A are incomparable, then the congruence induced by the ideal aA is not perfect.

Lemma 1.6. [5] *A semilattice is perfect if and only if it is a chain.* □

Let $S = [Y; S_\alpha]$. Assume that S is perfect. In the light of Lemmas 1.4 and 1.6, Y is a chain. Moreover, from [5] we can extract the following result. We give a simple proof for the sake of completeness.

Corollary 1.7. *Let $S = [Y; S_\alpha]$ be a perfect semigroup. Then Y is a chain and the following statements hold:*

(a) *if S does not have a zero, then each S_α is simple and $Y \cong S/\mathcal{J}$;*

(b) *if S contains a zero 0, then Y has a least element 0_Y, S_α is a simple semigroup for $\alpha \neq 0_Y$, and either $S_{0_Y} = \{0\}$ (then $Y \cong S/\mathcal{J}$) or S_{0_Y} is a 0-simple semigroup whose zero is not adjoined (and $J_a = a\eta \setminus \{0\}$ if $a \neq 0$).*

Proof. (a). Suppose first that S has no a zero element. As $a^2 \in S^1a^2S^1$, $a \in S^1a^2S^1$ (Lemma 1.5) and so S is intra-regular. Thus every S_α is a simple semigroup and $Y \cong S/\mathcal{J}$ (Lemma 1.2).

(b). Let now S contains a zero 0, say $0 \in S_{0_Y}$. Because $S_{0_Y}S_\alpha \subseteq S_{0_Y}$ for all $\alpha \in Y$, then $S_{0_Y}S_\alpha = S_{0_Y}$ for all $\alpha \in Y$ (since S is perfect). This implies that Y has a least element 0_Y.

Since Y is a chain and every S_α is a semigroup, then the condition $a^2 = 0$ implies that $a \in S_{0_Y}$. Thus S_α is a simple semigroup for all $\alpha \neq 0_Y$.

If $S_{0_Y} \neq \{0\}$, then $S_{0_Y}^2 = S_{0_Y} \neq \{0\}$, since it is clear that S_{0_Y} is an ideal of S, i.e., S_{0_Y} is *not null*. Suppose that $A \subseteq S_{0_Y}$ is a non-zero ideal of S. Then A is completely prime (by Lemma 1.5). It follows that A is a non-zero completely prime ideal of S_{0_Y}. Hence the partition $\{A, S_{0_Y} \setminus A\}$ of S_{0_Y} induces a semilattice congruence on S_{0_Y}. On the other hand, it is well-known that every η-class of S has no semilattice congruences except the universal relation. In particular, S_{0_Y} possesses this property. It follows that $A = S_{0_Y}$, i.e., S_{0_Y} is a 0-minimal ideal of S. Finally, observe that if 0 is adjoined to S_{0_Y}, then the partition

$$\{S_\alpha\,(\alpha \neq 0_Y), S_{0_Y} \setminus \{0\}, \{0\}\}$$

of S induces a semilattice congruence on S which is properly contained in the least semilattice congruence η, a contradiction, so S_{0_Y} is a 0-minimal ideal of S whose zero is not adjoined. Consequently, S_{0_Y} is a 0-simple semigroup whose zero is not adjoined (Lemma 1.1). Clearly, $J_a = a\eta \setminus \{0\}$ if $a \neq 0$. □

2. The main results

Remark that if ρ is a semilattice congruence on an eventually regular semigroup S, then every ρ-class of S is eventually regular.

Theorem 2.1. *Every eventually regular perfect semigroup S is regular.*

Proof. Suppose first that S has no a zero. Then S is a semilattice Y of simple semigroups S_α $(\alpha \in Y)$, where each S_α is a $\mathcal{J}$-class of S (cf. Corollary 1.7). Since each S_α is an idempotent $\mathcal{J}$-class, then it contains an idempotent element of S (because S is idempotent-surjective). In the light of Lemma 1.3, S is regular.

Let S has a zero. In view of Corollary 1.7, Y has a least element 0_Y. Put $A = S \backslash S_{0_Y}$. It is evident that the semigroup A is a semilattice of simple semigroups. Take any $a \in A$. Then the elements a and a^2 belong to the same *simple* subsemigroup B of A. Hence $a \in Ba^2B \subseteq Aa^2A$. Thus A is intra-regular. By the above A is regular. Finally, consider a 0-simple semigroup S_{0_Y} (see Corollary 1.7). This semigroup is also eventually regular, so S_{0_Y} is regular (by Lemma 1.3). Consequently, S is a regular semigroup. □

A semigroup is called *completely regular* if it is a union of groups. Recall from [9] that a semigroup is completely regular if and only if it is a semilattice of completely simple semigroups.

Corollary 2.2. *Let $S = [Y; S_\alpha]$ be a perfect group-bound semigroup. Then S is regular, Y is a chain and the following statements hold*:

(*a*) *if S does not have a zero, then every S_α is a completely simple semigroup* (*and $Y \cong S/\mathcal{J}$*)*, that is, S is completely regular*;

(*b*) *if S contains a zero, say 0, then Y has a least element 0_Y, S_α is completely simple for $\alpha \neq 0_Y$, and either $S_{0_Y} = \{0\}$* (*then clearly $Y \cong S/\mathcal{J}$*) *or S_{0_Y} is a completely 0-simple semigroup whose zero 0 is not adjoined* (*and then $J_a = a\eta \setminus \{0\}$ if $a \neq 0$*).

In the former case, S is a completely regular semigroup with 0 adjoined.

Proof. (*a*). Indeed, every S_α is a simple (regular) group-bound semigroup, so each S_α is a completely simple semigroup.

(*b*). It is sufficient to show that if $S_{0_Y} \neq \{0\}$, then S_{0_Y} is a completely 0-simple semigroup. In that case, S_{0_Y} is a 0-simple (regular) group-bound semigroup. Thus S_{0_Y} is completely 0-simple semigroup. □

Corollary 2.3. *Every perfect group-bound semigroup is completely semisimple.* □

Finally, we shall show that an eventually regular perfect semigroup satisfying one of the following minimal conditions is group-bound (note that any group-bound semigroup meets both of these conditions). We shall say that a semigroup S *satisf ies the condition* $\min_L^*$ (resp. $\min_R^*$) if and only if for every $\mathcal{J}$-class J of S, the set of all $\mathcal{L}$-classes (resp. $\mathcal{R}$-classes) contained in J has a minimal element (for more details cf. Section 6.6 [2]). Recall only that a regular semigroup satisfies $\min_L^*$ if and only if it meets $\min_R^*$.

Proposition 2.4. *Let S be an eventually regular perfect semigroup satisfying* $\min_L^*$ *or* $\min_R^*$. *Then S is completely semisimple. In particular, S is group-bound.*

Proof. Indeed, in that case, S is regular (Theorem 2.1), so every η-class of S is a regular subsemigroup of S. In view of the above remark, S satisfies $\min_L^*$ and $\min_R^*$ (cf. also Corollary 1.7). As S is semisimple, S is completely semisimple (see Theorem 6.45 in [2]). In particular, S is group-bound. □

References

[1] **S. Bulman-Fleming and K. McDowell**, *A non-regular perfect semigroup*, Semigroup Forum **35** (1987), 119–121.

[2] **A.H. Clifford and G.B. Preston**, *The algebraic theory of semigroups*, Vol. I & II, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R.I. (1961 & 1967).

[3] **P.M. Edwards**, *Eventually regular semigroups*, Bull. Austral. Math. Soc. **28** (1983), 23–38.

[4] **V.A. Fortunatov**, *Perfect semigroups that decompose into a commutative band of rectangular groups*, (Russian), Studies in algebra (Izdat. Saratov, Univ.) **2** (1970), 67–78.

[5] **V.A. Fortunatov**, *Perfect semigroups*, (Russian), Izv. Vysš. Učebn. Zaved. Matematika **3** (1972), 80–89.

[6] **F.S. Goberstein**, *Perfect Completely Semisimple Inverse Semigroups*, Semigroup Forum **45** (1992), 395–397.

[7] **H. Hamilton**, *Perfect completely regular semigroups*, Math. Nachr. **123** (1984), 169–176.

[8] **H. Hamilton and T. Tamura**, *Finite inverse perfect semigroups and their congruences*, J. Austral. Math. Soc. **32A** (1982), 114–128.

[9] **J.M. Howie**, *An introduction to semigroup theory*, Academic Press, London (1976).

[10] **J.M. Howie**, *Fundamentals of semigroup theory*, Oxford University Press, New York (1995).

[11] **P.R. Jones et al.**, *On the lattice of full eventually regular subsemigroups*, Comm. Algebra **33** (2005), 2587–2600.

[12] **V.V. Vagner**, *Algebraic topics of the general theory of partial connections in fiber bundles*, Izv. Vysš. Učebn. Zaved. Matematika **11** (1968), 26–32.

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