Quasigroups and Related Systems **22** (2014), 159 − 164

Some computational results concerning the spectrum of sets of latin squares

Rafael A. Arce-Nazario, Francis N. Castro, Javier Córdova, Kenneth Hicks, Gary L. Mullen, and Ivelisse M. Rubio

Abstract. We discuss some computational problems concerning the distribution of orthogonal pairs in sets of latin squares of small orders.

1. Introduction

A *latin square of order* n is an $n \times n$ array in which each of n distinct symbols appears exactly once in each row and each column. Two latin squares of order n are *orthogonal* if when superimposed, each of the possible n^2 ordered pairs occurs exactly once. We refer to [1] Chapter III, along with [2], [3], and [6] for discussions of latin squares and their applications.

Given a pair L_1, L_2 of latin squares of order n, let $r = N(L_1, L_2)$ denote the number of distinct ordered pairs which occur when L_1 and L_2 are superimposed. If r distinct ordered pairs occur, we say that the latin squares L_1 and L_2 are r*-orthogonal*.

We note that $N(L_1, L_2) = N(L_2, L_1)$ and we clearly have $n \leqslant r \leqslant n^2$ for any pair of latin squares of order n. The upper bound $r = n^2$ is obtained if L_1 and L_2 are orthogonal. The lower bound $r = n$ can always be obtained, for example, by letting $L_1 = L_2$. The *spectrum* for latin squares of order n is the set of all possible r values that can occur between the above bounds of n and n^2. Theorem 3.104, page 190 of [1] gives the following spectrum for pairs of latin squares.

Theorem 1.1. *For a positive integer* n*, a pair of* r*-orthogonal latin squares of order* n *exists if and only if* $r \in \{n, n^2\}$ *or* $n + 2 \leqslant r \leqslant n^2 - 2$*, except when*

1. $n = 2$ *and* $r = 4$*;*
2. $n = 3$ *and* $r \in \{5, 6, 7\}$*;*
3. $n = 4$ *and* $r \in \{7, 10, 11, 13, 14\}$*;*
4. $n = 5$ *and* $r \in \{8, 9, 20, 22, 23\}$*;*
5. $n = 6$ *and* $r \in \{33, 36\}$.

2010 Mathematics Subject Classification: 05B15
Keywords: Latin squares, mutually orthogonal latin squares.

2. The spectrum for more than two squares

Assume that the latin squares L_1 and L_2 are r-orthogonal. Because of the following argument, we can assume that the first row is in the standard order $1, 2, \dots, n$. Assume that we apply a permutation ϕ_1 to the entries of L_1 in order to permute the symbols to put the first row in standard order. Similarly, apply a permutation ϕ_2 to the entries of L_2 to also obtain a new square whose first row is in standard order. Then we have a pair (a, b) occuring when the squares L_1 and L_2 are superimposed if and only if the pair $(\phi_1(a), \phi_2(b))$ occurs when the two permuted squares are superimposed. We can thus assume that the first row in each square is in standard order.

We may also assume that the left column in one of the squares is also in standard order (such a latin square is then said to be *reduced*). From the above argument, we can assume the first rows of both squares are in standard order. Now interchange the rows of the first square by applying a permutation to rows 2 through n of both squares so that this square now has both the first row, and the first column, in standard order (and is thus a reduced latin square). Now apply the same permutation to rows 2 through n of the second square. We will then have one reduced square, and the second square will have its first row in standard order. Moreover the resulting two squares will still be r-orthogonal.

We now consider a set of $t \geqslant 2$ latin squares of order n. Let $r_n(t)$ denote the number of distinct pairs which arise when each possible pair of distinct squares is checked to determine their level of r-orthogonality. We clearly have

$$\binom{t}{2} n \leqslant r_n(t) \leqslant \binom{t}{2} n^2.$$

Moreover, $r_n(n-1) = \binom{n-1}{2} n^2$ if and only if we have a complete set of $n-1$ mutually orthogonal latin squares of order n. Equivalently, this bound is achieved if and only if there is a projective plane of order n; see Theorem 3.20, page 162 of [1]. The function $r_n(n-1)$ thus provides a measure of how close one is to having a complete set of $n-1$ mutually orthogonal latin squares (MOLS) of order n; or equivalently, a projective plane of order n. In [7] the conjecture that projective planes of order n and complete sets of MOLS of order n exist if and only if n is a prime power is proposed as the "Next Fermat Problem." If there is a complete set of $n-1$ MOLS of order n, then $r_n(t) = \binom{t}{2} n^2$ for each $2 \leqslant t \leqslant n-1$.

In [5] the authors used neofields to study the case $r_n(n-1)$ where $n > 2$ was an even positive integer. For any such n, they constructed $n-1$ latin squares of order n, with the property that any two distinct squares had an r value of $r = 5n - 4$. As a result, $r_6(2) \geqslant 26$ (it is known from Theorem 1 that $r_6(2) = 34$), $r_6(3) \geqslant 78, r_6(4) \geqslant 156$, and $r_6(5) \geqslant 260$. Later in [4] these values for $n = 6$ were improved to $r_6(3) \geqslant 94, r_6(4) \geqslant 178$, and $r_6(5) \geqslant 295$.

In the following we will improve these values and provide considerable computational data for latin squares of small orders. For the sake of completeness, we include some data for squares of orders $n < 6$.

From our earlier discussion, when testing t squares, we may assume that one square is reduced and the remaining $t-1$ squares each have their first row in standard order (such squares are said to be *semi-reduced*). Thus the total number of tests for t latin squares of order n will be

$$l_n((n-1)!l_n)^{t-1} = [(n-1)!]^{t-1} l_n^t,$$

where l_n is the number of reduced latin squares of order n. For small values of n, our calculations were first based upon this brute-force method.

During our calculations, we observed that the use of isotopy classes can save considerable computing time and thus allow us to handle larger values of t. For instance, the execution time for computing the maximum orthogonality for $n = 6$, $t = 5$ was decreased to 305 CPU-days by using isotopy classes, a 22× speedup vs. ignoring the use of isotopy classes. Two latin squares of order n are *i*sotopic if one can be obtained from the other by applying a permutation ϕ to the rows of one of the squares, then applying a permutation ψ to the columns of the resulting square, and finally applying a permutation δ to the symbols of the resulting square.

Representative squares for each of the isotopy classes of squares of order at most six are given on pages 129-137 of [2]. The cardinalities of the isotopy classes were given to us by Ian Wanless.

Theorem 2.1. *Assume that A is a latin square of order n and that after considering all of the pairs (A, X) where X runs through the set of all semi-reduced latin squares of order n, we obtain the nonzero r values $r_1, \ldots, r_k$. Assume that B is a latin square of order n which is isotopic to A, and that after considering all of the pairs (B, X) where X runs through the set of all semi-reduced latin squares of order n, we obtain the nonzero r values $r'_1, \ldots, r'_m$. Then $k = m$ and $r_i = r'_i$ for $i = 1, \ldots, k$.*

Proof. Let ϕ, ψ, δ be the row, column and symbol permutations respectively such that when applied to the latin square A we obtain the latin square B. Then, for each entry $a_{i,j}$ in A, we obtain the entry $b_{k,l} = b_{\phi(i),\psi(j)} = \delta\left(a_{i,j}\right)$ in B. Consider the square X' with entries $x'_{k,l} = x'_{\phi(i),\psi(j)} = \delta'\left(x_{i,j}\right)$, where $x_{i,j}$ is an entry in X and δ' is the permutation $\delta'\left(x_{\phi^{-1}(1),\psi^{-1}(1)}\right) = 1$, $\delta'\left(x_{\phi^{-1}(1),\psi^{-1}(2)}\right) = 2$, $\cdots$, $\delta'\left(x_{\phi^{-1}(1),\psi^{-1}(n)}\right) = n$. Then X' is a semi-reduced latin square. Now, $(a_{i,j}, x_{i,j})$ is a pair occurring when A and X are superimposed if and only if $\left(b_{k,l}, x'_{k,l}\right)$ is a pair occurring when B and X' are superimposed. This implies that there is a correspondence between pairs (A, X) with r distinct ordered pairs and pairs (B, X') with r distinct ordered pairs. □

We briefly discuss our search procedure when determining the spectrum of orthogonality for t squares. With 9408 distinct reduced and 1128960 distinct semi-reduced latin squares of order 6, the number of cases to check at level t with a brute-force approach is $(9408)(1128960)^{t-1}$. Hence, at $n = 6$, increasing t by one requires about 10^6 times longer to finish a complete calculation. For example, for $n = 6$ and $t = 5$, it would take many months on a standard personal computer (or years on a single-CPU machine) to do this calculation.

Consider the case $t = 3$ and assume the results for the number of pairs for $t = 2$ are already stored in computer memory. If the maximum in the spectrum for $t = 2$ is m_2, then clearly the maximum of the spectrum for $t = 3$ is $m_3 < 3 * m_2$. To see if this limit is reached, one can scan the list of combinations of squares L_i and L_j which, at level $t = 2$, have a number of pairs equal to m_2 and look for a third square, L_k which has n_{ik} pairs with square L_i and n_{jk} pairs with square L_j. The maximum for this case is then $m_2 + n_{ik} + n_{jk}$. This provides a lower bound for m_3.

Proceeding in a similar manner, choose the next lowest value in the spectrum for $t = 2$, call it n_{ij}. At level $t = 3$, it is only necessary to look for a square L_k such that $n_{ij} + n_{ik} + n_{jk}$ is above the lower bound found above. As n_{ij} decreases, this requires the value $n_{ik} + n_{jk}$ to increase. Hence there are many fewer cases to calculate.

Extrapolating this algorithm to level $t = 4$, one starts with the maximum number of pairs found for $t = 3$, with three squares, and scans for a fourth square that maximizes the total number of pairs, which provides a lower bound for m_4. Next, we scan the squares for the next lowest in the spectrum for $t = 3$ and check for a fourth square that could exceed this bound. Similarly, the same algorithm applies to higher values of t.

The above algorithm was described in terms of finding the maximum of the spectrum of total pairs at level t, but a similar algorithm can be used to deduce the minimum number of pairs. It should also be understood that the algorithm can be adapted to find the second lowest (or second highest), in the spectrum at level t. With care, the complete spectrum at level t is obtained.

At the website `http://emmy.uprrp.edu/latinsquares/` we provide considerable more detail for squares of small orders. For example, we provide histograms which indicate the frequencies with which the various values of r occur in the spectrum. Also, for $n \leqslant 6$ we provide an example of a set of t latin squares of order n which achieve each of the values of r listed in the spectrum for squares of order n considered t at a time.

2.1. Squares of orders 2 and 3

For $n = 2$, there are only two different latin squares, and the spectrum in this case is simply the value $r = 2$.

For $n = 3$ and $t = 2$, the spectrum only contains the values $r = 3$ and $r = 9$.

2.2. Squares of order 4

For $n = 4$ and $t = 2$ the spectrum is:

$$4, 6, 8, 9, 12, 16$$

For $n = 4$ and $t = 3$, the spectrum is:

$$12, 16, 20, 21, 22, 26, 27, 28, 30, 32, 33, 34, 36, 40, 48$$

2.3. Squares of order 5

In the following, when listing say $x - y$ in a spectrum, we mean that x, y and every value between x and y are included in the spectrum.

For $n = 5$ and $t = 2$ the spectrum is:

$$5, 7, 10 - 19, 21, 25$$

For $n = 5$ and $t = 3$ the spectrum is:

$$15, 19, 24, 25, 27, 29 - 59, 63, 75$$

For $n = 5$ and $t = 4$ the spectrum is:

$$30, 36, 38, 43, 45, 46, 48 - 120, 122, 124, 126, 130, 132, 150$$

2.4. Squares of order 6

In this section we provide similar data for latin squares of order 6. As a result, we will improve the values listed in [4] for $n = 6$ with $t = 3, 4, 5$.

For $n = 6$ and $t = 2$, the spectrum is given by

$$6, 8 - 32, 34$$

The value $r = 36$ is of course missing from the spectrum since there is no pair of MOLS of order 6.

For $n = 6$ and $t = 3$ the spectrum is

$$18, 22, 24 - 94, 96$$

For $n = 6$ and $t = 4$ the spectrum is

$$36, 39, 42, 44 - 184, 188$$

For $n = 6$ and $t = 5$ the spectrum is

$$60, 68, 72, 74, 75, 76, 77, 78, 80 - 298, 300$$

As a result of the above data we are now able to improve three values from [4]. In the following table, the values in the K/M column come from [5]; those in the D/S column come from [4] and the values in the last column come from the various spectrums listed above.

t	K/M	D/S	$M_6(t)$
2	26	34	34
3	78	91	96
4	156	178	188
5	260	295	300

Acknowledgement. The authors would like to thank Christian Rodríguez for creating the database and webpage for the latin squares spectrums.

References

[1] **C.J. Colbourn and J.H. Dinitz, Editors**, Handbook of Combinatorial Designs, Sec. Ed., Chapman and Hall/CRC Press, Boca Raton, FL (2007).

[2] **J. Dénes and A.D. Keedwell**, Latin Squares and their Applications, Academic Press, New York (1974).

[3] **J. Dénes and A.D. Keedwell**, *Latin Squares*, Annals of Disc. Math., North-Holland, Amsterdam. **46** (1991).

[4] **J.H. Dinitz and D.R. Stinson**, *On the maximum number of different ordered pairs of symbols in sets of latin squares*, J. Combin. Designs. **13** (2005), 1 – 15.

[5] **A.D. Keedwell and G.L. Mullen**, *Sets of partially orthogonal latin squares and projective planes*, Discrete Math. **288** (2005), 49 – 60.

[6] **C.F. Laywine and G.L. Mullen**, Discrete Mathematics Using Latin Squares, Wiley, New York (1998).

[7] **G.L. Mullen**, *A candidate for the next Fermat problem*, Math. Intelligencer. **17** (1995), 18 – 22.

Received March 12, 2014

R.A.Arce-Nazario, I.Rubio
Department of Computer Science, University of Puerto Rico, Río Piedras, PO Box 70377, San Juan, PR 00936-8377
E-mails: rafael.arce@upr.edu, iverubio@gmail.com

F.N.Castro
Department of Mathematics, University of Puerto Rico, Río Piedras, PO Box 70377, San Juan, PR 00936-8377
E-mail: franciscastr@gmail.com

J.Córdova
Department of Computer Sciences, University of Puerto Rico, Arecibo, PO Box 4010, Arecibo, PR 00614-4010
E-mail: javier.cordova@upr.edu

K.Hicks
Department of Physics and Astronomy, Ohio University, Athens, OH 45701
E-mail: hicks@phy.ohiou.edu

G.L.Mullen
Department of Mathematics, The Pennsylvania State University, University Park, PA 16802
E-mail: mullen@math.psu.edu