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Autotopisms of some quasigroups

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Abstract. We present one method of construction of some autotopisms for quasigroups satisfying the identity $\alpha(x)\cdot\beta(x\cdot y)=\gamma(y)$.

Denote by S_n the set of all permutations of the set $Q=\{1,2,\ldots,n\}$. The triplet $A=(\omega,\varphi,\psi)$, where $\omega,\varphi,\psi\in S_n$, is called an *autotopism* of a quasigroup $(Q,\cdot)$ if

$$\omega(x\cdot y)=\varphi(x)\cdot\psi(y)$$

holds for all $x,y\in Q$.

The set of all autotopisms of a quasigroup of order n form a group. The order of this group is a divisor of $(n!)^3$ but it cannot exceed $(n!)^2$. Moreover, two components of an autotopism determine the third one uniquely (see [1] or [5]). There are quasigroups that have only one (trivial) autotopism. Such quasigroups are called *super rigid*. The smallest super rigid quasigroups has 7 elements [3].

In this note we will consider quasigroups satisfying the identity

$$\alpha(x)\cdot\beta(x\cdot y)=\gamma(y), \tag{1}$$

where $\alpha,\beta,\gamma\in S_n$. Such triplet of permutations will be denoted by $R=(\alpha,\beta,\gamma)$.

Note that parastrophes of a quasigroup satisfying (1) are pairwise isotopic [4].

Theorem 1. *A quasigroup $(Q,\cdot)$ satisfying the identity (1) has an autotopism of the form $(\gamma\beta,\alpha^2,\beta\gamma)$.*

Proof. Indeed, (1) implies

$$\beta(\alpha(x)\cdot\beta(x\cdot y))=\beta\gamma(y).$$

Multiplying this identity by $\alpha^2(x)$ we obtain

$$\alpha^2(x)\cdot\beta(\alpha(x)\cdot\beta(x\cdot y))=\alpha^2(x)\cdot\beta\gamma(y).$$

From this, applying (1) to the left side, we get

$$\gamma\beta(x\cdot y)=\alpha^2(x)\cdot\beta\gamma(y). \tag{2}$$

So, $A=(\gamma\beta,\alpha^2,\beta\gamma)$ is an autotopism of $(Q,\cdot)$. □

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Theorem 2. *A quasigroup* $(Q,\cdot)$ *satisfying* (1) *satisfies the identity*

$$\alpha_k(x)\cdot\beta_k(x\cdot y)=\gamma_k(y) \tag{3}$$

with $\alpha_k=\alpha^{3^k}$, $\beta_k=\beta(\gamma\beta)^{\frac{3^k-1}{2}}$, $\gamma_k=\gamma(\beta\gamma)^{\frac{3^k-1}{2}}$, *where* $k=0,1,\ldots,p-1$ *and* $\alpha_p=\alpha$, $\beta_p=\beta$, $\gamma_p=\gamma$.

Proof. Since a quasigroup $(Q,\cdot)$ satisfying (1) has an autotopism $A=(\gamma\beta,\alpha^2,\beta\gamma)$, from (1) we obtain

$$\gamma\beta\gamma(y)=\gamma\beta(\alpha(x)\cdot\beta(x\cdot y))=\alpha^2(\alpha(x))\cdot\beta\gamma(\beta(x\cdot y))=\alpha^3(x)\cdot\beta\gamma\beta(x\cdot y),$$

which means that in this quasigroup

$$\alpha_1(x)\cdot\beta_1(x\cdot y)=\gamma_1(y),$$

where $\alpha_1=\alpha^3$, $\beta_1=\beta\gamma\beta$, $\gamma_1=\gamma\beta\gamma$.

Thus, $(Q,\cdot)$ has an autotopism $A_1=(\gamma_1\beta_1,\alpha_1^2,\beta_1\gamma_1)$ and satisfies the identity

$$\alpha_2(x)\cdot\beta_2(x\cdot y)=\gamma_2(y),$$

where $\alpha_2=\alpha_1^3=\alpha^{3^2}$, $\beta_2=\beta(\gamma\beta)^{\frac{3^2-1}{2}}$, $\gamma_2=\gamma(\beta\gamma)^{\frac{3^2-1}{2}}$, and so on. □

Corollary. *A quasigroup satisfying the identity* (1) *has an autotopism of the form* $A_k=(\omega_k,\varphi_k,\psi_k)$ *with* $\omega_k=\gamma_k\beta_k=(\gamma\beta)^{3^k}$, $\varphi_k=\alpha_k^2=\alpha^{2\cdot 3^k}$, $\psi_k=\beta_k\gamma_k=(\beta\gamma)^{3^k}$, *where* $k=0,1,\ldots,p-1$ *and* $\omega_p=\omega$, $\varphi_p=\varphi$, $\psi_p=\psi$. *Moreover, then* $\alpha_{k+1}=\varphi_k\alpha_k$, $\beta_{k+1}=\psi_k\beta_k$, $\gamma_{k+1}=\omega_k\gamma_k$.

Example. A quasigroup determined by the table

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	8	5	6	4	3	1	7
3	3	6	1	8	7	2	5	4
4	4	5	7	2	1	8	6	3
5	5	1	6	7	8	4	3	2
6	6	3	4	5	2	7	8	1
7	7	4	8	1	3	5	2	6
8	8	7	2	3	6	1	4	5

is an isotope of a quasigroup defined in [2]. This quasigroup satisfies (1) with $\alpha=(1287465.3.)$, $\beta=(18.46.2.357.)$, $\gamma=(175.28.34.6.)$, where $(175.28.34.6.)$ means that this permutation is a composition of cycles $(175),(28)$ and (34).

Let $R=(\alpha,\beta,\gamma)$, where α,β,γ are as in the above. According to Theorem 1 this quasigroup has an autotopism $A=(\omega,\varphi,\psi)$ such that

$$\omega=\gamma\beta=(1287463.5.),\quad \varphi=\alpha^2=(1845276.3.),\quad \psi=\beta\gamma=(1364582.7.).$$

By Theorem 2, this quasigroup satisfies (3) with $R_1 = (\alpha_1, \beta_1, \gamma_1)$, where, in view of Corrollary, $\alpha_1, \beta_1, \gamma_1$ have the form

$$\alpha_1 = \varphi\alpha = (1758624.3.),\quad \beta_1 = \psi\beta = (12.38.4.576.),\quad \gamma_1 = \omega\gamma = (14.275.36.8.).$$

Then we compute $A_1 = (\omega_1, \varphi_1, \psi_1)$ and $R_2 = (\alpha_2, \beta_2, \gamma_2)$:

$$\begin{cases} \omega_1 = \gamma_1\beta_1 = (1738624.5.), \\ \varphi_1 = \alpha_1^2 = (1564782.3.), \\ \psi_1 = \beta_1\gamma_1 = (1426835.7.), \end{cases} \quad \text{and} \quad \begin{cases} \alpha_2 = \varphi_1\alpha_1 = (1845276.3.), \\ \beta_2 = \psi_1\beta_1 = (16.24.3.578.), \\ \gamma_2 = \omega_1\gamma_1 = (1.23.475.68.). \end{cases}$$

$A_2 = (\omega_2, \varphi_2, \psi_2)$ and $R_3 = (\alpha_3, \beta_3, \gamma_3)$:

$$\begin{cases} \omega_2 = \gamma_2\beta_2 = (1843276.5.), \\ \varphi_2 = \alpha_2^2 = (1426857.3.), \\ \psi_2 = \beta_2\gamma_2 = (1652348.7.), \end{cases} \quad \text{and} \quad \begin{cases} \alpha_3 = \varphi_2\alpha_2 = (1564782.3.), \\ \beta_3 = \psi_2\beta_2 = (157.28.34.6.), \\ \gamma_3 = \omega_2\gamma_2 = (18.2.3.46.375.). \end{cases}$$

$A_3 = (\omega_3, \varphi_3, \psi_3)$ and $R_4 = (\alpha_4, \beta_4, \gamma_4)$:

$$\begin{cases} \omega_3 = \gamma_3\beta_3 = (1364782.5.), \\ \varphi_3 = \alpha_3^2 = (1672548.3.), \\ \psi_3 = \beta_3\gamma_3 = (1285463.7.), \end{cases} \quad \text{and} \quad \begin{cases} \alpha_4 = \varphi_3\alpha_3 = (1426857.3.), \\ \beta_4 = \psi_3\beta_3 = (14.257.36.8.), \\ \gamma_4 = \omega_3\gamma_3 = (12.38.4.567.). \end{cases}$$

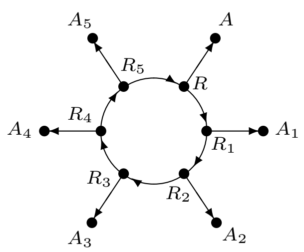
$A_4 = (\omega_4, \varphi_4, \psi_4)$ and $R_5 = (\alpha_5, \beta_5, \gamma_5)$:

$$\begin{cases} \omega_4 = \gamma_4\beta_4 = (1426837.5.), \\ \varphi_4 = \alpha_4^2 = (1287465.3.), \\ \psi_4 = \beta_4\gamma_4 = (1538624.7.), \end{cases} \quad \text{and} \quad \begin{cases} \alpha_5 = \varphi_4\alpha_4 = (1672548.3.), \\ \beta_5 = \psi_4\beta_4 = (1.23.457.68.), \\ \gamma_5 = \omega_4\gamma_4 = (16.24.3.587.). \end{cases}$$

$A_5 = (\omega_5, \varphi_5, \psi_5)$ and $R_6 = (\alpha_6, \beta_6, \gamma_6)$:

$$\begin{cases} \omega_5 = \gamma_5\beta_5 = (1672348.5.), \\ \varphi_5 = \alpha_5^2 = (1758624.3.), \\ \psi_5 = \beta_5\gamma_5 = (1843256.7.), \end{cases} \quad \text{and} \quad \begin{cases} \alpha_6 = \varphi_5\alpha_5 = (1287465.3.) = \alpha, \\ \beta_6 = \psi_5\beta_5 = (18.46.2.357.) = \beta, \\ \gamma_6 = \omega_5\gamma_5 = (175.28.34.6.) = \gamma. \end{cases}$$

Relationships between A_i and R_i we can present by the following graph.



The set autotopisms $A, A_1, A_2, A_3, A_4, A_5$ together with the identity autotopism $E = (\varepsilon, \varepsilon, \varepsilon)$ forma a cyclic group of order 7. The group of all autotopisms of this quasigroup has 42 elements.

Note also that in this quasigroup the identity (1) also is satisfied with $\alpha = \varepsilon$ (the identity permutation) and $\beta = \gamma = (13.48.26.57.)$. So, in this case $R = (\varepsilon, \beta, \beta)$, and consequently $A = (\varepsilon, \varepsilon, \varepsilon)$, $R_1 = R$, $A_1 = A$.

Remark. A similar results can be obtained for quasigroups satisfying one of the identities

$$\alpha(x) \cdot \beta(yx) = \gamma(y), \tag{4}$$

$$\beta(xy) \cdot \alpha(x) = \gamma(y), \tag{5}$$

$$\beta(yx) \cdot \alpha(x) = \gamma(y), \tag{6}$$

$$\beta(xy) = \gamma(y) \cdot \alpha(x), \tag{7}$$

where α, β, γ are fixed permutations of the set Q, used in [4] to the description of isotopy classes of parastrophes.

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