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Sequentially dense flatness of semigroup acts

Leila Shahbaz

Abstract. s-dense monomorphisms and injectivity with respect to these monomorphisms were first introduced and studied by Giuli for acts over the monoid $(\mathbb{N}^{\infty}, \min)$. Ebrahimi, Mahmoudi, Moghaddasi, and Shahbaz generalized these notions to acts over a general semigroup. In this paper, we study flatness with respect to the class of s-dense monomorphisms. The theory of flatness properties of acts over monoids has been of major interest over the past some decades, but so far there are not any papers published on this subject that relate specifically to the class of s-dense monomorphisms. We give some sufficient conditions for s-dense flatness of semigroup acts. Also, we characterize a large number of semigroups over which s-dense flatness coincides with flatness. This gives a useful criterion for flatness of acts over such semigroups. In fact it is shown that the study of s-dense flatness is also useful in the study of ordinary flatness of acts.

1. Introduction

One of the very useful notions in many branches of mathematics as well as in computer science is the notion of an action of a semigroup or a monoid on a set. Let S be a semigroup. Recall that a right S-*act* or S-*system* denoted by A_S, is a set A together with a function $\lambda : A \times S \to A$, called the *action* of S (or the S-action) on A, such that for each $a \in A$ and $s, t \in S$ (denoting $\lambda(a, s)$ by as) $a(st) = (as)t$. If S is a monoid with an identity e, we add the condition $xe = x$. Analogously, a left S-act ${}_SA$ is defined.

A morphism $f : A_S \to B_S$ between right S-acts A_S, B_S is called an S-*map* if, for each $a \in A$, $s \in S$, $f(as) = f(a)s$.

Since id_A and the composite of two S-maps are S-maps, we have the category **Act**-S (S-**Act**) of all right (left) S-acts and S-maps between them (for more information about acts see [1] and [7]).

The study of flatness properties of acts over monoids was first considered in the early 1970's by Mati Kilp and Bo Stenström as a way to generalize the notions of flatness of modules to the non-additive setting. Since then many researchers continued working in this subject that all culminated in [7].

The tensor functors are of as great importance in the theory of acts as they are in the theory of modules.

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Let $A \in \mathbf{Act}\text{-}S$, $B \in S\text{-}\mathbf{Act}$, and let υ be the smallest equivalence relation on the set $A \times B$ generated by the pairs $((as, b), (a, sb))$ for $a \in A, b \in B, s \in S$.

Define $A_S \otimes {}_S B := (A \times B)/\upsilon$, and $a \otimes b := [(a, b)]_\upsilon \in A_S \otimes {}_S B$, $a \in A, b \in B$.

In [7] the following results are proved for acts over monoids, but for semigroup acts the proofs are similar.

Proposition 1.1. *Take $B \in S\text{-}\mathbf{Act}$ and $A = \coprod_{i \in I} A_i \in \mathbf{Act}\text{-}S$ with the injections $u_i : A_i \to A$ where $A_i, i \in I$, are right S-acts. Then*

$$(\coprod_{i \in I} A_i) \otimes B \cong \coprod_{i \in I}^{\mathbf{Set}} (A_i \otimes B)$$

with the injections $u_i \otimes id_B, i \in I$, where $u_i \otimes id_B(a \otimes b) = u_i(a) \otimes id_B(b)$.

Analogously, if $B = \coprod_{i \in I} B_i \in S\text{-}\mathbf{Act}$ with the injections $u_i : B_i \to B$ where $B_i, i \in I$, are left S-acts and $A \in \mathbf{Act}\text{-}S$ then

$$A \otimes (\coprod_{i \in I} B_i) \cong \coprod_{i \in I}^{\mathbf{Set}} (A \otimes B_i)$$

with the injections $id_A \otimes u_i, i \in I$. □

Definition 1.2. Let A be a right S-act, ${}_E\mathbf{2} = \{0, 1\}$ the left E-act for $E = \{1\}$ and $\mathbf{2}^A = Hom({}_E A_S, {}_E\mathbf{2})$ the left S-act where for any $\varphi \in \mathbf{2}^A$ and for any $s \in S$ the mapping $s\varphi$ is defined by $(s\varphi)(a) = \varphi(as)$ for any $a \in A$. The left S-act $\mathbf{2}^A$ is called the *character act* of A.

Definition 1.3. For $A \in \mathbf{Act} - \mathbf{S}$ we have that $A \otimes - : \mathbf{S} - \mathbf{Act} \longrightarrow \mathbf{Set}$ given by $M \mapsto A \otimes M$ and $(g : M \to M') \mapsto (id_A \otimes g : A \otimes M \to A \otimes M')$ is a covariant functor.

Theorem 1.4. *Let A be a right S-act. The functor $A \otimes -$ preserves the monomorphism $i : {}_S N \to {}_S M$ if and only if $\mathbf{2}^A$ is injective relative to the monomorphism i.* □

2. s-dense flatness

In this section, we recall the class of s-dense monomorphisms needed to define s-dense flatness and then flatness with respect to this class of monomorphisms is studied. The notion of s-dense monomorphisms was first defined in [6] and [8] for acts over the monoid $(\mathbb{N}^\infty, \min)$, and then generalized and studied in some other papers.

Definition 2.1. A subact A of a left S-act B is said to be *s-dense* in B if $Sb \subseteq A$ for each $b \in B$. An S-map $f : A \to B$ is said to be *s-dense* if $f(A)$ is an s-dense subact of B.

Notice that in the case where S is a monoid, the only s-dense subact of an S-act B is B itself, and the only s-dense monomorphisms are isomorphisms. So, this notion makes more sense for semigroup acts than for monoid acts, or for acts over the semigroup parts of the monoids of the form $T = S^1$, in which an identity is adjoined to the semigroup S.

Definition 2.2. A right S-act A is called *s-dense flat* or *s-flat* if the functor $A \otimes -$ takes s-dense monomorphisms of left S-acts to monomorphisms.

Lemma 2.3. *Let $\{A_i : i \in I\}$ be a family of right S-acts and $A_S = \coprod_{i \in I} A_i$. Then A_S is s-flat if and only if each A_i is s-flat.*

Proof. It is similar to the case of usual flatness. □

Lemma 2.4. *A right S-act A_S is s-flat if the functor $A \otimes -$ takes all s-dense inclusions of left S-acts into inclusions, i.e., if ${}_SN$ is an s-dense subact of ${}_SM$ and elements $a \otimes m$ and $a' \otimes m'$ are equal in $A \otimes M$ then they are equal already in $A \otimes N$.*

Proof. Let $f : {}_SN \to {}_SM$ be an s-dense monomorphism. Assume that $a_1 \otimes n_1$, $a_2 \otimes n_2 \in A \otimes N$ are such that $(id_A \otimes f)(a_1 \otimes n_1) = (id_A \otimes f)(a_2 \otimes n_2)$. Thus $a_1 \otimes f(n_1) = a_2 \otimes f(n_2)$ in $A \otimes M$. It follows by hypothesis that $a_1 \otimes f(n_1) = a_2 \otimes f(n_2)$ already in $A \otimes Imf$. Let $g : Imf \to N$ be an S-map such that $g \circ f = id_N$. Then $a_1 \otimes n_1 = (id_A \otimes gf)(a_1 \otimes n_1) = (id_A \otimes g)((id_A \otimes f)(a_1 \otimes n_1)) = (id_A \otimes g)((id_A \otimes f)(a_2 \otimes n_2)) = (id_A \otimes gf)(a_2 \otimes n_2) = a_2 \otimes n_2$. Thus $id_A \otimes f$ is a monomorphism. Hence A is s-flat. □

Now we show the relation between s-flatness and s-injectivity (injectivity with respect to s-dense monomorphisms).

Theorem 2.5. *Let A be a right S-act. The functor $A \otimes -$ takes the s-dense monomorphism $i : {}_SN \to {}_SM$ to a monomorphism if and only if $\mathbf{2}^A$ is s-injective relative to the s-dense monomorphism i.*

Proof. The proof is similar to the case of usual flatness. □

Recall the following proposition from [9].

Proposition 2.6. *For a semigroup S, the following are equivalent.*

(*i*) *All right (left) S-acts are s-injective.*

(*ii*) *S has a left (right) identity element.* □

Recall that a right S-act A is called *principally weakly flat* if the functor $A \otimes -$ preserves all embeddings of principal left ideals into S.

Remark 2.7. Each flat act is s-flat, but the converse is not true in general. For example, let $S = (\mathbb{N}, .)$. Then $A_{\mathbb{N}} = \mathbb{N} \sqcup^{\mathbb{N}\setminus\{1\}} \mathbb{N} = \{(1,x)\} \dot{\cup} \{\mathbb{N} \setminus \{1\}\} \dot{\cup} \{(1,y)\}$ is not principally weakly flat by Example III.14.4 of [7] and so it is not flat. But since S is a monoid, each S-act is s-injective by Proposition 2.6. So, $\mathbf{2}^{A_{\mathbb{N}}}$ is s-injective and hence $A_{\mathbb{N}}$ is s-flat by the above theorem.

Now, we apply the relationship between s-flatness and s-injectivity to obtain sufficient conditions for s-flatness similar to the Baer-Skornjakov criterion for s-injectivity.

First we recall the following theorem from [9].

Theorem 2.8. *For a right S-act A, the following are equivalent.*

(i) *A is s-injective.*

(ii) *For every s-dense monomorphism $h : B \to cS^1$ to a cyclic act and every S-map $f : B \to A$ there exists an S-map $g : cS^1 \to A$ such that $gh = f$.*

(iii) *Every S-map $f : cS \to A$ from a cyclic act can be extended to $\overline{f} : cS^1 \to A$.*

(iv) *Every S-map $f : S \to A$ can be extended to an S-map $\overline{f} : S^1 \to A$.*

(v) *For every s-dense monomorphism $h : B \to B \cup cS^1$ to a singly generated extension of B and every S-map $f : B \to A$ there exists an S-map g from $B \cup cS^1$ to A such that $gh = f$.* □

Proposition 2.9. *Let A be a right S-act. Then the following conditions are equivalent.*

(i) *A is s-flat.*

(ii) *The functor $A \otimes -$ takes all s-dense embeddings of left S-acts into cyclic left S-acts to monomorphisms.*

(iii) *The functor $A \otimes -$ takes an inclusion ${}_ScS \hookrightarrow {}_ScS^1$ to a monomorphism.*

(iv) *The functor $A \otimes -$ takes all s-dense monomorphisms $h : {}_SB \to {}_S(B \cup cS^1)$ into a singly generated extension of ${}_SB$ to monomorphisms.*

(v) *The functor $A \otimes -$ takes an inclusion ${}_SS \to {}_SS^1$ to a monomorphism.*

Proof. $(i) \Rightarrow (ii)$, $(i) \Rightarrow (iii)$, $(i) \Rightarrow (iv)$, $(i) \Rightarrow (v)$ are clear.

$(ii) \Rightarrow (i)$ Let the functor $A \otimes -$ take all s-dense embeddings of left S-acts into cyclic left S-acts to monomorphisms. By Theorem 2.5 we get that ${}_S\mathbf{2}^A$ is s-injective relative to all s-dense embeddings of left S-acts into cyclic left S-acts.

Now by Theorem 2.8, ${}_S\mathbf{2}^A$ is s-injective. Applying once more Theorem 2.5, one gets that A is s-flat.

$(iii) \Rightarrow (i)$ Let the functor $A \otimes -$ take ${}_ScS \hookrightarrow {}_ScS^1$ to a monomorphism. By Theorem 2.5, ${}_S\mathbf{2}^A$ is s-injective relative to ${}_ScS \hookrightarrow {}_ScS^1$ and so by Theorem 2.8, it is s-injective. Applying once more Theorem 2.5, one gets that A is s-flat.

$(iv) \Rightarrow (i)$ Let the functor $A \otimes -$ take all s-dense monomorphisms $h : {}_SB \to {}_S(B \cup cS^1)$ into a singly generated extension of ${}_SB$ to monomorphisms. By Theorem 2.5, ${}_S\mathbf{2}^A$ is s-injective relative to all s-dense monomorphisms $h : {}_SB \to {}_S(B \cup cS^1)$. Thus by Theorem 2.8, ${}_S\mathbf{2}^A$ is s-injective. Applying once more Theorem 2.5, one gets that A is s-flat.

$(v) \Rightarrow (i)$ Let the functor $A \otimes -$ take an inclusion ${}_SS \hookrightarrow {}_SS^1$ to a monomorphism. By Theorem 2.5, ${}_S\mathbf{2}^A$ is s-injective relative to an inclusion ${}_SS \hookrightarrow {}_SS^1$. Thus by Theorem 2.8, ${}_S\mathbf{2}^A$ is s-injective. Applying once more Theorem 2.5, one gets that A is s-flat. □

Now we characterize semigroups over which all S-acts are s-flat.

Definition 2.10. A semigroup S is called *right absolutely s-flat* if all right S-acts are s-flat.

Proposition 2.11. *Let S be a semigroup with a right identity element. Then S is right absolutely s-flat.*

Proof. Since S has a right identity element thus each left S-act is s-injective by Proposition 2.6. Then for every S-act A_S, ${}_S\mathbf{2}^A$ is s-injective. Thus A_S is s-flat by Theorem 2.5. □

Now, we characterize a large number of semigroups over which s-dense flatness coincides with flatness. This gives a useful criterion for flatness of acts over such semigroups.

Theorem 2.12. *If S is a(n)*

- (i) *semigroup for which $(Id_r(S), \cap, \cup)$ is a Boolean algebra, or*
- (ii) *left (right) zero semigroup, or*
- (iii) *cyclic semigroup, or*
- (iv) *zero semigroup, or*
- (v) *idempotent semigroup each of whose proper right ideals is generated by a central idempotent, or*
- (vi) *lattice considered as a semigroup with $\wedge$ as its binary operation each of whose proper right ideals is a complete sublattice, or*
- (vii) *finite chain considered as a semigroup, or*

(*viii*) *Clifford semigroup each of whose proper non-empty ideals is principal*

then each s-flat S-act is flat.

Proof. Let A be an s-flat right S-act. Then by Theorem 2.5, ${}_S\mathbf{2}^A$ is s-injective. Since by [3] each s-injective act over one of the above semigroups is injective thus ${}_S\mathbf{2}^A$ is injective. Now, by Theorem 1.4, A is flat. □

Theorem 2.13. *Each s-flat projection algebra* (S*-act over the monoid* $(\mathbb{N}^\infty, \min)$) *is flat.*

Proof. The proof is similar to the proof of the above theorem, because every s-injective projection algebra is injective by Theorem 3.19 of [8]. □

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Department of Mathematics, University of Maragheh, Maragheh 55181-83111, Iran
E-mails: leilashahbaz@yahoo.com, l_shahbaz@maragheh.ac.ir