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On dual ordered semigroups

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Abstract. We introduce and study dual ordered semigroups. Obtained results generalize the results on semigroups without order.

1. Introduction

A dual ring credited to Baer [1] and Kaplansky [6] have been widely studied (see [3], [4], [5], [8]). Using only the multiplication properties of the elements of the ring, Schwarz [10] introduced and studied dual semigroups. The author proved fundamental structure theorem concerning such semigroups. The purpose of this paper is to define and study dual ordered semigroups. The results obtain extend the results on semigroup without order.

For the rest of this section, we recall some definitions and results used throughout the paper.

A semigroup $(S,\cdot)$ together with a partial order $\leqslant$ (on S) that is compatible with the semigroup operation, meaning that for $x,y,z\in S$,

$$x\leqslant y\Rightarrow zx\leqslant zy,\quad xz\leqslant yz,$$

is called an *ordered semigroup* (see [2], [4]). An element 0 of S is called a *zero* element of S if $0x=x0=0$ for all $x\in S$ and $0\leqslant x$ for all $x\in S$.

For non-empty subsets A,B of an ordered semigroup $(S,\cdot,\leqslant)$, let

$$AB=\{xy\mid x\in A, y\in B\},$$

$$(A]=\{x\in S\mid x\leqslant a \text{ for some } a\in A\}.$$

If $x\in S$, we write Ax for $A\{x\}$ and xA for $\{x\}A$. Note that

(1) $A\subseteq (A]$;

(2) $A\subseteq B$ implies $(A]\subseteq (B]$;

(3) $((A](B]]=(AB]$.

Let $(S,\cdot,\leqslant)$ be an ordered semigroup. A non-empty subset A of S is called a *left* (*respectively, right*) *ideal* [7] of S if

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(i) $SA \subseteq A$ (respectively, $AS \subseteq A$), and

(ii) for $x \in A$ and $y \in S$, $y \leqslant x$ implies $y \in A$.

The second condition means that $A = (A]$.

A non-empty subset A of S is called a (*two-sided*) *ideal* of S if A is both a left and a right ideal of S. Note that if A and B are ideals of S then $(AB]$ is an ideal of S. Intersection of ideals of S is an ideal of S if it is non-empty. Union of ideals of S is an ideal of S. Finite intersection of ideals of S is an ideal of S.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup with zero 0. Analogously to [10], if A is a non-empty subset of S, then the *left annihilator* of A, denoted by $l(A)$, is defined by

$$l(A) = \{x \in S \mid xA = 0\}.$$

Dually, the *right annihilator* of A, denoted by $r(A)$, is defined by

$$r(A) = \{x \in S \mid Ax = 0\}.$$

It is easy to see that $l(A)A = 0$, $Ar(A) = 0$.

Lemma 1.1. *The following hold for an ordered semigroup* $(S, \cdot, \leqslant)$*:*

(1) *for* $\emptyset \neq A \subseteq S$, $l(A)$ *is a left ideal of* S *and* $r(A)$ *is a right ideal of* S;

(2) *for* $\emptyset \neq A \subseteq S$, $A \subseteq r(l(A))$ *and* $A \subseteq l(r(A))$;

(3) *for* $\emptyset \neq A_1 \subseteq A_2 \subseteq S$, $l(A_2) \subseteq l(A_1)$ *and* $r(A_2) \subseteq r(A_1)$;

(4) *if* M *is a right* (*left*) *ideal of* S, *then* $r(M)$ (*respectively,* $l(M)$) *is a* (*two-sided*) *ideal of* S.

Proof. (1). We prove that $l(A)$ is a left ideal of S. Dually, we have $r(A)$ is a right ideal of S. Clearly, $l(A) \neq \emptyset$. If $x \in S$, $y \in l(A)$, then $(xy)A = x(yA) = 0$, and so $xy \in l(A)$. Let $x \in l(A)$ and $y \in S$ such that $y \leqslant x$. Then $yA \subseteq xA = 0$, and hence $y \in l(A)$.

(2). Since $l(A)A = 0$, $A \subseteq r(l(A))$. Similarly, $A \subseteq l(r(A))$.

(3). If $x \in l(A_2)$, since $A_1 \subseteq A_2$, then $xA_1 \subseteq xA_2 = 0$. Thus $x \in l(A_1)$. Similarly, $r(A_2) \subseteq r(A_1)$.

(4). Assume that M is a right ideal of S. Since

$$M(Sr(M)) = (MS)r(M) \subseteq Mr(M) = 0,$$

we have $Sr(M) \subseteq r(M)$. Since

$$M(r(M)S) = (Mr(M))S = 0S = 0,$$

we get $r(M)S \subseteq r(M)$. If $x \in r(M), y \in S$ such that $y \leqslant x$, then $My \subseteq Mx = 0$, and thus $y \in r(M)$. It is now conclude that $r(M)$ is an ideal of S. By the similar arguments we obtain that if M is a left ideal of S, then $l(M)$ is an ideal of S. □

Lemma 1.2. *Let $\{A_\alpha \mid \alpha \in \Lambda\}$ be a collection of subsets of an ordered semigroup $(S, \cdot, \leqslant)$. Then*

(1) $l(\bigcup_{\alpha\in\Lambda} A_\alpha) = \bigcap_{\alpha\in\Lambda} l(A_\alpha)$,

(2) $r(\bigcup_{\alpha\in\Lambda} A_\alpha) = \bigcap_{\alpha\in\Lambda} r(A_\alpha)$.

Proof. The proof is straightforward. □

2. Dual ordered semigroups

The notion of dual ordered semigroups is defined as follows:

Definition 2.1. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then S is said to be *dual* if $l(r(L)) = L$ for every left ideal L of S, and $r(l(R)) = R$ for every right ideal R of S.

Lemma 2.2. *Let $(S, \cdot, \leqslant)$ be a dual ordered semigroup. Let $\{R_\alpha \mid \alpha \in \Lambda\}$ be a family of right ideals of S, and let $\{L_\alpha \mid \alpha \in \Lambda\}$) be a family of left ideals of S. Then*

(1) $l(\bigcap_{\alpha\in\Lambda} R_\alpha) = \bigcup_{\alpha\in\Lambda} l(R_\alpha)$,

(2) $r(\bigcap_{\alpha\in\Lambda} L_\alpha) = \bigcup_{\alpha\in\Lambda} r(L_\alpha)$.

Proof. By Lemma 1.2, we obtain

$$\bigcup_{\alpha\in\Lambda} l(R_\alpha) = l(r(\bigcup_{\alpha\in\Lambda} l(R_\alpha))) = l(\bigcap_{\alpha\in\Lambda} r(l(R_\alpha))) = l(\bigcap_{\alpha\in\Lambda} R_\alpha)$$

and

$$\bigcup_{\alpha\in\Lambda} r(L_\alpha) = r(l(\bigcup_{\alpha\in\Lambda} r(L_\alpha))) = r(\bigcap_{\alpha\in\Lambda} l(r(L_\alpha))) = r(\bigcap_{\alpha\in\Lambda} L_\alpha).$$

□

Corollary 2.3. *Let $(S, \cdot, \leqslant)$ be a dual ordered semigroup.*

(1) *If L_1, L_2 are left ideals of S such that $L_1 \cap L_2 = 0$, then $r(L_1) \cup r(L_2) = S$.*

(2) *If R_1, R_2 are right ideals of S such that $R_1 \cap R_2 = 0$, then $l(R_1) \cup l(R_2) = S$.*

Lemma 2.4. *Let $(S, \cdot, \leqslant)$ be a dual ordered semigroup. Let L be a left ideal of S and R be a right ideal of S. Then*

(1) *$l(S) = r(S) = 0$.*

(2) *If $L \neq S$, then $r(L) \neq 0$.*

(3) *If $R \neq S$, then $l(R) \neq 0$.*

(4) *If $L \neq 0$, then $r(L) \neq S$.*

(5) *If $R \neq 0$, then $l(R) \neq S$.*

Proof. By Lemma 1.1,

$$r(S) = r(S \cup l(0)) = r(S) \cap r(l(0)) = r(S) \cap \{0\} = 0.$$

Similarly, we have $l(S) = 0$. Then (1) holds. If $r(L) = 0$, then $L = l(r(L)) = l(0) = S$. Similarly, if $l(R) = 0$. then $R = r(l(R)) = r(0) = S$. Hence (2) and (3) hold. For (4) and (5), $r(L) = S$ implies $L = l(r(L)) = l(S) = 0$. Similarly, if $l(R) = S$, then $R = r(l(R)) = r(S) = 0$. □

Corollary 2.5. *In a dual ordered semigroup $(S, \cdot, \leqslant)$, $(xS] = 0$ or $(Sx] = 0$ implies $x = 0$.*

Proof. This follows from Lemma 2.4(1). □

Lemma 2.6. *Let $(S, \cdot, \leqslant)$ be a dual ordered semigroup.*

(1) *If L is a minimal left ideal of S, then $r(L)$ is a maximal right ideal of S.*

(2) *If M is a minimal two-sided ideal of S, then $r(M)$ and $l(M)$ are maximal two-sided ideals of S.*

Proof. (1). Assume that L is a minimal left ideal of S. Let A be a proper right ideal of S such that $r(L) \subseteq A$. Then $0 \neq l(A) \subseteq l(r(L)) = L$, and thus $l(A) = L$. Hence $A = r(l(A)) = r(L)$.

(2). Assume that M is a minimal ideal of S. Let A be a proper two-sided ideal of S such that $r(M) \subseteq A$. Then $0 \neq l(A) \subseteq l(r(M)) = M$, and thus $l(A) = M$. Hence $A = r(l(A)) = r(M)$. Similarly, we have $l(M)$ is a maximal ideal of S. □

Lemma 2.7. *If $(S, \cdot, \leqslant)$ is a dual ordered semigroup, then $x \in (xS]$ and $x \in (Sx]$ for every $x \in S$.*

Proof. Let $x \in S$. We prove that $x \in (xS]$ ($x \in (Sx]$ can be proved analogously). Since $(xS] = r(l((xS]))$, it suffices to show that $x \in r(l((xS]))$, i.e., that $l((xS])x = 0$. If $a \in l((xS])$, then $a(xS] = 0$, and so $ax \in l(S)$. By Lemma 2.4, $ax = 0$. It is clear that $0 \in l((xS])x$, and hence $l((xS])x = 0$. □

Lemma 2.8. *If A is a non-empty subset of a dual ordered semigroup $(S, \cdot, \leqslant)$, then $A \subseteq (SA]$ and $A \subseteq (AS]$.*

Proof. This follows by Lemma 2.7. □

Corollary 2.9. *If L (respectively, R, M) is a left (respectively, right, two-sided) ideal of a dual ordered semigroup $(S, \cdot, \leqslant)$, then $L = (SL]$ (respectively, $R = (RS]$, $M = (SM] = (MS]$).*

Corollary 2.10. *Let $(S, \cdot, \leqslant)$ be a dual ordered semigroup. Then $S = (S^2]$.*

Definition 2.11. Let $(S, \cdot, \leqslant)$ be a dual ordered semigroup. A left (respectively, right, two-sided) ideal $L \neq 0$ of S is said to be *nilpotent* if there is an integer $\varrho > 0$ such that $L^\varrho = 0$.

Lemma 2.12. *If L is a nilpotent left ideal of a dual ordered semigroup $(S, \cdot, \leqslant)$, then $(LS]^\varrho = 0$. Further $(L \cup LS]^{2\varrho} = 0$.*

Proof. Assume that L is a nilpotent left ideal of a dual ordered semigroup S. Then there is an integer $\varrho > 0$ such that $L^\varrho = 0$. We have

$$(LS]^\varrho = (LS](LS] \cdots (LS] \subseteq (LSLS \cdots LS] \subseteq (LLL \cdots LS] = (L^\varrho S] = (0S] = 0.$$

□

Lemma 2.13. *Let $(S, \cdot, \leqslant)$ be a dual ordered semigroup. Then $(S^n] = S$ for every integer $n > 0$.*

Proof. It is clear that $(S] = S$. Assume that $(S^k] = S$ for $k \geq 1$. We have

$$(S^{k+1}] = (SS^k] = (S(S^k]] = (S^2] = S$$

and hence $(S^n] = S$ for every integer $n > 0$. □

Note, by Lemma 2.13, that a dual ordered semigroup cannot be nilpotent.

Lemma 2.14. *Let I be an ideal of a dual ordered semigroup $(S, \cdot, \leqslant)$. If L is a nilpotent left ideal of S contained in I, then I contains a nilpotent ideal of S.*

Proof. Since $0 \neq L \subseteq I$, it follows that $(LS] \subseteq (IS] \subseteq (I] = I$. Then $L \cup (LS] \neq 0$ is an ideal of S contained in I. By Lemma 2.12, $L \cup (LS]$ is nilpotent. □

The union of all nilpotent two-sided ideals of an ordered semigroup $(S, \cdot, \leqslant)$, called the *radical* of S, will be denoted by N. And, for a two-sided ideal A of S, $N \cap A = 0$ means A does not contain a nilpotent subideal of S.

Lemma 2.15. *Let $(S, \cdot, \leqslant)$ be a dual ordered semigroup and A a two-sided ideal of S with the property $N \cap A = 0$.*

(1) $r(A) = l(A)$.

(2) *If $A \cap r(A) = A \cap l(A) = 0$, then $A \cup r(A) = A \cup l(A) = S$.*

(3) *If L is a left (right, two-sided) ideal of A, then L is also a left (right, two-sided) ideal of S.*

(4) *If L is a left (right, two-sided) ideal of $r(A)$, then L is also a left (right, two-sided) ideal of S.*

Proof. (1). Note that $Ar(A) = 0$. We will show that $A \cap r(A) = 0$. Suppose that $A \cap r(A) \neq 0$. Since $0 \neq A \cap r(A) \subseteq A$ and $0 \neq A \cap r(A) \subseteq r(A)$, we have $(A \cap r(A))^2 \subseteq Ar(A) = 0$. By Lemma 2.14, A contains a nilpotent two-sided ideal of S. This is a contradiction. $A \cap r(A) = 0$. Since $r(A)A \subseteq r(A) \cap A = 0$, we get $r(A) \subseteq l(A)$. Similarly, $l(A) \subseteq r(A)$. Hence $l(A) = r(A)$.

(2). This follows by Corollary 2.3.

(3). Let L be a left ideal of A. Then

$$r(A)L \subseteq r(A)A = l(A)A = 0,$$

and so

$$SL = (A \cup r(A))L = AL \cup r(A)L = AL \cup \{0\} \subseteq L.$$

Let $x \in L, y \in S$ be such that $y \leqslant x$. Then $x \in A$, so $y \in A$. Thus $y \in L$. Therefore L is a left ideal of S.

(4). Assume that L is a left ideal of $r(A)$. Then $AL \subseteq Ar(A) = 0$ and thus

$$SL = (A \cup r(A))L = AL \cup r(A)L \subseteq \{0\} \cup L \subseteq L.$$

If $x \in L$, and $y \in S$ such that $y \leqslant x$, then $y \in r(A)$ and thus $y \in L$. Hence L is a left ideal of S. □

Theorem 2.16. *Let $(S, \cdot, \leqslant)$ be a dual ordered semigroup and A a two-sided ideal of S such that $A \cap N = 0$. Then A and $r(A)$ are dual ordered semigroups.*

Proof. We will show that A is a dual ordered semigroup. For a non-empty subset M of A, we let $l'(M)$ and $r'(M)$ denote the left and the right annihilators of M in A, respectively. Let R be a right ideal of A. Since $A \cap r(A) = 0$ and $l(A) = r(A)$, it follows that $l(R) = l(A) \cup l'(R)$. By Lemma 1.2 (2), $r(l(R)) = r(l(A)) \cap r(l'(R))$. Since R is a right ideal of S, $R = A \cap r(l'(R))$. Since $l'(R) \subseteq A$ and $r(A)l'(R) \subseteq r(A)A = 0$, we get $r(l'(R)) = r(A) \cup r'(l'(R))$. Hence, we have

$$\begin{aligned} R &= A \cap \{r(A) \cup r'(l'(R))\} \\ &= (A \cap r(A)) \cup (A \cap r'(l'(R))) \\ &= \{0\} \cup (A \cap r'(l'(R))) \\ &= \{0\} \cup r'(l'(R)) = r'(l'(R)). \end{aligned}$$

Similarly, $l'(r'(L)) = L$ for any left ideal L of A. Hence A is a dual ordered semigroup.

We will show that $r(A)$ is a dual ordered semigroup. Denote the left and right annihilators of $A \subseteq r(A)$ in $r(A)$ by $l''(A)$ and $r''(A)$, respectively. Assume that R is a right ideal of $r(A)$. Since $A \cap r(A) = 0$ and $l(A) = r(A)$, it follows that $l(R) = A \cup l''(R)$. By Lemma 1.2 (2), $r(l(R)) = r(A) \cap r(l''(R))$. Since R is a right ideal of S, $R = r(A) \cap r(l''(R))$. Since $l''(R) \subseteq r(A)$ and $l''(R)A = 0$, $r(l''(R)) = A \cup r''(l''(R))$. We get

$$\begin{aligned} R &= r(A) \cap \{A \cup r''(l''(R))\} \\ &= (r(A) \cap A) \cup (r(A) \cap r''(l''(R))) \\ &= \{0\} \cup (r(A) \cap r''(l''(R))) \\ &= \{0\} \cup r''(l''(R)) = r''(l''(R)). \end{aligned}$$

Similarly, $l''(r''(L)) = L$ for any left ideal L of $r(A)$. Hence $r(A)$ is a dual ordered semigroup. □

Theorem 2.17. *Let $(S, \cdot, \leqslant)$ be a dual ordered semigroup with the radical N. For any two-sided ideal I of S, let $S' = \bigcup\{I \mid I \cap N = 0\}$ and $r(S') = S''$. Then S admits a unique decomposition into a sum of two two-sided ideals $S = S' \cup S''$, where $S'S'' = S''S' = S' \cap S'' = 0$. The summands having the following properties*:

(1) *if $N = 0$, then $S' = S$ and $S'' = 0$;*

(2) *if $N \neq 0$, then S' is either zero or it is a dual ordered semigroup without nilpotent ideals, S'' is a dual ordered semigroup with the radical N in which each two-sided ideal has a non-zero intersection with N.*

Proof. By Lemma 2.15 and $S' \cap N = 0$, it follows that $r(S') = l(S')$ and $S' \cap S'' = S' \cap r(S') = 0$. According to the definition we have $S'S'' = S'r(S') = 0$ and $S''S' = r(S')S' = l(S')S' = 0$. Since $S' \cap S'' = 0$, $S' \cup S'' = S$ by Lemma 2.15. If $N = 0$, then $S' = S$ and so $S'' = r(S') = r(S) = 0$. We have (1).

Next, we show that (2) holds. Let $N \neq 0$ and $S' \neq 0$. Since $S' \cap N = 0$, S' cannot contain nilpotent ideals of S, S' is a dual ordered semigroup by Theorem 2.16. By Theorem 2.16, $S'' = r(S')$ is a dual ordered semigroup. Since $S' \cap N = 0$ and $S = S' \cup S''$, S'' contains the radical N. By the definition of S' and $S' \cap S'' = 0$, it follows that each two-sided ideal $I \subseteq S''$ has a non-zero intersection with N. □

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