On generalized bi-$\Gamma$-ideals in $\Gamma$-semigroups

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Abstract. We study generalized bi-$\Gamma$-ideals, prime, semiprime and irreducible generalized bi-$\Gamma$-ideals in $\Gamma$-semigroups.

1. Introduction

Let $S$ and $\Gamma$ be two nonempty sets. Then a triple of the form $(S, \Gamma, \cdot)$ is called a $\Gamma$-semigroup, where $\cdot$ is a ternary operation $S \times \Gamma \times S \to S$ such that $(x \cdot y \cdot \beta \cdot z)$ for all $x, y, z \in S$ and all $\alpha, \beta \in \Gamma$.

We will denote $(S, \Gamma, \cdot)$ by $S$ and $a \cdot \gamma \cdot b$ by $a\gamma b$.

Definition 1.1. A nonempty subset $B$ of $S$ is called
- a sub-$\Gamma$-semigroup of $S$ if $a\gamma b \in B$, for all $a, b \in B$ and $\gamma \in \Gamma$,
- a generalized bi-$\Gamma$-ideal of $S$ if $B \Gamma S \Gamma B \subseteq B$,
- a bi-$\Gamma$-ideal of $S$ if $B \Gamma S \Gamma B \subseteq B$ and $B \Gamma B \subset B$.

A $\Gamma$-semigroup $S$ is called a gb-simple if it does not contain the proper generalized bi-$\Gamma$-ideal.

Definition 1.2. A generalized bi-$\Gamma$-ideal $B$ of a $\Gamma$-semigroup $S$ is
- prime if $B_1 \Gamma B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$,
- strongly prime if $B_1 \Gamma B_2 \cap B \Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$,
- irreducible if $B_1 \cap B_2 = B$ implies $B_1 = B$ or $B_2 = B$,
- strongly irreducible if $B_1 \cap B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any generalized bi-$\Gamma$-ideals $B_1$ and $B_2$ of $S$.

A quasi $\Gamma$-ideal is prime if it is prime as a bi-$\Gamma$-ideal.

Definition 1.3. A generalized bi-$\Gamma$-ideal $B$ of $S$ is
- semiprime if $B_1 \Gamma B_1 \subseteq B$ implies that $B_1 \subseteq B$ for any bi-$\Gamma$-ideal $B_1$ of $S$.

Other definition one can find in [1] and [2].

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2. Properties of generalized bi-$\Gamma$-ideals

Lemma 2.1. The smallest generalized bi-$\Gamma$-ideal of a $\Gamma$-semigroup $S$ containing a nonempty subset $T$ of $S$ has the form $T \cup TTSTT$.

Proof. Let $B = T \cup TTSTT$. Then $T \subseteq B$. So,

$$B \Gamma S \Gamma B = (T \cup TTSTT) \Gamma S \Gamma (T \cup TTSTT)$$

$$\subseteq [T(T \Gamma ST)(T \cup TTSTT)] \cup [TTSSTT(T \Gamma ST)(T \cup TTSTT)]$$

$$\subseteq [T(T \Gamma ST)T \cup T(T \Gamma ST)TTSTT] \cup [TTSTT(T \Gamma ST)T \cup TTSTT(T \Gamma ST)TTSTT]$$

$$\subseteq [TTSTT \cup TTSTT] \cup [TTSTT \cup TTSTT]$$

$$= TTSTT \subseteq T \cup TTSTT = B.$$ 

Hence $B = T \cup TTSTT$ is a generalized bi-$\Gamma$-ideal of $S$.

To prove that $B$ is the smallest generalized bi-$\Gamma$-ideal of $S$ containing $T$ suppose that $G$ is a generalized bi-$\Gamma$-ideal of $S$ containing $T$. Then $TTSTT \subseteq G \Gamma STG \subseteq G$. Therefore, $B = T \cup TTSTT \subseteq G$. Hence $B$ is the smallest generalized bi-$\Gamma$-ideal of $S$ containing $T$. \qed

The smallest generalized bi-$\Gamma$-ideal of $S$ containing $T$ will be denoted by $(T)$.

Lemma 2.2. Suppose that $A$ is a sub-$\Gamma$-semigroup of a $\Gamma$-semigroup $S$, $s \in S$ and $(s \Gamma A \Gamma s) \cap A \neq \emptyset$. Then $(s \Gamma A \Gamma s) \cap A$ is a generalized bi-$\Gamma$-ideal of $A$.

Proof. Indeed,

$$(s \Gamma A \Gamma s \cap A) \Gamma A \Gamma (s \Gamma A \Gamma s \cap A) \subseteq [(s \Gamma A \Gamma s) \Gamma A \cap A \Gamma A] \Gamma (s \Gamma A \Gamma s \cap A)$$

$$\subseteq [(s \Gamma A \Gamma s) \Gamma A \cap A] \Gamma (s \Gamma A \Gamma s \cap A)$$

$$\subseteq [[(s \Gamma A \Gamma s) \Gamma A] \Gamma (s \Gamma A \Gamma s)] \cap [A \Gamma (s \Gamma A \Gamma s \cap A)]$$

$$\subseteq [(s \Gamma A \Gamma s) \cap (A \Gamma s \Gamma A \Gamma s)] \cap A$$

$$\subseteq (s \Gamma A \Gamma s) \cap A.$$ 

Hence $(s \Gamma A \Gamma s) \cap A$ is a generalized bi-$\Gamma$-ideal of $A$. \qed

Theorem 2.3. For a $\Gamma$-semigroup $S$ the following assertions are equivalent:

(i) $S$ is a gb-simple $\Gamma$-semigroup,

(ii) $s \Gamma STs = S$ for all $s \in S$,

(iii) $s = S$ for all $s \in S$.

Proof. (i) $\Rightarrow$ (ii). Let $S$ be a gb-simple $\Gamma$-semigroup and $s \in S$. Then $s \Gamma STs$ is a generalized bi-$\Gamma$-ideal of $S$. As $S$ is a gb-simple $\Gamma$-semigroup, $s \Gamma STs = S$.

(ii) $\Rightarrow$ (iii). If $s \Gamma STs = S$ for all $s \in S$, then, $(s) = \{s\} \cup s \Gamma STs = \{s\} \cup S = S$.

(iii) $\Rightarrow$ (i). Let $(s) = S$, for all $s \in S$, and assume $B$ is a generalized bi-$\Gamma$-ideal of $S$ and $s \in B$. Then $(s) \subseteq B$. By our hypothesis, we obtain $S = (s) \subseteq B \subseteq S$. So, $S = B$. Hence $S$ is a gb-simple $\Gamma$-semigroup. \qed
**Theorem 2.4.** A bi-$\Gamma$-ideal $B$ of a $\Gamma$-semigroup $S$ is a minimal generalized bi-$\Gamma$-ideal of $S$ if and only if $B$ is a gb-simple $\Gamma$-semigroup.

*Proof.* Let $B$ be a minimal generalized bi-$\Gamma$-ideal of $S$. By our hypothesis, $B$ is a $\Gamma$-semigroup. Suppose $D$ is a generalized bi-$\Gamma$-ideal of $B$. Then $D \Gamma B \Gamma D \subseteq D \subseteq B$. As $B$ is a generalized bi-$\Gamma$-ideal of $S$, we obtain $D \Gamma B \Gamma D$ is a generalized bi-$\Gamma$-ideal of $S$. As $B$ is a minimal generalized bi-$\Gamma$-ideal of $S$, we obtain $D \Gamma B \Gamma D = B$. So, we have $B = D$. Therefore, $B$ is a gb-simple $\Gamma$-semigroup.

Conversely, let $B$ be a gb-simple $\Gamma$-semigroup. Suppose $D$ is a generalized bi-$\Gamma$-ideal of $B$ so that $D \subseteq B$. Then $D \Gamma B \Gamma D \subseteq D \Gamma S \Gamma D \subseteq D$. So $D$ is a generalized bi-$\Gamma$-ideal of $B$. As $B$ is a gb-simple $\Gamma$-semigroup, we obtain $B = D$. Hence $B$ is a minimal generalized bi-$\Gamma$-ideal of $S$. □

**Theorem 2.5.** Every generalized bi-$\Gamma$-ideal of a $\Gamma$-semigroup $S$ is a bi-$\Gamma$-ideal of $S$ if and only if $xoy \in \{x, y\} \Gamma S \Gamma \{x, y\}$, for every $x, y \in S$ and $\alpha \in \Gamma$.

*Proof.* Suppose $S$ is a $\Gamma$-semigroup in which every generalized bi-$\Gamma$-ideal is a bi-$\Gamma$-ideal. Then, for every $x, y \in S$, the generalized bi-$\Gamma$-ideal generated by subset $\{x, y\}$ is given by $\{x, y\} \cup \{x, y\} \Gamma S \Gamma \{x, y\}$ which is a bi-$\Gamma$-ideal of $S$, so we have $xoy \in \{x, y\} \Gamma S \Gamma \{x, y\}$.

Conversely, if $x, y$ are elements of a generalized bi-$\Gamma$-ideal $B$ of $S$, then we have $xoy \in B \Gamma S \Gamma B \subseteq B$. Hence $B$ is a bi-$\Gamma$-ideal of $S$. □

3. Prime and irreducible generalized bi-$\Gamma$-ideals

**Proposition 3.1.** A semiprime generalized bi-$\Gamma$-ideal of $S$ is a quasi-$\Gamma$-ideal of $S$.

*Proof.* Suppose that $B$ is semiprime and let $x \in (ST \cap B)S$. Then $x \Gamma ST x \subseteq (B \Gamma S) \Gamma ST (ST \Gamma B) = B \Gamma ST B \subseteq B$ and since $B$ is semiprime, we obtain $x \in B$. Hence $B = ST \cap B \Gamma S$. □

**Proposition 3.2.** A $\Gamma$-semigroup $S$ is regular if and only if every generalized bi-$\Gamma$-ideal of $S$ is semiprime.

*Proof.* Let $S$ be regular and suppose that $B$ is any generalized bi-$\Gamma$-ideal of $S$. If $b \notin B$, then $b \in s \Gamma ST s$, so we obtain $s \Gamma ST s \notin B$ and hence $B$ is semiprime. Conversely, if every generalized bi-$\Gamma$-ideal of $S$ is semiprime, then so is $B = s \Gamma ST s$ for any $s \in S$. As $s \Gamma ST s \subseteq B$, we obtain $b \in B$ and hence $S$ is regular. □

**Proposition 3.3.** The intersection of any nonempty family of prime generalized bi-$\Gamma$-ideals of a $\Gamma$-semigroup is a semiprime bi-$\Gamma$-ideal.

*Proof.* Suppose that $S$ is a $\Gamma$-semigroup and $\mathcal{P} = \{P \mid P$ is a prime generalized bi-$\Gamma$-ideal of $S\}$. As $0 \in P$, for all $P \in \mathcal{P}$, we obtain $0 \in \bigcap \mathcal{P}$. Thus $\bigcap \mathcal{P} \neq \emptyset$. Suppose $q \in (\bigcap \mathcal{P}) \Gamma ST (\bigcap \mathcal{P})$. Then $q = q_1 \alpha s \beta q_2$, for some $q_1, q_2 \in \bigcap \mathcal{P}, s \in S$.
and $\alpha, \beta, \gamma \in \Gamma$. Thus $q = q_1\alpha^s\beta^q_2 \in P\Gamma \Sigma P \subseteq P$, for all $P \in \mathcal{P}$. Therefore, $q \in \bigcap \mathcal{P}$. So $(\bigcap \mathcal{P})\Gamma \Sigma (\bigcap \mathcal{P}) \subseteq \bigcap \mathcal{P}$. Therefore, $\bigcap \mathcal{P}$ is a generalized bi-$\Gamma$-ideal of $S$. Suppose $B$ be a generalized bi-$\Gamma$-ideal of $S$ such that $B^2 \subseteq \bigcap \mathcal{P}$. We have $B^2 \subseteq P$, for all $P \in \mathcal{P}$. As $P$ is a prime generalized bi-$\Gamma$-ideal of $S$, we obtain $B \subseteq P$, for all $P \in \mathcal{P}$. Thus $B \subseteq \bigcap \mathcal{P}$. Hence $\bigcap \mathcal{P}$ is a semiprime generalized bi-$\Gamma$-ideal of $S$. \hfill \square

**Proposition 3.4.** A prime generalized bi-$\Gamma$-ideal is a prime one-sided $\Gamma$-ideal.

**Proof.** Let $\Sigma B \not\subseteq B$ and $B\Sigma S \not\subseteq B$. Since $B$ is prime, it follows that $B\Gamma \Sigma B = (B\Sigma)\Gamma \Sigma (B\Sigma) \not\subseteq B$, which is a contradiction. Hence $B$ is a prime one-sided $\Gamma$-ideal. \hfill \square

**Corollary 3.5.** A quasi-$\Gamma$-ideal of $S$ is a prime one-sided $\Gamma$-ideal of $S$. \hfill \square

**Proposition 3.6.** A generalized bi-$\Gamma$-ideal $B$ of a $\Gamma$-semigroup $S$ is prime if and only if $RTL \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$, where $R$ and $L$ are right and left $\Gamma$-ideal of $S$.

**Proof.** If $B$ is prime and $RTL \subseteq B$ with $R \not\subseteq B$, then for every $r \in R \setminus B$, $r\Gamma \Sigma I \subseteq B$, for all $I \in L$, therefore $L \subseteq B$. Conversely, if $B$ is not prime, there exists $a, b \not\in B$ such that $a\Gamma \Sigma b \subseteq B$. But then $(a\Gamma)\Gamma (\Sigma b) \subseteq B$ and $a\Gamma S, S\Gamma b \not\subseteq B$. \hfill \square

**Proposition 3.7.** If a bi-$\Gamma$-ideal $B$ of $S$ is prime, then

$$I(B) = \{s \in B \mid S\Gamma s \Sigma S \subseteq B\}$$

is a prime $\Gamma$-ideal of $S$.

**Proof.** Suppose $B$ is prime and let $J_1 \Gamma J_2 \subseteq I(B)$, for two-sided ideals $J_1$ and $J_2$. Then, since $J_1 \Gamma J_2 \subseteq B$, by Proposition 3.6, $J_1 \subseteq B$ or $J_2 \subseteq B$. Now $I(B)$ is the largest $\Gamma$-ideal in $B$, it follows that $J_1 \subseteq I(B)$ or $J_2 \subseteq I(B)$. \hfill \square

**Theorem 3.8.** Every strongly irreducible, semiprime generalized bi-$\Gamma$-ideal of a $\Gamma$-semigroup $S$ is a strongly prime generalized bi-$\Gamma$-ideal.

**Proof.** Let $B$ be a strongly irreducible semiprime generalized bi-$\Gamma$-ideal of $S$. Suppose that $B_1, B_2$ are generalized bi-$\Gamma$-ideals of $S$ such that $B_1 \Gamma B_2 \cap B_2 \Gamma B_1 \subseteq B$. As $(B_1 \cap B_2)^2 \subseteq B_1 \Gamma B_2$ and $(B_1 \cap B_2)^2 \subseteq B_2 \Gamma B_1$, it follows that $(B_1 \cap B_2)^2 \subseteq B_1 \Gamma B_2 \cap B_2 \Gamma B_1 \subseteq B$. As $B$ is semiprime, we obtain $B_1 \cap B_2 \subseteq B$ and since $B$ is strongly irreducible, we obtain $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence $B$ is a strongly prime generalized bi-$\Gamma$-ideal of $S$. \hfill \square

**Theorem 3.9.** For any generalized bi-$\Gamma$-ideal $B$ of a $\Gamma$-semigroup $S$ and any $s \in S \setminus B$ there exists an irreducible generalized bi-$\Gamma$-ideal $J$ of $S$ such that $B \subseteq J$ and $s \not\in J$. 

Generalized bi-$\Gamma$-ideals

Proof. Suppose $GB_B = \{B_1 \mid B_1$ is a generalized bi-$\Gamma$-ideal of $S$ and $B \subseteq B_1$ and $s \notin B_1\}$. Obviously, $B \in GB_B$ and so $GB_B \neq \emptyset$. We have $GB_B$ is a partially ordered set under inclusion. Suppose $C$ is a chain of $GB_B$. Suppose $c \in (\bigcup C)\Gamma ST(\bigcup C)$. Then $c = c\alpha s\beta c''$ for some $c', c'' \in \bigcup C$, $s \in S$ and $\alpha, \beta \in \Gamma$. Therefore, $c' \in B_1$ and $c'' \in B_2$, for some $B_1, B_2 \in C$. As $C$ is a chain of $GB_B$, we obtain $B_1$ and $B_2$ are comparable. Thus $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$; so $c', c'' \in B_1$ or $c', c'' \in B_2$. As $B_1$ and $B_2$ are generalized bi-$\Gamma$-ideals of $S$, it follows that $c = c\alpha s\beta c'' \in B_1\Gamma STB_1 \subseteq B_1 \subseteq \bigcup C$ or $c = c\alpha s\beta c'' \in B_2\Gamma STB_2 \subseteq B_2 \subseteq \bigcup C$. Therefore, $c \in \bigcup C$, so $\bigcup C$ is a generalized bi-$\Gamma$-ideal of $S$. As $s \notin C$, for all $c \in C$, we obtain $s \notin \bigcup C$. Obviously, $B \subseteq \bigcup C$. Therefore, $\bigcup C \subseteq GB_B$. We have $C \subseteq \bigcup C$, for any $c \in C$. Therefore $\bigcup C$ is an upper bound $C$ in $GB_B$. By Zorn’s Lemma, there exists a maximal element $J \in GB_B$. Therefore, $J$ is a generalized bi-$\Gamma$-ideal of $S$ such that $B \subseteq J$ and $b \notin J$. Suppose $P$ and $Q$ are generalized bi-$\Gamma$-ideals of $S$ such that $P \cap Q = J$. Let $P \neq J$ and $Q \neq J$. Then $J = P \cap Q \subseteq P$ and $J = P \cap Q \subseteq Q$. So $B \subseteq J \subseteq P$ and $B \subseteq J \subseteq Q$. If $s \notin P$, then $C \in GB_B$. This is a contradiction since $J$ is a maximal element of $GB_B$, therefore $s \notin P$. In a similar fashion, we obtain $s \in Q$. Thus $s \in P \cap Q = J$ which is not possible. Therefore, $P = J$ or $Q = J$. Hence $J$ is an irreducible generalized bi-$\Gamma$-ideal. □

Theorem 3.10. For a $\Gamma$-semigroup $S$ the following statements are equivalent:

(i) $S$ is regular and intra-regular $\Gamma$-semigroup.

(ii) $B^*B = B$ for every generalized bi-$\Gamma$-ideal $B$ of $S$.

(iii) $B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$ for all generalized bi-$\Gamma$-ideals $B_1$ and $B_2$ of $S$.

(iv) Every generalized bi-$\Gamma$-ideal of $S$ is semiprime.

(v) Every proper generalized bi-$\Gamma$-ideal $B$ of $S$ is the intersection of irreducible semiprime generalized bi-$\Gamma$-ideals of $S$ containing $B$.

Proof. It follows by Theorem 3.9 [3]. □

Theorem 3.11. A generalized bi-$\Gamma$-ideal of a regular and intra-regular $\Gamma$-semigroup is strongly irreducible if and only if it is strongly prime.

Proof. Follows by Proposition 3.10 [3]. □

Theorem 3.12. In a $\Gamma$-semigroup $S$ each generalized bi-$\Gamma$-ideal is strongly prime if and only if $S$ is regular, intra-regular and the set of generalized bi-$\Gamma$-ideals of $S$ is a totally ordered under inclusion.

Proof. If each generalized bi-$\Gamma$-ideal of $S$ is strongly prime, then each generalized bi-$\Gamma$-ideal of $S$ is semiprime. Hence, by Theorem 3.10, $S$ is a regular and intra-regular $\Gamma$-semigroup. Thus the set of all its generalized bi-$\Gamma$-ideals is partially ordered by inclusion. If $B_1$ and $B_2$ are generalized bi-$\Gamma$-ideals of $S$, then $B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$, by Theorem 3.10. As $B_1 \cap B_2$ is a strongly prime generalized bi-$\Gamma$-ideal, we obtain $B_1 \subseteq B_1\cap B_2$ or $B_2 \subseteq B_1\cap B_2$. If $B_1 \subseteq B_1\cap B_2$, then $B_1 \subseteq B_2$. If $B_2 \subseteq B_1\cap B_2$, then $B_2 \subseteq B_1$. □
If $B_2 \subseteq B_1 \cap B_2$, then $B_2 \subseteq B_1$. Thus the set of all generalized bi-$\Gamma$-ideals of $S$ is totally ordered by inclusion.

The converse statement is a consequence of Theorem 3.12 in [3].

**Theorem 3.13.** If the set of all generalized bi-$\Gamma$-ideals of a $\Gamma$-semigroup $S$ is a totally ordered by inclusion, then $S$ is both regular and intra-regular if and only if each generalized bi-$\Gamma$-ideal of $S$ is prime.

**Proof.** By Theorem 3.13 in [3], each generalized bi-$\Gamma$-ideal of $S$ is prime.

Conversely, if each generalized bi-$\Gamma$-ideal of $S$ is prime, then it is semiprime. Theorem 3.10 completes the proof.

**Theorem 3.14.** For a $\Gamma$-semigroup $S$ the following statements are equivalent:

(i) The set of all generalized bi-$\Gamma$-ideals of $S$ is totally ordered by inclusion.

(ii) Every generalized bi-$\Gamma$-ideal of $S$ is strongly irreducible.

(iii) Every generalized bi-$\Gamma$-ideal of $S$ is irreducible.

**Proof.** (i) $\Rightarrow$ (ii). Let $B, B_1, B_2$ be generalized bi-$\Gamma$-ideals of $S$ such that $B_1 \cap B_2 \subseteq B$. Then by (i) we obtain $B_1 \subseteq B$ or $B_2 \subseteq B_1$. Therefore $B_1 = B_1 \cap B_2 \subseteq B$ or $B_2 = B_1 \cap B_2 \subseteq B$. Hence $S$ is strongly irreducible.

(ii) $\Rightarrow$ (iii). Let $B, B_1, B_2$ be generalized bi-$\Gamma$-ideals of $S$ such that $B_1 \cap B_2 = B$ for some strongly irreducible generalized bi-$\Gamma$-ideal $B$. Then $B \subseteq B_1$ and $B \subseteq B_2$. By the hypothesis, we obtain $B_1 \subseteq B$ or $B_2 \subseteq B$. So $B_1 = B$ or $B_2 = B$. Hence $B$ is irreducible.

(iii) $\Rightarrow$ (i). Suppose that $B_1, B_2$ are generalized bi-$\Gamma$-ideals of $S$. Then $B_1 \cap B_2$ is also a generalized bi-$\Gamma$-ideal of $S$ and by the assumption, $B_1 = B_1 \cap B_2 \subseteq B_2$ or $B_2 = B_1 \cap B_2 \subseteq B_1$. Therefore $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. This proves (i). 

**References**


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