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The filter theory in quotients of complete lattices

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Abstract. We study a partitioning filter F of a distributive complete lattice $(L, \vee, \wedge)$. Specifically, the properties and possible basic structures of the quotient L/F are investigated.

1. Introduction

P. J. Allen [1] introduced the notion of a Q-ideal and a construction process was presented by which one can build the quotient structure of a semiring modulo a Q-ideal. The present authors introduce the notion of a Q-filter F in the distributive complete lattice L and constructed the quotient semiring L/F. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of this semirings as well as relations between semirings of various extensions [2, 3, 4]. In this paper, we extend the definition and some results given in [1] and [2] to a more general Q-filter case.

An *upper bound* of a subset X of a poset $(L, \leqslant)$ is an element $a \in L$ containing every $x \in X$. The *least upper bound* is an upper bound contained in every other upper bound; it is denoted l.u.b. X or sup$X$ (sup$X$ is unique if it is exists). The notions of *lower bound* of X and *greatest lower bound* (g.l.b. X or inf$X$) of X are defined dually (inf$X$ is unique if it is exists). A *lattice* is a poset $(L, \leqslant)$ in which every couple elements x, y has a g.l.b. (called the *meet* of x and y, and written $x \wedge y$) and a l.u.b. (called the *join* of x and y, and written $x \vee y$). A lattice L is *complete* when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonempty complete lattice contains a least element 0 and greatest element 1. A lattice L is called a *distributive lattice* if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L. First we need the following well-known lemma.

Lemma 1.1. In a complete lattice L we have
(1) $a \wedge a = a, \ a \vee a = a$,
(2) $a \wedge b = b \wedge a, \ a \vee b = b \vee a$,
(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c), \ a \vee (b \vee c) = (a \vee b) \vee c$,
(4) $a \wedge 0 = 0$ and $a \vee 0 = a$,
(5) $a \vee b = 0$ implies $a = b = 0$,

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(6) $a \vee 1 = 1$ and $a \wedge 1 = a$.

2. Quotient of lattices

Let $(L, \vee, \wedge)$ be a distributive complete lattice with a least element 0 and greatest element 1. Then $(L, \vee)$ and $(L, \wedge)$ are commutative semigroups, connected by $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$, and there exist $0, 1 \in L$ such that $r \vee 0 = r$ and $r \wedge 0 = 0 \wedge r = 0$ and $r \vee 1 = 1 \vee r = r$ for all $r \in L$. Thus L is a commutative semiring with nonzero identity.

Remark 2.1. *Throughout this paper we shall assume, unless otherwise stated, that $(L, \vee, \wedge)$ is a distributive complete lattice semiring with a least element 0 and greatest element 1.*

Definition 2.2. Let L be as in Remark 2.1. A nonempty subset F of L is called a *filter* if it is closed under $\wedge$ and satisfies the condition $a \vee b \in F$ for all $a \in F$ and $b \in L$ (so $1 \in F$ and $\{1\}$ is a filter of L. Moreover, $0 \in F$ if and only if $L = F$).

Let L be as in Remark 2.1. A filter F of L is called *subtractive* if $x, x \wedge y \in F$ imply $y \in Y$ (so $\{1\}$ is a subtractive filter of L). If F is a filter of L and $x \wedge y \in F$ $(x, y \in L)$, then $x \vee (x \wedge y) = x \wedge (x \vee y) = x \in F$. Similarly, $y \in F$. Thus we have the following lemma:

Lemma 2.3. *Let L be as in Remark 2.1. Then every filter of L is subtractive.*

Definition 2.4. Let L be as in Remark 2.1. A filter F of L is called a *partitioning filter* (or a Q-filter denoted by F_Q) if there exists a subset Q of L such that

(1) $L = \bigcup\{q \wedge F : q \in Q\}$, where $a \wedge F = \{a \wedge t : t \in F\}$ for all $a \in L$,

(2) for $q_1, q_2 \in Q$ $(q_1 \wedge F) \cap (q_2 \wedge F) \neq \emptyset$ if and only if $q_1 = q_2$.

Example 2.5. Let $A = \{1, 2, 3\}$. Then the set $L = \{X : X \subseteq A\}$ forms a distributive complete lattice under set inclusion with greatest element A and least element $\emptyset$. It is clear that $F = \{A, \{1, 2\}\}$ is a Q-filter, where $Q = \{\{3\}, \{1, 3\}, \{2, 3\}, A\}$ (note that if $x, y \in L$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$).

Proposition 2.6. *Let L be as in Remark 2.1. If F is a filter of L and $x \in L$, then there exists a unique $q \in Q$ such that $x \wedge F \subseteq q \wedge F$. In particular, $x = q \wedge a$ for some $a \in F$.*

Proof. Let $x \in L$. Since $\{q \wedge F\}_{q \in Q}$ is a partition of L, there exists $q \in Q$ such that $x \in q \wedge F$. If $y \in x \wedge F$, there exists $a \in F$ such that $y = x \wedge a$. Since $x \in q \wedge F$, there exists $b \in F$ such that $x = q \wedge b$; hence $y = x \wedge a = q \wedge a \wedge b \in q \wedge F$. Thus $x \wedge F \subseteq q \wedge F$. The uniqueness follows from (2) of Definition 2.4. □

If F is a Q-filter of L and $q, q' \in Q$, then $q \vee q' \in (q \wedge F) \vee (q' \wedge F)$ and $(q \wedge F) \vee (q' \wedge F) \neq \emptyset$. So, on $L/F = \{q \wedge F : q \in Q\}$ we can define the binary operations $\bar{\vee}$ and $\bar{\wedge}$ as follows:

(1) $(q_1 \wedge F)\bar{\vee}(q_2 \wedge F) = q_3 \wedge F$, where q_3 is the unique element in Q such that $(q_1 \vee q_2) \wedge F \subseteq q_3 \wedge F$,

(2) $(q_1 \wedge F)\bar{\wedge}(q_2 \wedge F) = q_3 \wedge F$, where q_3 is the unique element in Q such that $(q_1 \wedge q_2) \wedge F \subseteq q_3 \wedge F$ (note that $q_1 \wedge F = q_2 \wedge F$ if and only if $q_1 = q_2$).

Proposition 2.7. *Let L be as in Remark* 2.1. *If F is a Q-filter of L, then $(L/F, \bar{\vee})$ and $(L/F, \bar{\wedge})$ are commutative monoids.*

Proof. Clearly, $\bar{\vee}$ and $\bar{\wedge}$ are well-defined and they are commutative operations. Now we show that

$$(q_1 \wedge F)\bar{\vee}[(q_2 \wedge F)\bar{\vee}(q_3 \wedge F)] = [(q_1 \wedge F)\bar{\vee}(q_2 \wedge F)]\bar{\vee}(q_3 \wedge F).$$

There exists the unique element q' of Q such that $(q_1 \wedge F)\bar{\vee}[(q_2 \wedge F)\bar{\vee}(q_3 \wedge F)] = (q_1 \wedge F)\bar{\vee}(q' \wedge F)$, where

$$(q_2 \vee q_3) \wedge F \subseteq q' \wedge F. \tag{1}$$

Also we have $(q_1 \wedge F)\bar{\vee}(q' \wedge F) = t_1 \wedge F$, where t_1 is the unique element of Q such that $(q_1 \vee q') \wedge F \subseteq t_1 \wedge F$, and set $e = q_1 \vee q_2 \vee q_3$. Now (1) gives

$$e \in (q_1 \vee q_2 \vee q_3) \wedge F \subseteq (q_1 \wedge F) \vee (q_2 \vee q_3) \wedge F \subseteq (q_1 \wedge F) \vee (q' \wedge F) \subseteq t_1 \wedge F. \tag{2}$$

By assumption, $[(q_1 \wedge F)\bar{\vee}(q_2 \wedge F)]\bar{\vee}(q_3 \wedge F) = (t_2 \wedge F)\bar{\vee}(q_3 \wedge F) = t_3 \wedge F$, where t_2 and t_3 are the unique elements of Q such that $(q_1 \vee q_2) \wedge F \subseteq (t_2 \wedge F$ and $(t_2 \vee q_3) \wedge F \subseteq t_3 \wedge F$. It follows that

$$e \in (q_1 \vee q_2 \vee q_3)F \subseteq (q_1 \vee q_2) \wedge F \vee (q_3 \wedge F) \subseteq (t_2 \wedge F) \vee (q_3 \wedge F \subseteq t_3 \wedge F. \tag{3}$$

Now (2) and (3) give $t_1 = t_3$, and so $\bar{\vee}$ is an associative operation.

Next, we will show that $(L/F, \bar{\vee})$ has a zero element. By Proposition 2.6, there is a unique element $q_0 \in Q$ such that $0 \wedge F \subseteq q_0 \wedge F$; so $0 = q_0 \wedge a$ for some $a \in F$. We show that $q_0 \wedge F$ is the zero in L/F. If $q \wedge F \in L/F$, then $(q \wedge F)\bar{\vee}(q_0 \wedge F = q' \wedge F$, where q' is the unique element of Q such that $(q \vee q_0) \wedge F \subseteq q' \wedge F$, so $q \vee q_0 = q' \wedge c$ for some $c \in F$. Thus $q \wedge a = q' \wedge c \wedge a$; hence $q \wedge a \in (q \wedge F) \cap (q' \wedge F)$. It follows that $q = q'$, and so $(q \wedge F)\bar{\vee}(q_0 \wedge F) = q \wedge F$. Similarly, $(q_0 \wedge F)\bar{\vee}(q \wedge F) = q \wedge F$. By an argument like that case $\bar{\vee}$ above, $\bar{\wedge}$ is an associative operation. Finally, let $q_e \in Q$ be a unique element such that $1 \wedge F \subseteq q_e \wedge F$; so $1 = q_e \wedge d$ for some $d \in F$. We show that $q_e \wedge F$ is the identity in L/F. Let $q \wedge F \in L/F$ and $(q \wedge F)\bar{\wedge}(q_e \wedge F) = q' \wedge F$, where q' is the unique element of Q such that $(q \wedge q_e) \wedge F \subseteq q' \wedge F$. Since $1 \wedge F \subseteq q_e \wedge F$, we have $q \wedge F \subseteq (q \wedge q_e) \wedge F \subseteq q' \wedge F$; thus $q = q'$. It follows that $(q \wedge F)\bar{\wedge}(q_e \wedge F) = q \wedge F$ for all $q \wedge F \in L/F$. Similarly, $(q_e \wedge F)\bar{\wedge}(q \wedge F) = q \wedge F$. □

Theorem 2.8. *Let L be as in Remark* 2.1. *If F is a Q-filter of L, then $(L/F, \bar{\vee}, \bar{\wedge})$ is a commutative semiring with identity.*

Proof. Assume that $q_1 \wedge F, q_2 \wedge F, q_3 \wedge F \in L/F$; we show that

$$(q_1 \wedge F)\bar{\wedge}[(q_2 \wedge F)\bar{\vee}(q_3 \wedge F)] = [(q_1 \wedge F)\bar{\wedge}(q_2 \wedge F)]\bar{\vee}[(q_1 \wedge F)\bar{\wedge}(q_3 \wedge F)].$$

There exists a unique element q_{23} of Q such that $(q_1 \wedge F)\bar{\wedge}[(q_2 \wedge F)\bar{\vee}(q_3 \wedge F)] = (q_1 \wedge F)\bar{\wedge}(q_{23} \wedge F)$, where

$$(q_2 \vee q_3) \wedge F \subseteq q_{23} \wedge F, \tag{4}$$

so $q_1 \wedge [(q_2 \vee q_3) \wedge F] \subseteq (q_1 \wedge q_{23}) \wedge F$. Also we have $(q_1 \wedge F)\bar{\wedge}(q_{23} \wedge F) = q' \wedge F$, where q' is the unique element of Q such that $(q_1 \wedge q_{23}) \wedge F \subseteq q' \wedge F$. Now (4) gives

$$q_1 \wedge (q_2 \vee q_3) \in q_1 \wedge [(q_2 \vee q_3) \wedge F] \subseteq (q_1 \wedge q_{23}) \wedge F \subseteq q' \wedge F. \tag{5}$$

By assumption, $[(q_1 \wedge F)\bar{\wedge}(q_2 \wedge F)]\bar{\vee}[(q_1 \wedge F)\bar{\wedge}(q_3 \wedge F)] =$

$$(q_{12} \wedge F)\bar{\vee}(q_{13} \wedge F) = q'' \wedge F,$$

where q_{12}, q_{13} and q'' are the unique elements of Q such that $(q_1 \wedge q_2) \wedge F \subseteq q_{12} \wedge F$, $(q_1 \wedge q_3) \wedge F \subseteq q_{13} \wedge F$, and $(q_{12} \vee q_{13}) \wedge F \subseteq q'' \wedge F$. Thus $[(q_1 \wedge q_2) \wedge F] \vee [(q_1 \wedge q_3) \wedge F] \subseteq q'' \wedge F$. Now by (5), $q_1 \wedge (q_2 \vee q_3) = (q_1 \wedge q_2) \vee (q_1 \wedge q_3) \in (q' \wedge F) \cap (q'' \wedge F)$; hence $q' = q''$, and so we have equality. Thus $\bar{\wedge}$ distributes over $\bar{\vee}$ from the left. Likewise, $\bar{\wedge}$ distributes over $\bar{\vee}$ from the right. Assume that $q_0 \wedge F$ is the zero in L/F and let $(q \wedge F)\bar{\wedge}(q_0 \wedge F) = q' \wedge F$, where q' is the unique element of Q such that $(q \wedge q_0) \wedge F \subseteq q' \wedge F$. But $0 \wedge F \subseteq (q_0 \wedge q) \wedge F \subseteq q' \wedge F$, hence $q_0 = q'$. Thus $(q \wedge F)\bar{\wedge}(q_0 \wedge F) = q_0 \wedge F$ for all $q \wedge F \in L/F$. Similarly, $(q_0 \wedge F)\bar{\wedge}(q \wedge F) = q_0 \wedge F$ for all $q \wedge F \in L/F$. Now the assertion follows from Proposition 2.7. □

Theorem 2.9. *Assume that L is as in Remark* 2.1 *and let F be a partitioning filter of L with respect to two subsets Q_1 and Q_2 of L. Then*

(1) *L/F_{Q_1} and L/F_{Q_2} are equal as sets,*

(2) *$L/F_{Q_1} \cong L/F_{Q_2}$.*

Proof. (1). Let $q_1 \wedge F \in L/F_{Q_1}$. Since $q_1 \in L$, there exists a unique $q_2 \in Q_2$ such that $q_1 \wedge F \subseteq q_2 \wedge F$ by Proposition 2.6. Again there exists a unique $q_1' \in Q_1$ such that $q_2 \wedge F \subseteq q_1' \wedge F$. It follows that $q_1 \wedge F = q_2 \wedge F = q_1' \wedge F \in R/I_{Q_2}$. Thus $L/F_{Q_1} \subseteq L/F_{Q_2}$. Likewise, $L/F_{Q_2} \subseteq L/F_{Q_1}$.

(2). Define $\varphi : L/F_{Q_1} \to L/F_{Q_2}$ by $\varphi(q \wedge F) = q' \wedge F$, where q' is the unique element of Q_2 such that $q \wedge F \subseteq q' \wedge F$. Clearly, φ is well-defined.

Let $q_1 \wedge F, q_2 \wedge F \in L/F_{Q_1}$. Then

$$\varphi((q_1 \wedge F)\bar{\vee}(q_2 \wedge F)) = \varphi(q_3 \wedge F) = q_4 \wedge F, \tag{6}$$

where $q_3 \in Q_1$ is the unique element such that $(q_1 \vee q_2) \wedge F \subseteq q_3 \wedge F$ and $q_4 \in Q_2$ is the unique element such that $q_3 \wedge F \subseteq q_4 \wedge F$. Now $q_1 \vee q_2 \in q_3 \wedge F \subseteq q_4 \wedge F$. Also,

$$\varphi(q_1 \wedge F)\bar{\vee}\varphi(q_2 \wedge F) = (q_5 \wedge F)\bar{\vee}(q_6 \wedge F) = q_7 \wedge F, \tag{7}$$

where $q_5, q_6 \in Q_2$ are unique elements such that $q_1 \wedge F \subseteq q_5 \wedge F$, $q_2 \wedge F \subseteq q_6 \wedge F$, and $q_7 \in Q_2$ is the unique element such that $(q_5 \vee q_6) \wedge F \subseteq q_7 \wedge F$. Now $q_1 \vee q_2 \in (q_4 \wedge F) \cap (q_7 \wedge F)$. Thus $q_4 = q_7$. Therefore, by (6) and (7), $\varphi((q_1 \wedge F)\bar{\vee}(q_2 \wedge F)) = \varphi(q_1 \wedge F)\bar{\vee}\varphi(q_2 \wedge F)$. Similarly, it can be shown that $\varphi((q_1 \wedge F)\bar{\wedge}(q_2 \wedge F)) = \varphi(q_1 \wedge F)\bar{\wedge}\varphi(q_2 \wedge F)$.

Let $q_2 \wedge F \in L/F_{Q_2}$. Since $q_2 \in R$, there is a unique element q_1 of Q_1 such that $q_2 \wedge F \subseteq q_1 \wedge F$ by Proposition 2.6. But then there exists a unique $q_2' \in Q_2$ such that $q_1 \wedge F \subseteq q_2' \wedge F$. Now $q_2 = q_2'$ gives $q_2 \wedge F = q_2' \wedge F$, and hence $\varphi(q_1 \wedge F) = q_2 \wedge F$. Thus φ is onto. Suppose that $\varphi(q_1 \wedge F) = \varphi(q_2 \wedge F) = q \wedge F$ say, where $q \in Q_2$ is a unique such that $q_1 \wedge F \subseteq q \wedge F$ and $q_2 \wedge F \subseteq q \wedge F$. Since $q \in L$, there exists a unique $q' \in Q_1$ such that $q \wedge F \subseteq q' \wedge F$; hence $q_1 = q' = q_2$. So $q_1 \wedge F = q_2 \wedge F$. Thus $\varphi : L/F_{Q_1} \to L/F_{Q_2}$ is an isomorphism. □

Lemma 2.10. *Assume that L is as in Remark 2.1 and let F be a Q-filter of L.*

(1) *There exists a unique $q_e \in Q$ such that $F = q_e \wedge F$. In particular, $q_e \wedge F$ is the identity element of L/F.*

(2) *If F' is a filter of L with $F \subseteq F'$, then F is a $F' \cap Q$-filter of F'.*

Proof. (1). Since $1 \in L$, by Proposition 2.6, there exists a unique $q_e \in Q$ such that $F = 1 \wedge F \subseteq q_e \wedge F$; hence $1 = q_e \wedge a$ for some $a \in F$. Now it suffices to show that $q_e \wedge F \subseteq F$. Let $x \in q_e \wedge F$. Then $x = q_e \wedge b$ for some $b \in F$; so $x = (q_e \wedge b) \wedge 1 = q_e \wedge b \wedge a \in F$. Finally, by an argument like that in Proposition 2.7, $q_e \wedge F$ is the identity element of L/F.

(2). It suffices to show that $F' = \cup\{q \wedge F : q \in Q \cap F'\}$. Since the inclusion $\cup\{q \wedge F : q \in Q \cap F'\} \subseteq F'$ is clear, we will prove the reverse inclusion. Let $x \in F'$. By Proposition 2.6, $x = q \wedge a$ for some $q \in Q$ and $a \in F \subseteq F'$. Then $q \in Q \cap F'$ since F' is a subtractive filter of L, and so we have equality. □

Theorem 2.11. *Assume that L is as in Remark 2.1 and let F be a Q-filter of L.*

(1) *If F' is a subtractive filter of L and $F \subseteq F'$, then $F'/F = \{q \wedge F : q \in Q \cap F'\}$ is a subtractive filter of L/F.*

(2) *If F' is a subtractive filter of L/F, then $F' = J/F$ for some subtractive filter J of L.*

Proof. (1). Let q_e be the unique element in Q such that $q_e \wedge F$ is the identity in L/F. First, we show that $q_e \wedge F \in F'/F$. Let $a \wedge F \in F'/F \subseteq L/F$, where $a \in F' \cap Q$. Then $(a \wedge F)\bar{\wedge}(q_e \wedge F) = a \wedge F$, where $(q_e \wedge a) \wedge F \subseteq a \wedge F$; hence $a \wedge q_e = a \wedge c \in F'$ for some $c \in F$. Thus $q_e \in F' \cap Q$ since F' is subtractive; so $q_e \wedge F \in F'/F$. Next, suppose that $q_1 \wedge F, q_2 \wedge F \in F'/F$; we show that $(q_1 \wedge F)\bar{\wedge}(q_2 \wedge F) \in F'/F$. Since F is a Q-filter, there is a unique element $q_3 \in Q$ with $(q_1 \wedge F)\bar{\wedge}(q_2 \wedge F) = q_3 \wedge F$, where $(q_1 \wedge q_2) \wedge F \subseteq q_3 \wedge F$, so $q_1 \wedge q_2 = q_3 \wedge b \in F'$ for some $b \in F$; hence $q_3 \in F' \cap Q$ since F' is a subtractive filter of L. Therefore, $(q_1 \wedge F)\bar{\wedge}(q_2 \wedge F) \in F'/F$. Now it is enough to show that if $r \wedge F \in L/F$ and $a \wedge F \in F'/F$ (for some $r \in Q$, $a \in F' \cap Q$), then $(r \wedge F)\bar{\vee}(a \wedge F) \in F'/F$. There exists a unique element $q_4 \in Q$ such that $(r \wedge F)\bar{\wedge}(a \wedge F) = q_4 \wedge F$, where

$r \vee a \in (r \vee a) \wedge F \subseteq q_4 \wedge F$, so $r \vee a = q_4 \wedge d \in F'$ for some $d \in F$. It follows that $q_4 \in F' \cap Q$; hence $q_4 \wedge F \in F'/F$. Thus F'/F is a filter of L/F. Finally, assume that $t \wedge F \in F'/F$ and $(t \wedge F)\bar{\wedge}(s \wedge F) = u \wedge F \in F'/F$, where $t, u \in F' \cap Q$, $s \in Q$, and $(t \wedge s) \wedge F \subseteq u \wedge F$. Then $t \wedge s = u \wedge d \in F'$ for some $d \in F$; thus $s \in F' \cap Q$ since F' is a subtractive filter. Therefore, $s \wedge F \in F'/F$, as needed.

(2). Assume that q_e is the unique element in Q such that $q_e \wedge F$ is the identity in L/F and set $J = \{ r \in L \ : \ \exists q \in Q \ \ s.t \ \ r \in q \wedge F \ , \ q \wedge F \in F'\}$. The proof can now be broken down into a sequence of steps.

i) $F \subseteq J$. Let $a \in F$. By Proposition 2.7, $a \in F = q_e \wedge F \in F'$, so $a \in J$. Thus $F \subseteq J$. Since $1 \in F$, $1 \in J$.

ii) J is a filter of L. For if r, s in J, there are elements $q_1, q_2 \in Q$ such that $q_1 \wedge F, q_2 \wedge F \in F'$, $r = q_1 \wedge c, s = q_2 \wedge d$ for some $c, d \in F$, and $(q_1 \wedge F)\bar{\wedge}(q_2 \wedge F) = q_3 \wedge F \in F'$, where $q_3 \in Q$ is the unique element such that $(q_1 \wedge q_2) \wedge F \subseteq q_3 \wedge F$; hence $r \wedge s \in (q_1 \wedge q_2) \wedge F \subseteq q_3 \wedge F \in F'$. Thus $r \wedge s \in J$. Similarly, if $r \in J$ and $t \in L$, then there are elements $q_1, q_2 \in Q$ such that $r \in q_1 \wedge F \in F'$ and $t \in q_2 \wedge F$. Since F' is a filter of R/I, $(q_1 \wedge F)\bar{\vee}(q_2 \wedge F) = q_3 \wedge F \in F'$, where $r \wedge t \in (q_1 \vee (q_2) \wedge F \subseteq q_3 \wedge F$; thus $r \vee t \in J$.

iii) J is a subtractive filter of L. Let $a, a \wedge b \in J$. Then there are elements q_1, q_2, and q_3 of Q such that $a \in q_1 \wedge F \in F'$, $ab \in q_2 \wedge F \in F'$ and $b \in q_3 \wedge F$, so $a = q_1 \wedge c$, $a \wedge b = q_2 \wedge d$ and $b = q_3 \wedge f$ for some $c, d, f \in F$; hence $a \wedge b \in (q_4 \wedge F) \cap (q_2 \wedge F)$, where q_4 is a unique element of Q such that $(q_1 \wedge F)\bar{\wedge}(q_3 \wedge F) = q_4 \wedge F$; hence $q_2 = q_4$. Therefore, $q_3 \wedge F \in F'$ since F' is a subtractive filter; so $b \in J$. Thus J is a subtractive filter of L. Finally, we can see that $F' = J/F = \{q \wedge F : q \in J \cap Q\}$. □

Definition 2.12. Let L be as in Remark 2.1. L is called an *L-domain*, if $a \vee b = 1$ $(a, b \in L)$, then either $a = 1$ or $b = 1$. A proper filter F of L is called *prime* if $x \vee y \in F$, then $x \in F$ or $y \in F$.

Theorem 2.13. *Assume that L is as in Remark 2.1 and let F be a Q-filter of L.*

(1) *If P is a filter of L with $F \subseteq P$, then P is a prime filter of L if and only if P/F is a prime filter of L/F.*

(2) *F is a prime filter of L if and only if L/F is a L-domain.*

Proof. (1). Assume that P is a prime filter of L and let $q_1 \wedge F, q_2 \wedge F \in L/F$ be such that $(q_1 \wedge F)\bar{\vee}(q_2 \wedge F)) \in P/F$, where $q_1, q_2 \in Q$. There exists a unique $q_3 \in Q \cap P$ such that $q_1 \vee q_2 \in (q_1 \vee q_2) \wedge F \subseteq q_3 \wedge F \in P/F$; so $q_1 \vee q_2 = q_3 \wedge c$ for some $c \in F$; hence $q_1 \vee q_2 \in P$. Then P prime gives $q_1 \in P$ or $q_2 \in P$; thus either $q_1 \wedge F \in P/F$ or $q_2 \wedge F \in P/$.

Conversely, suppose that P/F is a prime filter and let $x, y \in L$ such that $x \vee y \in P$. Then there exist $q_4, q_5 \in Q$ such that $x \in q_4 \wedge F$ and $y \in q_5 \wedge F$; so $x = q_4 \wedge e$ and $y = q_5 \wedge f$ for some $e, f \in F$. Let q be the unique element in Q such that $(q_4 \wedge F)\bar{\vee}(q_5 \wedge F) = q \wedge F$, where $(q_4 \vee q_5) \wedge F \subseteq q \wedge F$. It follows that $x \vee y = q \wedge d \in P$ for some $d \in F$; so $q \in P$ since P is a subtractive filter; hence $(q_4 \wedge f)\bar{\vee}(q_5 \wedge F) = q \wedge F \in P/F$. Now P/F is a prime filter gives either

$q_4 \wedge F \in P/F$ or $q_5 \wedge F \in P/F$. Therefore, either $q_4 \in P$ (so $x \in P$) or $q_5 \in P$ (so $y \in P$). Thus P is a prime filter of L.

(2). Let q_e be the unique element in Q such that $q_e \wedge F$ is the identity in L/F. Let F be a prime filter of L and $q_1 \wedge F, q_2 \wedge F$ be elements of L/F such that $(q_1 \wedge F)\bar{\vee}(q_2 \wedge F) = q_e \wedge F$, where $(q_1 \vee q_2) \wedge F \subseteq q_e \wedge F = F$. Hence $(q_1 \vee q_2) \wedge a = (q_1 \wedge a) \vee (q_2 \wedge a) \in F$ for every $a \in F$. Since P is a prime filter, either $q_1 \wedge a \in F$ or $q_2 \wedge a \in F$; hence $(q_1 \wedge F) \cap (q_e \wedge F) \neq \emptyset$ or $(q_2 \wedge F) \cap (q_e \wedge F) \neq \emptyset$. This implies that $q_1 \wedge F = q_e \wedge F$ or $q_2 \wedge F = q_e \wedge F$.

Conversely, assume that L/F is a L-domain and let $a \vee b \in F$ for some $a, b \in L$. Since F is a partitioning filter, there exist $q_1, q_2 \in Q$ such that $a \in q_1 \wedge F$ and $b \in q_2 \wedge F$. There exists a unique $q_3 \in Q$ such that $(q_1 \wedge F)\bar{\vee}(q_2 \wedge F = q_3 \wedge F$, where $a \vee b \in (q_1 \wedge F) \vee (q_2 \wedge F) = (q_1 \vee q_2) \wedge F \subseteq q_3 \wedge F$; hence $q_3 = q_e$ since $a \vee b \in (q_3 \wedge F) \cap (q_e \wedge F)$. As L/F is a L-domain, $q_1 \wedge F = q_e \wedge F$ or $q_2 \wedge F = q_e \wedge F$. Thus $a \in F$ or $b \in F$, and the proof is complete. □

Let L be as in Remark 2.1. If A is an arbitrary nonempty subset of L, then the set $T(A)$ consisting of all elements of L of the form $(a_1 \wedge a_2 \wedge \cdots \wedge a_n) \vee x$ (with $a_i \in A$ for all $1 \leqslant i \leqslant n$ and $x \in L$) is a filter of L containing A (let $u = (a_1 \wedge a_2 \wedge \cdots \wedge a_n) \vee x, v = (b_1 \wedge b_2 \wedge \cdots \wedge b_m) \vee y \in T(A)$ and $z \in L$. An inspection will show that $u \wedge v = (\wedge_{i=1}^n a_i \wedge (\wedge_{i=1}^m b_i) \vee t \in T(A)$ for some $t \in L$ and $u \vee z = ((\wedge_{i=1}^n a_i) \vee (r \vee z) \in T(A)$; hence $T(A)$ is a filter of L).

Theorem 2.14. *Let L be as in Remark* 2.1. *If F is a maximal filter of L, then F is a prime filter.*

Proof. Let $a \vee b \in F$, $a \notin F$ and $b \notin F$. As F is a maximal filter, $T(F \cup \{a\}) = T(F \cup \{b\}) = L$ since $F \subsetneqq T(F \cup \{a\}) \subseteq L$ and $F \subsetneqq T(F \cup \{b\}) \subseteq L$. Since $0 \in L$, we split the proof into three cases for $T(F \cup \{a\})$.

Case 1: There exist $m_1, ..., m_n \in F$ and $r \in L$ such that $(m_1 \wedge m_2 \wedge ... \wedge m_n) \vee r = 0$. Since F is a filter, we have $0 \in F$ which is a contradiction.

Case 2: $a \vee r = 0$ for some $r \in L$. So $b = b \vee a \vee r$; hence $b \in F$, a contradiction.

Case 3: There exist $m, n \in F$, $r, s \in L$ and a positive integers t, k such that $(m \wedge \bigwedge_{i=1}^t a) \vee r = (m \wedge a) \vee r = 0$ and $(n \wedge \bigwedge_{i=1}^k b) \vee s = (n \wedge b) \vee s = 0$; hence $m \wedge a = 0 = n \wedge b$. It follows tha $m \wedge n \wedge a = m \wedge n \wedge b = 0$. Thus $(m \wedge n) \wedge (a \vee b)) = (m \wedge n \wedge a) \vee (m \wedge n \wedge b) = 0$. As $(m \wedge n) \wedge (a \vee b) \in F$, we obtain $0 \in F$, a contradiction. Thus F is a prime filter of L. □

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