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Filter theory on hyper residuated lattices

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Abstract. We apply the hyper structures to residuated lattices and introduce the notion of hyper residuated lattice which is a generalization of the residuated lattice and verified some related results. Finally, we state and prove some theorems about filters and deductive systems.

1. Introduction

Residuated lattices, introduced by Ward and Dilworth [12], are a common structure among algebras associated with logical systems. In this definition to any bounded lattice $(\mathcal{L}, \vee, \wedge, 0, 1)$, a multiplication '$*$' and an operation '$\rightarrow$' are equpped such that $(\mathcal{L}, *, 1)$ is a commutative monoid and the pair $(*, \rightarrow)$ is an adjoint pair, i.e.,

$$x * y \leqslant z \text{ if and only if } x \leqslant y \rightarrow z, \ \forall x, y, z \in \mathcal{L}.$$

The main examples of residuated lattices are MV-algebras introduced by Chang [4] and BL-algebras introduced by Hájek [9]. The hyperstructure theory was introduced by Marty [10], at the 8th Congress of Scandinavian Mathematicians. In his definition, a function $f : A \times A \longrightarrow P^*(A)$, of the set $A \times A$ into the set of all non-empty subsets of A, is called a *binary hyperoperation*, and the pair (A, f) is called a *hypergroupoid*. If f is associative, A is called a *semihypergroup*, and it is said to be *commutative* if f is commutative. Also, an element $1 \in A$ is called the *unit* or the *neutral element* if $a \in f(1, a)$, for all $a \in A$. Since then many researchers have worked on this area. R. A. Borzooei et al. introduced and studied hyper K-algebras [2] and S. Ghorbani et al. [8], applied the hyperstructures to MV-algebras and introduced the concept of hyper MV-algebra, which is a generalization of MV-algebra. In [11], Mittas et al. applied the hyperstructures to lattices and introduced the concepts of a hyperlattice and supperlattice: A *superlattice* is a partially ordered set $(S; \leqslant)$ endowed with two binary hyperoperations $\vee$ and $\wedge$ satisfying the following properties: for all $a, b, c \in S$,

(SL1) $a \in (a \vee a) \cap (a \wedge a)$,
(SL2) $a \vee b = b \vee a, \quad a \wedge b = b \wedge a$,
(SL3) $(a \vee b) \vee c = a \vee (b \vee c), \quad (a \wedge b) \wedge c = a \wedge (b \wedge c)$,
(SL4) $a \in ((a \vee b) \wedge a) \cap ((a \wedge b) \vee a)$,

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(SL5) $a \leqslant b$ implies $b \in a \vee b$ and $a \in a \wedge b$,

(SL6) if $a \in a \wedge b$ or $b \in a \vee b$ then $a \leqslant b$.

Hyperstructures have many applications to several sectors of both pure and applied sciences. A short review of the theory of hyperstructures appear in [5]. In [6] a wealth of applications can be found, too. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy set and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence and probabilities.

It is well know, the class of MV-algebras, BL-algebras, and Heyting algebras are proper subclass of the class of residuated lattices. In this paper, as an application of hyperstructures to residuated lattices, we introduce the notion of a hyper residuated lattice. We define the concepts of (weak) filter and (weak) deductive system, and verify their properties, as mentioned in the abstract. In fact, we want to construct a hyper structure, which is more general than hyper MV-algebra and hyper K-algebra.

2. Hyper residuated lattices

Throughout this paper, L will denote a hyper residuated lattice, unless otherwise stated.

Let $(X, \leqslant)$ be a partially ordered set and A, B be two subsets of X. Then we write

- $A \ll B$, if there exist $a \in A$ and $b \in B$ such that $a \leqslant b$.
- $A \leqslant B$ if for any $a \in A$, there exists $b \in B$ such that $a \leqslant b$.

Definition 2.1. [13] By a *hyper residuated lattice* we mean a non-empty set L endowed with four binary hyperoperations $\vee$, $\wedge$, $\odot$, $\rightarrow$ and two constants 0 and 1 satisfying the following conditions:

(HRL1) $(L, \leq, \vee, \wedge, 0, 1)$ is a bounded superlattice,

(HRL2) $(L, \odot, 1)$ is a commutative semihypergroup with 1 as the identity,

(HRL3) $a \odot c \ll b$ if and only if $c \ll a \rightarrow b$.

L is called *nontrivial* if $0 \neq 1$. An element $a \in L$ is called *scalar* if $|a \odot x| = 1$, for all $x \in L$.

Example 2.2. (i) Let $S = [0, 1]$. Then S with the natural ordering is a partially ordered set. Define the hyperoperations $\vee$, $\wedge$, $\odot$, and $\rightarrow$ on S as follows:

$$a \odot b = a \wedge b = \min\{a, b\}, \qquad b \vee a = a \vee b = \begin{cases} S, & a = b, \\ S - \{a\}, & a < b, \\ S - \{b\}, & b < a \end{cases}$$

$$a \to b = \begin{cases} 1, & a \leqslant b, \\ b, & a > b. \end{cases}$$

Then, it is easy to check that $(S, \vee, \wedge, \odot, \to, 0, 1)$ satisfies the properties (HRL1) $-$(HRL3) and so is a hyper residuated lattice.

(ii) Let $L = [0,1]$ and $\odot$, $\vee$ be the hyperoperations in (i). Define two hyperoperations $\wedge$ and $\to$ on L as follows:

$$a \wedge b = \{x \in L | x \leqslant a, x \leqslant b\}, \qquad a \to b = \begin{cases} \{1\}, & a \leqslant b, \\ [b,1], & a > b. \end{cases}$$

It is not difficult to check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper residuated lattice.

(iii) Let $(L = \{0, a, b, 1\}, \leq)$ be a chain such that $0 < a < b < 1$. Define the hyperoperations $\vee$ and $\wedge$ on L as given in the tables 1 and 2:

Table 1

$\vee$	0	a	b	1
0	{0,a,b,1}	{a,b,1}	{b,1}	{1}
a	{a,b,1}	{a,1,b}	{b,1}	{1}
b	{b,1}	{b,1}	{b,1}	{1}
1	{1,0}	{1}	{1}	{1}

Table 2

$\wedge$	0	a	b	1
0	{0}	{0}	{0}	{0}
a	{0}	{0,a}	{0,a}	{0,a}
b	{0}	{0,a}	{0,b,a}	{0,b,a}
1	{0}	{0,a}	{0,b,a}	{0,a,b,1}

Then $(L, \vee, \wedge, 0, 1)$ is a bounded hypper lattice. Let $x \odot y = \wedge$ and define the hyperoperations $\to$ and $\rightsquigarrow$ on L as given in the tables 3 and 4.

Table 3

$\to$	0	a	b	1
0	{1}	{1}	{1}	{1}
a	{a,b,1}	{1,a}	{1}	{1}
b	{a,1}	{a}	{b,1}	{1}
1	{0,1}	{a}	{1,b}	{1}

Table 4

$\rightsquigarrow$	0	a	b	1
0	{1}	{1,b}	{1,b}	{1,b}
a	{a,b,1}	{1}	{1}	{1}
b	{a,b,1}	{a}	{1,b}	{1,b}
1	{0,a,1}	{1,a}	{1}	{1}

Routine calculations show that $(L, \vee, \wedge, \odot, \to, 0, 1)$ and $(L, \vee, \wedge, \odot, \rightsquigarrow, 0, 1)$ are hyper residuated lattices. $\square$

Proposition 2.3. *In any hyper residuated lattice L, for all $x, y, z \in L$ and $A, B, C \subseteq L$, the following hold:*

(1) $1 \ll A$ *implies* $1 \in A$, *for all non-empty subsets* A *of* L,

(2) $x \leqslant y$ *implies* $1 \in x \to y$, *and if* 1 *is a scalar, the converse hold,*

(3) $1 \in x \to x$, $1 \in x \to 1$, $1 \in 0 \to x$, *if* 1 *is a scalar,* $x \in 1 \to x$,

(4) $A \ll B \to C$ *if and only if* $A \odot B \ll C$ *if and only if* $B \ll A \to C$,

(5) $0 \in x \odot 0$, $x \ll \neg\neg x$, *where* $\neg x = x \to 0$,

(6) $x \odot (x \to y) \ll y$, $x \odot (x \to y) \ll x$,

(7) $x \ll y \to (x \odot y)$,

(8) $x \odot y \ll x$, $x \odot y \ll y$. *Particularly,* $0 \in x \odot 0$,

(9) $A \odot B \ll A$, $A \odot B \ll B$,

(10) $A \ll x \ll B$ *implies* $A \ll B$. *Moreover, if* $A \cap B \neq \emptyset$, *then* $A \ll B$ *and* $B \ll A$.

(11) $x \leqslant y$ *implies* $x \odot z \ll y \odot z$,

(12) $x \leqslant y$ *implies* $z \to x \ll z \to y$,

(13) $x \leqslant y$ *and* $x \leqslant z$ *imply* $x \ll y \wedge z$,

(14) $y \leqslant x$ *and* $z \leqslant x$ *imply* $y \vee z \ll x$,

(15) $x \to y \subseteq \{u \,|\, u \odot x \ll y\}$,

(16) $x \leqslant y$ *implies* $y \to z \ll x \to z$,

(17) *If* y' *is a scalar of* L, *then* $(x \to y') \odot (y' \to z) \ll x \to z$,

(18) $x \to (y \to z) \ll (x \odot y) \to z$,

(19) $(x \odot y) \to z \ll x \to (y \to z)$.

Proof. The proofs of $(3) - (7)$, (9), (11), (14), (15) and (19) are straightforward.

(1). If $A \subseteq L$ is such that $1 \ll A$, then $1 \ll a$, for some $a \in A$ whence $1 = a \in A$.

(2). Assume that $a \leqslant b$. From $a \in a \odot 1$ it follows that $a \odot 1 \ll b$ whence $1 \ll a \to b$. Thus, $1 \in a \to b$, by (1). Conversely, if 1 is a scalar, $1 \in a \to b$ implies that $\{a\} = a \odot 1 \ll b$, i.e., $a \leqslant b$.

(8). Since, $y \leqslant 1 \in x \to x$, so $x \odot y \ll x$. Similarly, it follows that $x \odot y \ll y$.

(10). Assume $A \ll x$ and $x \ll B$. Then $a \ll x$ and $x \ll b$, for some $a \in A$ and $b \in B$, whence $a \leqslant b$, i.e., $A \ll B$. The proof of other part is easy.

(12). Let $x \leqslant y$. Since, $z \odot (z \to x) \ll x$, by (6), $z \odot (z \to x) \ll y$ and so $z \to x \ll z \to y$.

(13). From $x \leqslant a$ and $x \leqslant b$ it follows that $x \in x \wedge a$ and $x \in x \wedge b$ whence $x \in x \wedge b \subseteq (x \wedge a) \wedge b = x \wedge (a \wedge b)$. Hence, there exists $u \in a \wedge b$ such that $x \in x \wedge u$ and so $x \leqslant u$ means that $x \ll a \wedge b$.

(16). Let $x \leqslant y$ and $z \in L$. By (15), we have

$$\begin{aligned} y \to z \subseteq \{u \in L \,|\, y \odot u \ll z\} &= \{u \in L \,|\, y \ll u \to z\} \subseteq \{u \in L \,|\, x \ll u \to z\} \\ &= \{u \in L \,|\, u \ll x \to z\}, \end{aligned}$$

whence $y \to z \ll x \to z$.

(17). Let y' be a scalar element of L, $u \in x \to y'$ and $v \in y' \to z$. Then $u \ll x \to y'$ and so $u \odot x \ll y'$. By a similar way, $v \odot y' \ll z$. Hence there exists

$a \in u \odot x$ such that $a \leqslant y'$ and so by (11), $a \odot v \ll y' \odot v$. Hence $v \odot (u \odot x) \ll v \odot y'$. Since $v \odot y' \ll z$ and $|v \odot y'| = 1$, then we get that $(v \odot u) \odot x = v \odot (u \odot x) \ll z$. Hence there exists $b \in u \odot v$ such that $x \odot b \ll z$ and so $b \ll x \to z$. Since $b \in u \odot v \subseteq (x \to y') \odot (y' \to z)$, then $(x \to y') \odot (y' \to z) \ll x \to z$.

(18). Let $u \in x \to (y \to z)$. Then there exists $a \in y \to z$ such that $u \in x \to a$. Then we get that

$$\begin{aligned} u \ll x \to a &\Rightarrow u \odot x \ll a \\ &\Rightarrow u \odot x \ll y \to z, \\ &\Rightarrow (u \odot x) \odot y \ll z, \qquad \text{by (4)}, \\ &\Rightarrow u \odot (x \odot y) \ll z, \\ &\Rightarrow u \ll (x \odot y) \to z, \qquad \text{by (4)}. \end{aligned}$$

Hence, $x \to (y \to z) \ll (x \odot y) \to z$. □

The next theorem shows that if there exists a hyper residuated lattice of order n, then there exists a hyper residuated lattice of order $n+1$.

Theorem 2.4. *Each hyper residuated lattice of order n can be extend to a hyper residuated lattice of order $n+1$, for any $n \in \mathbb{N}$.*

Proof. Let L be a hyper residuated lattice of order n, for $n \in \mathbb{N}$ and $e \notin L$. Set $\overline{L} = L \cup \{e\}$ and define a relation $\leqslant'$ on $\overline{L}$ by

$$\begin{aligned} z &\leqslant' y \quad \Leftrightarrow z \leqslant y, \text{ for all } z, y \in L, \\ x &\leqslant' e \quad \text{for all } x \in L'. \end{aligned}$$

Then $(\overline{L}, \leqslant')$ is a poset and 0 and e are the minimum and the maximum elements of $\overline{L}$, respectively. We define the binary hyperoperations $\vee', \wedge', \odot'$ and $\to'$ on $\overline{L}$ by

$$a \vee' b = \begin{cases} a \vee b & \text{if } a, b \in L, \\ \{e\} & \text{if } a = e \text{ or } b = e. \end{cases} \qquad a \to' b = \begin{cases} (a \to b) \cup \{e\} & \text{if } a, b \in L, 1 \in a \to b, \\ a \to b & \text{if } a, b \in L, 1 \notin a \to b, \\ \{e\} & \text{if } b = e, \\ \{b\} & \text{if } a = e. \end{cases}$$

$$a \odot' b = \begin{cases} a \odot b & \text{if } a, b \in L, \\ \{a\} & \text{if } a \in L \text{ and } b = e, \\ \{b\} & \text{if } b \in L \text{ and } a = e, \\ \{e\} & \text{if } a = b = e. \end{cases} \qquad a \wedge' b = \begin{cases} a \wedge b & \text{if } a, b \in L, \\ \{b\} & \text{if } b \in L \text{ and } a = e, \\ \{a\} & \text{if } a \in L \text{ and } b = e, \\ \{e\} & \text{if } a = b = e. \end{cases}$$

Routine calculation shows that (HRL1) and (HRL2) hold. We shall prove (HRL3). Let $x, y, z \in \overline{L}$.

(1). Let $x, y, z \in L$ and $1 \notin y \to z$. Then by definitions of $\odot'$ and $\leqslant'$, we get

$$x \odot' y \ll' z \Leftrightarrow x \odot y \ll z \Leftrightarrow x \ll y \to z \Leftrightarrow x \ll' y \to' z.$$

(2). Let $x, y, z \in L$ and $1 \in y \to z$. If $x \odot' y \ll' z$, then by definition of $\to'$, $e \in y \to' z$ and so $x \ll' y \to' z$. Now, let $x \ll' y \to' z$. Since $1 \in y \to z$, then $x \ll y \to z$ and so $x \odot y \ll z$. Hence $x \odot' y = x \odot y \ll' z$.

(3). Let $x, y \in L$ and $z = e$. Since $y \to' z = \{e\}$ and $u \ll' e$, for all $u \in L'$, then $x \odot' y \ll' z$ implies $x \ll' y \to' z$. Now, let $x \ll' y \to' z$. Since $z = e$, then clearly, $x \odot' y \ll' z$.

(4). Let $x, z \in L$ and $y = e$. Then $x \odot' y = \{x\}$ and $y \to' z = \{z\}$. Therefore, $x \odot' y \ll' z$ if and only if $x \ll' y \to' z$.

(5). Let $y, z \in L$ and $x = e$. Then $x \odot' y = \{y\}$. If $x \odot' y = \{y\} \ll' z$, then $y \ll' z$. Since $y, z \in L$ we get $y \ll z$ and so $1 \in y \to z$. Hence $e \in y \to' z$ and so $x \ll' y \to' z$. Now, let $x \ll' y \to' z$. Then by definition of $\leq'$, we have $e \in y \to' z$ and so $1 \in y \to z$ or $z = e$. Since $y \in L$, then $y \neq e$ and so $1 \in y \to z$. Therefore, $x \odot' y \ll' z$.

(6). Let $x = y = e$ and $z \in L$. Then $x \odot' y = \{e\}$ and $y \to' z = \{z\}$. Hence $x \odot' y = \{e\} \ll' z$ and $x \ll' y \to' z = \{z\}$ are impossible.

(7). Let $x = z = e$ and $y \in L$. Then $x \odot' y = \{y\}$ and $y \to' z = \{e\}$. Therefore, $x \odot' y \ll' z$ if and only if $x \ll' y \to' z$.

An analogous result holds for $y = z = e$.

(8). For $x = y = z = e$, it is obvious.

Therefore, $(\overline{L}, \vee', \wedge', \odot', \to', 0, e)$ is a hyper residuated lattice of order $n+1$. □

Corollary 2.5. *For any $n \geqslant 4$ and $n \in \mathbb{N}$, there exists at least one hyper residuated lattice of order n.*

Proof. By Example 2.2 and Theorem 2.4, the proof is clear. □

3. (Weak) Filters and deductive systems

In this section, we introduce the concepts of (weak) filters and (weak) deductive systems in hyper residuated lattices and we give some related results. Then we introduced special kinds of weak deductive systems in hyper residuated lattices and verify the relation between them.

Definition 3.1. [13] Let F be a non-empty subset of L satisfying

(F) $x \leqslant y$ and $x \in F$ imply $y \in F$.

then F is called a

- *filter* if $x \odot y \subseteq F$, for all $x, y \in F$,
- *weak filter* if $F \ll x \odot y$, for all $x, y \in F$.

A filter F of L is said to be *proper* if $F \neq L$ and this is equivalent to that $0 \notin F$

Remark 3.2. Clearly, any filter is a weak filter. Moreover, $1 \in F$, for any (weak) filter F of L.

Example 3.3. In any hyper residuated lattice L, $\{1\}$ is a weak filter and L is a filter of L. Of course, in Example 2.2(i), $\{1\}$ is a filter and in Example 2.2(iii), $\{a, b, 1\}$ and $\{b, 1\}$ are weak filters of L. But, $\{1, b\}$ is not a filter. □

The next theorem gives an equivalent condition for weak filters.

Theorem 3.4. A non-empty subset F of L is a weak filter if and only if it satisfies (F) and $(x \odot y) \cap F \neq \emptyset$, for all $x, y \in F$.

Proof. Straightforward. □

Definition 3.5. Let D be a non-empty subset of L. D is called a

- *deductive system* if for all $x, y \in L$,
  (DS) $1 \in D$,
  (HDS) $x \in D$ and $x \to y \subseteq D$ imply $y \in D$,
- *weak deductive system* if (DS) holds and for all $x, y \in L$,
  (WHDS) $x \in D$ and $D \ll x \to y$ imply $y \in D$.

A deductive system D is said to be proper if $D \neq L$.

Example 3.6. In Example 2.2(ii), for any $a \in (0, 1]$, $D = [a, 1]$ is a deductive system of L, which is not a weak deductive system of L, since $[a, 1] \ll a \to y$, for any $y \leqslant a$ and $y \notin [a, 1]$). Moreover, in Example 2.2(i), for any $a \in S$, $D = [a, 1]$ is a weak deductive system of S. □

Proposition 3.7. *Let L be a hyper residuated lattice. Then*

(i) *every weak deductive system satisfies (F);*

(ii) *if D is a non-empty subset of L satisfying (F), then D is a weak deductive system of L if and only if $(x \to y) \cap D \neq \emptyset$ and $x \in D$ imply $y \in D$.*

Proof. (i). Let F be a weak hyper deductive system of L, $x \leqslant y$ and $x \in F$, for $x, y \in L$. Then by Proposition 2.3(2), $1 \in x \to y$, and so $F \ll x \to y$. Now, from (WHDS) it follows that $y \in F$. Thus, (F) holds.

(ii). ($\Rightarrow$) It follows from Proposition 2.3,(10).

($\Leftarrow$) Let D be a non-empty subset of L satisfying the given conditions. Obviously, $1 \in D$. Now, let $x \in D$ and $D \ll x \to y$. Then there exist $d \in D$ and $u \in x \to y$ such that $d \leqslant u$ and so by (F), $u \in D$. Hence $D \cap (x \to y) \neq \emptyset$ and so $y \in D$. Therefore, D is a weak hyper deductive system of L. □

Now, we give the connection between (weak) filters and (weak) deductive systems.

Theorem 3.8. *Let L be a hyper residuated lattice. Then*

(i) *every weak deductive system is a weak filter,*

(ii) *every filter is a deductive system.*

Proof. (i). Let F be a weak deductive system of L. Then by Proposition 3.7(i), (F) holds. Now, let $x, y \in F$. By Proposition 2.3(7), $y \ll x \to (x \odot y)$ and so $y \leqslant u$, for some $u \in x \to (x \odot y)$. Hence $u \in F$. But $u \in x \to (x \odot y)$ implies that $u \in x \to v$, for some $v \in x \odot y$, and hence $F \ll x \to v$. Since, $x \in F$, so $v \in F$ and hence, $F \ll x \odot y$.

(ii). Assume that F is a filter of L. Since, F is non-empty, then there exists $x \in L$ such that $x \in F$. From $x \ll 1$ and (F), it follows that $1 \in F$. Thus, (DS) holds. Now, let $x \in F$ and $x \to y \subseteq F$, for $x, y \in L$. Then, $x \odot (x \to y) = \cup_{u \in x \to y} x \odot u \subseteq F$. On the other hand, from Proposition 2.3(6), we know that $x \odot (x \to y) \ll y$. Hence, there exists $v \in x \odot (x \to y)$ such that $v \leqslant y$, and since $v \in F$, so $y \in F$. □

Example 3.9. Consider the residuated lattice L given in the Example 2.2(iii). It is not difficult to check that $F = \{b, 1\}$ is a weak filter of L but it is not a weak deductive system. Because $F \ll \{a, b\} = b \rightsquigarrow 0$, $b \in F$ while $0 \notin F$. □

Definition 3.10. A non-empty subset A of L is said to be

- $S_{\odot}$*-reflexive* if $(x \odot y) \cap A \neq \emptyset$ implies $x \odot y \subseteq A$, for all $x, y \in L$,
- $S_{\to}$*-reflexive* if $(x \to y) \cap A \neq \emptyset$ implies $x \to y \subseteq A$, for all $x, y \in L$.

Clearly, any $S_{\odot}$-reflexive weak filter of L is a filter.

Example 3.11. (i) Let $(L, \vee, \wedge, \odot, \to, 0, 1)$ be the hyper residuated lattice in Example 2.2(iii). Then $A = \{0\}$ is a $S_{\to}$-reflexive subset of $(L, \vee, \wedge, \odot, \to, 0, 1)$.

(ii) Let $(L; \leq, \vee, \wedge, 0, 1)$ be the bounded super lattice defined in Example 2.2(iii). Consider the following tables:

Table 5

$\odot$	0	a	b	1
0	{0}	{0}	{0}	{0}
a	{0}	{a,0}	{a}	{a}
b	{0}	{a}	{b}	{b}
1	{0}	{a}	{b}	{1}

Table 6

$\to$	0	a	b	1
0	{1}	{1}	{1}	{1}
a	{0,a}	{1}	{1}	{1}
b	{0}	{0,a}	{1}	{1}
1	{0}	{a}	{b}	{1}

It is not difficult to check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper residuated lattice. Let $F_1 = \{1\}, F_2 = \{1, b\}$. Then F_1 and F_2 are $S_{\odot}$-reflexive (weak) filters of L and F_1 is a $S_{\to}$-reflexive deductive system of L. □

Theorem 3.12. *Every* $S_{\odot}$*-reflexive weak filter is a weak deductive system.*

Proof. Let F be an $S_{\odot}$-reflexive weak filter of L. Obviously $1 \in F$. Now, let $x, y \in L$ be such that $x \in F$ and $F \ll x \to y$. Then there exist $a \in F$ and $b \in x \to y$ such that $a \leqslant b$. Hence $b \in F$ and so by Theorem 3.4, $(x \odot b) \cap F \neq \emptyset$. Since F is $S_{\odot}$-reflexive, we get $x \odot b \subseteq F$. From $b \in x \to y$ it follows that $b \odot x \ll y$ and so $u \leqslant y$, for some $u \in b \odot x$. Since $x \odot b \subseteq F$, then $u \in F$ whence $y \in F$. Therefore, F is a weak deductive system of L. □

Theorem 3.13. *Every $S_{\to}$-reflexive deductive system is a filter.*

Proof. Let D be a $S_{\to}$-reflexive deductive system, $x \in D$ and $x \leqslant y$, for some $y \in L$. By Proposition 2.3(2), $1 \in x \to y$ and so $(x \to y) \cap D \neq \emptyset$. Since D is a $S_{\to}$-reflexive, we get $x \to y \subseteq D$ whence $y \in D$. Hence D satisfies (F). Now, let $x, y \in D$. If $u \in x \odot y$, then $x \odot y \ll u$ and so $x \ll y \to u$. From $x \in D$ it follows that $D \ll y \to u$ and so $D \cap (y \to u) \neq \emptyset$. Since D is $S_{\to}$-reflexive, then $y \to u \subseteq D$ whence $u \in D$. Hence, $x \odot y \subseteq D$ means that D is a filter of L. □

Proposition 3.14. *Let $\{F_i \mid i \in I\}$ be a family of non-empty subsets of L.*

(i) *If F_i is a filter (deductive system, weak deductive system), for all $i \in I$, then $\cap F_i$ is a filter (deductive system, weak deductive system) of L.*

(ii) *Assume that $\{F_i \mid i \in I\}$ be a chain. If F_i is a filter (weak filter, weak deductive system), for all $i \in I$, then $\cup F_i$ is a filter (weak filter, weak deductive system) of L.*

Proof. We only prove the case of weak deductive systems. The proof of the other cases is easy.

(i). Assume that F_i is a weak deductive system of L, for all $i \in I$. Clearly, $1 \in \cap F_i$. Let $x \in \cap F_i$ and $\cap F_i \ll x \to y$, for some $y \in L$. Then $x \in F_i$ and $F_i \ll x \to y$, for all $i \in I$. Hence $y \in F_i$, for all $i \in I$ and so $y \in \cap F_i$. Therefore, $\cap F_i$ is a weak deductive system of L.

(ii). Let $\{F_i \mid i \in I\}$ be a chain of weak deductive systems of L. Clearly, $1 \in \cup F_i$. Let $x \in \cup F_i$ and $\cup F_i \ll x \to y$, for some $y \in L$. Then, there exist $j, k \in I$ such that $x \in F_j$ and $F_k \ll x \to y$. Since F_i's forms a chain, so we can assume that $F_j \subseteq F_k$. Thus, $F_k \ll x \to y$ and $x \in F_k$ imply that $y \in F_k \subseteq \cup F_i$ proving $\cup F_i$ is a weak deductive system of L. □

The next example shows that Proposition 3.14(ii) may not be true for deductive systems, in general.

Example 3.15. Let $L = \{x_i \mid i \in \mathbb{N}\} \cup \{0, 1\}$ be a lattice, whose Hasse diagram is below (see Figure 1).

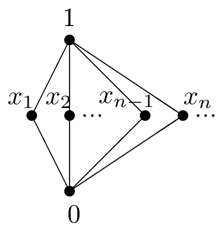

Figure 1: The Hasse diagram of L

Define binary hyperoperations $\vee, \wedge, \odot$ and $\rightarrow$ on L as follows:

$$a \vee b = \{c \in L \,|\, a \leqslant c \text{ and } b \leqslant c\}, \qquad a \wedge b = \{c \in L \,|\, c \leqslant a \text{ and } c \leq b\}$$

$a \odot b = a \wedge b$ and

$$a \rightarrow b = \begin{cases} \{1\} & \text{if } a \leqslant b, \\ \{x_i \,|\, i \in \mathbb{N}\} & \text{if } a = 1,\ b \in L - \{1\}. \\ \{x_j \,|\, j \in \mathbb{N}, j \leqslant i\} \cup \{1\} & \text{if } a, b \in \{x_i \,|\, i \in \mathbb{N}\},\ a = x_i,\ a \neq b, \\ \{x_j \,|\, j \in \mathbb{N}, j \leqslant i\} \cup \{1\} & \text{if } a \in \{x_i \,|\, i \in \mathbb{N}\},\ b = 0 \end{cases}$$

for all $a, b \in L$. Routine calculations show that $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a hyper residuated lattice. Let $D_i = \{1, x_1, \ldots, x_i\}$, for all $i \in \mathbb{N}$. It is easy to verify that D_i is a deductive system of L and $D_i \subseteq D_{i+1}$, for all $i \in \mathbb{N}$. But, $1 \in \cup_{i \in I} D_i$, $1 \rightarrow 0 = \{x_i \,|\, i \in \mathbb{N}\} \subseteq \cup_{i \in I} D_i$ and $0 \notin \cup_{i \in I} D_i$. Therefore, $\cup_{i \in I} D_i$ is not a deductive system of L. □

Definition 3.16. Let F be a proper (weak) filter of L. Then F is said to be *maximal* if $F \subseteq J \subseteq L$, implies $F = J$ or $J = L$, for all (weak) filters J of L.

Maximal (weak) deductive systems are defined analogously.

Example 3.17. Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be the hyper residuated lattice defined in the Example 3.11. Then $F = \{1, b\}$ is a maximal filter and $\{1, a, b\}$ is a maximal weak filter of L. □

Theorem 3.18. *In a hyper residuated lattice*

(i) *every proper (weak) filter of L is contained in a maximal (weak) filter of L,*

(ii) *every proper weak deductive system of L is contained in a maximal weak deductive system of L.*

Proof. (i). Let F be a proper (weak) filter of L and S be the collection of all proper (weak) filters of L containing F. Then $F \in S$ and $(S, \subseteq)$ is a poset. Let $\{F_i \,|\, i \in I\}$ be a chain in S. Then by Proposition 3.14(ii), $\cup F_i$ is a (weak) filter of L containing F. If $0 \in \cup F_i$, then there exists $i \in I$ such that $0 \in F_i$, which is impossible. Hence $\cup F_i$ is a proper (weak) filter of L containing F and so $\cup F_i \in S$. Hence any chain of elements of S has an upper bound in S. By Zorn's lemma, S has a maximal element such as M. We show that M is a maximal (weak) filter of L. Let $M \subseteq J \subseteq L$, for some (weak) filter J of L. If $J \neq L$, then $J \in S$. Since M is a maximal element of S we get $M = J$. Therefore, M is a maximal (weak) filter of L.

(ii). Similar to (i). □

From the fact that $\{1\}$ is a weak filter of any hyper residuated lattice, we conclude that

Corollary 3.19. *Every nontrivial hyper residuated lattice has a maximal weak hyper filter.*

4. (Positive) Implicative weak deductive systems

Definition 4.1. Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a hyper residuated lattice and D be a non-empty subset of L containing 1. Then D is called

- *an implicative weak deductive system* or simply $IWDS$ if for all $x, y, z \in L$

  $x \rightarrow (y \rightarrow z) \cap D \neq \emptyset$ and $x \rightarrow y \cap D \neq \emptyset$ imply $x \rightarrow z \cap D \neq \emptyset$,

- *a positive implicative weak deductive system* or simply $PIWDS$ if

  $x \rightarrow ((y \rightarrow z) \rightarrow y)) \cap D \neq \emptyset$ and $x \in D$ imply $y \in D$, for all $x, y, z \in L$.

Note: Clearly, if L is a residuated lattice, then the concept of implicative (positive implicative) filters are coincide by the concept of implicative (positive implicative) weak deductive systems.

Example 4.2. Let $L = \{a, b, c, 0, 1\}$ be a partially ordered set whose Hasse diagram depicted in Figure 2.

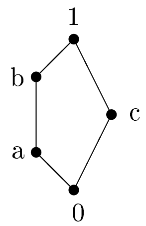

Figure 2: The Hasse diagram of L

Let $x \wedge y = \{u \in L \,|\, u \leqslant x, u \leqslant y\}$ and $x \vee y = \{u \in L \,|\, x \leqslant u, y \leqslant u\}$, for all $x, y \in L$. Now, consider the following tables:

Table 7

$\rightarrow$	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{c}	{1}	{1}	{c}	{1}
b	{c}	{a,b,c}	{1}	{c}	{1}
c	{a,b}	{a,b}	{b,a}	{1}	{1}
1	{0}	{a}	{b,a}	{c}	{1}

Table 8

$\odot$	0	a	b	c	1
0	{0}	{0}	{0}	{0}	{0}
a	{0}	{a}	{a}	{0}	{a}
b	{0}	{a}	{b,a}	{0}	{a,b}
c	{0}	{0}	{0}	{c}	{c}
1	{0}	{a}	{b,a}	{c}	{1}

It is easy to show that $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a hyper residuated lattice. Moreover, easy calculations show that

(i) $\{1, a\}$, and $\{1, a, b\}$ are implicative weak deductive systems.

(ii) $\{1, a\}$ is not a positive implicative weak deductive systems (since $1 \in 1 \rightarrow (\{a, c\} \rightarrow b) \subseteq 1 \rightarrow ((b \rightarrow a) \rightarrow b)$ and $b \notin \{1, a\}$).

(iii) $\{1, a, b\}$ is a positive implicative weak deductive system. □

Lemma 4.3. *Let $(L, \vee, \wedge, \odot, \to, 0, 1)$ be a hyper residuated lattice. Then L satisfies the following conditions: for all $a, b, c \in L$,*

(i) $a \to (b \to c) \leqslant b \to (a \to c)$,

(ii) $x \leqslant y$ *implies* $z \to x \leqslant z \to y$,

(iii) $a \to b \leqslant (b \to c) \to (a \to c)$.

Proof. (i). Let u be an arbitrary element of $a \to (b \to c)$. Then $u \ll (a \to (b \to c))$ and so $u \ll a \to x$, for some $x \in b \to c$. Hence $u \odot a \ll x$ and so $y \ll x$, for some $y \in u \odot a$. Since $x \in b \to c$, then we get $y \ll b \to c$ and so $y \odot b \ll c$. Hence $(u \odot b) \odot a = (u \odot a) \odot b \ll c$ and by Proposition 2.3(4), we get $u \odot b \ll a \to c$. Therefore, by Proposition 2.3(4), $u \ll b \to (a \to c)$ and so $a \to (b \to c) \leqslant b \to (a \to c)$.

(ii). Let $u \in z \to x$. Then $u \ll z \to x$, so $u \odot z \ll x$. Since $x \leqslant y$, then we get $u \odot z \ll y$ and so $u \ll z \to y$. Therefore, $z \to x \leqslant z \to y$.

(iii). We know that $(b \to c) \to (a \to c) \subseteq \cup\{u \to v | u \in b \to c,$ and $v \in a \to c\}$. Let $u \in b \to c$. Then $u \ll b \to c$. Thus $u \odot b \ll c$. Hence $b \ll u \to c$ and so there exists $t \in u \to c$ such that $b \leqslant t$. Now, by (i) and (ii), we get $a \to b \leqslant a \to t \subseteq (a \to (u \to c)) \leqslant u \to (a \to c)$. Since $u \in b \to c$, we conclude that $a \to b \leqslant (b \to c) \to (a \to c)$. □

Note that, in the proof of Lemma 4.3(iii) we proved that $a \to b \leqslant u \to (a \to x)$, for all $u \in b \to x$.

From now on, in this section, $(L, \vee, \wedge, \odot, \to, 0, 1)$ or simply L will denote a hyper residuated lattice satisfies $1 \odot x = \{x\}$, for all $x \in L$, unless otherwise stated.

Proposition 4.4. *Let D be a non-empty subset of L. Then*

(i) *for all $x \in L$, $x \in 1 \to x$ and x is a maximum element of $1 \to x$,*

(ii) *if D is a $PIWDS$ of L, then D is a weak deductive system,*

(iii) *if D is an $IWDS$ of L is an upset, then D is a weak deductive system.*

Proof. (i). Let $x \in L$. For any $u \in 1 \to x$, we have $u \ll 1 \to x$ and so $\{u\} = 1 \odot u \ll x$. Since $x \in 1 \odot x$, then we get $1 \odot x \ll x$. It follows that $x \ll 1 \to x$. Hence there exists $u \in 1 \to x$ such that $x \leqslant u$. So, $x \leqslant u \leqslant x$. Therefore, $x \in 1 \to x$.

(ii). Assume that D is a $PIWDS$ of L. Clearly, (DS) holds. Let $(x \to y) \cap D \neq \emptyset$ and $x \in D$. Then by Proposition 2.3(3), we have $x \to (1 \to y) \subseteq x \to ((y \to 1) \to y)$. Now, by (i) we get $x \to y \subseteq x \to (1 \to y)$ and so $(x \to ((y \to 1) \to y)) \cap D \neq \emptyset$. Since $x \in D$ and D is a positive implicative weak deductive system of L, we conclude that $y \in D$. Therefore, D is a weak deductive system of L.

(iii). Assume that D is an $IWDS$ of L. Clearly, (DS) holds. Let $(x \to y) \cap D \neq \emptyset$ and $x \in D$. Then by (i), $(1 \to (x \to y)) \cap D \neq \emptyset$ and $(1 \to x) \cap D \neq \emptyset$. Since D

is an implicative weak deductive system of L, then $(1 \to y) \cap D \neq \emptyset$. Since by (i) y is a maximum element of $1 \to y$ and D is an upset, then we get $y \in D$. □

Theorem 4.5. *Let D be a non-empty subset of L. Then*

(i) *D is a PIWDS of L if and only if D is a weak deductive system such that $((x \to y) \to x) \cap D \neq \emptyset$ implies $x \in D$, for all $x, y \in L$,*

(ii) *D is an IWDS of L if and only if $D_x = \{u \in L \mid (x \to u) \cap D \neq \emptyset\}$ is a weak deductive system of L, for all $x \in L$.*

Proof. (i). Let D be a $PIWDS$. Then by Proposition 4.4(ii), D is a weak deductive system. Now, let and $((x \to y) \to x) \cap D \neq \emptyset$. Then there exists $u \in ((x \to y) \to x) \cap D$. By Proposition 4.4(i), $u \in 1 \to u \subseteq (1 \to ((x \to y) \to x)) \cap D$. Since $1 \in D$ and D is a $PIWDS$, then we get $x \in D$. Conversely, let D be a weak deductive system such that $((x \to y) \to x) \cap D \neq \emptyset$ implies $x \in D$, for all $x, y \in L$. Let $(x \to ((y \to z) \to y)) \cap D \neq \emptyset$ and $x \in D$. Since D is a weak deductive system and $x \in D$, then $((y \to z) \to y) \cap D \neq \emptyset$ and so $y \in D$. Therefore, D is a $PIWSD$.

(ii). Let D be an $IWDS$ of L and $x \in L$. By Proposition 2.3(3), $1 \in D_x$. Now, let $(a \to b) \cap D_x \neq \emptyset$ and $a \in D_x$, for some $a, b \in L$. Then $(x \to a) \cap D \neq \emptyset$ and $(x \to (a \to b)) \cap D \neq \emptyset$. Since D is an $IWDS$, we get $(x \to b) \cap D \neq \emptyset$ and so $b \in D_x$. Hence D_x is a weak deductive system. Conversely, let $D_x = \{u \in L|(x \to u) \cap D \neq \emptyset\}$ is a weak deductive system of L, for all $x \in L$. If $(x \to (y \to z)) \cap D \neq \emptyset$ and $(x \to y) \cap D \neq \emptyset$, for some $x, y, z \in L$, then $y \in D_x$ and $(y \to z) \cap D_x \neq \emptyset$. Since D_x is a weak deductive system of L, then we conclude that $z \in D_x$ and so $(x \to z) \cap D \neq \emptyset$. Therefore, D is an $IWDS$ of L. □

Example 4.6. Let $P = \{1, 0, a, b\}$, $P' = \{1, 0, a, c\}$ and $\leqslant$ be the partially relation was defined in Example 4.2. Then $(P, \leqslant)$ and $(P', \leqslant)$ are two partially ordered sets. Consider the following tables.

Table 9

$\to$	0	a	b	1
0	{1}	{1}	{1}	{1}
a	{0}	{1,a}	{1}	{1}
b	{0}	{a}	{1,b}	{1}
1	{0}	{a}	{b,a}	{1}

Table 10

$\rightsquigarrow$	0	a	c	1
0	{1}	{1}	{1}	{1}
a	{c}	{1,c}	{c}	{1}
c	{a}	{a}	{1,a}	{1}
1	{0}	{a}	{c}	{1}

Easy calculations show that $(P, \vee, \wedge, \odot, \to 0, 1)$ and $(P', \vee, \wedge, \odot, \rightsquigarrow, 0, 1)$ are two hyper residuated lattices, where $\vee$, $\odot$ and $\wedge$ are the same as in L (Example 4.2) except restricted to P and P', respectively.

(i) Consider the hyper residuated lattice $(P, \vee, \wedge, \odot, \to 0, 1)$. If $D = \{1\}$, then $D_1 = \{1\}$, $D_a = \{1, a, b\}$, $D_b = \{1, b\}$ and $D_0 = P$. Since D_0, D_a, D_b and D_1 are weak deductive systems of $(P, \vee, \wedge, \odot, \to 0, 1)$, then by Theorem 4.5(ii), $\{1\}$ is an $IWDS$ of $(P, \vee, \wedge, \odot, \to 0, 1)$. Moreover, $a \notin \{1\}$ and $((a \to a) \to a) \cap \{1\} \neq \emptyset$. Hence by Theorem 4.5(i), $\{1\}$ is not $PIWDS$ of $(P, \vee, \wedge, \odot, \to 0, 1)$.

(ii) $\{1, a, b\}$ is a $PIWDS$ of P.

(iii) $\{1\}$ is a $PIWDS$ of $(P', \vee, \wedge, \odot, \rightsquigarrow, 0, 1)$. □

Open Problem: *Is there a PIWDS which is not IWDS?*

Example 4.7. Let $(S, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be the hyper residuated lattice in Example 2.2(ii). It is easy to show that $[a, 1]$ is a weak deductive system of S, for any $a \in [0, 1)$. Let $D = [a, 1]$. Then by definition of $\rightarrow$ we get

$$D_x = \begin{cases} \text{[x,1]} & \text{if } x \leqslant a, \\ D & \text{if } a \leqslant x \end{cases}$$

Hence D_x is a weak deductive system of S and so by Theorem 4.5(ii), D is an $IWDS$ of S. Now, let $D = (0, 1]$. Since $(0 \rightarrow y) \rightarrow 0 = 1 \rightarrow 0 = \{0\}$, for all $y \in [0, 1]$, then we get D is a $PIWDS$ of L. We show that $(0, 1]$ is the only proper $PIWDS$ of S. Let F be a $PIWDS$ of S. Then by Proposition 4.4 and Theorem 3.8, F is an upset. So $F = (a, 1]$ or $F = [a, 1]$, for some $a \in S - \{0\}$. Let $e, f \in (0, a)$ such that $f < e$. Then $(e \rightarrow f) \rightarrow e = f \rightarrow e = \{1\}$ and $((e \rightarrow f) \rightarrow e) \cap F \neq \emptyset$. Since $e \in S - F$, then by Theorem 4.5(i), D is not $PIWDS$ of S. □

Theorem 4.8. *Let D be a weak deductive system of L. Then the following are equivalent:*

(i) *D is an $IWDS$ of L,*

(ii) *$(y \rightarrow (y \rightarrow x)) \cap D \neq \emptyset$ implies $(y \rightarrow x) \cap D \neq \emptyset$, for all $x, y \in L$,*

(iii) *$(z \rightarrow (y \rightarrow (y \rightarrow x))) \cap D \neq \emptyset$ and $z \in D$ imply $(y \rightarrow x) \cap D \neq \emptyset$, for all $x, y \in L$,*

(iv) *$(x \rightarrow u) \cap D \neq \emptyset$ for any $x \in L$ and any $u \in x \odot x$.*

Proof. (i)⇒ (ii). Let D be an $IWDS$ of L and $(y \rightarrow (y \rightarrow x)) \cap D \neq \emptyset$. By Proposition 2.3(3), $(y \rightarrow y) \cap D \neq \emptyset$. Since D is an $IWDS$ of L, then $(y \rightarrow x) \cap D \neq \emptyset$.

(ii)⇒ (iii). Let (ii) holds, $(z \rightarrow (y \rightarrow (y \rightarrow x))) \cap D \neq \emptyset$ and $z \in D$. Since D is a weak deductive system, then $(y \rightarrow (y \rightarrow x)) \cap D \neq \emptyset$ and so $y \rightarrow x \in D$.

(iii)⇒ (i). Let (iii) holds, $(x \rightarrow (y \rightarrow z)) \cap D \neq \emptyset$ and $(x \rightarrow y) \cap D \neq \emptyset$. Since $x \rightarrow (y \rightarrow z) \leqslant y \rightarrow (x \rightarrow z)$ (by Lemma 4.3) and D is an upset (by Theorem 3.8), then we get there exists $u \in (y \rightarrow (x \rightarrow z)) \cap D$. Now,

$$\begin{aligned} u \ll y \rightarrow (x \rightarrow z) &\Rightarrow u \odot y \ll x \rightarrow z, \text{ by Proposition 2.3(4)} \\ &\Rightarrow y \ll u \rightarrow (x \rightarrow z), \text{ by Proposition 2.3(4)} \\ &\Rightarrow y \leqslant a, \text{ for some } a \in u \rightarrow (x \rightarrow z) \\ &\Rightarrow x \rightarrow y \leqslant x \rightarrow a, \text{ by Lemma 4.3(ii)} \\ &\Rightarrow x \rightarrow y \leqslant x \rightarrow (u \rightarrow (x \rightarrow z)) \\ &\Rightarrow (x \rightarrow (u \rightarrow (x \rightarrow z))) \cap D \neq \emptyset, \text{ since } x \rightarrow y \cap D \neq \emptyset \\ &\Rightarrow (u \rightarrow (x \rightarrow (x \rightarrow z))) \cap D \neq \emptyset, \text{ by Lemma 4.3(i).} \end{aligned}$$

Since $u \in D$, then by (iii), we conclude that $(x \to z) \cap D \neq \emptyset$. Therefore, D is an $IWDS$ of L.

(ii) $\Rightarrow$ (iv). Suppose that $x \in L$ and $u \in x \odot x$. Then $x \odot x \ll u$ and so $x \ll x \to u$. Hence by Proposition 2.3(3), $1 \in (x \to (x \to u)) \cap D$ and $1 \in (x \to x) \cap D$. Since D is an $IWDS$ of L, then $(x \to u) \cap D \neq \emptyset$.

(iv) $\Rightarrow$ (ii). Let $(y \to (y \to x)) \cap D \neq \emptyset$, for some $x, y \in L$. Then there exists $u \in (y \to (y \to x)) \cap D$. By Proposition 2.3(3), $1 \in u \to (y \to (y \to x))$ and so by Lemma 4.3(i), $1 \in y \to (y \to (u \to x))$. It follows that $1 \ll y \to (y \to t)$, for some $t \in u \to x$ and so $\{y\} = 1 \odot y \ll y \to t$. Hence $y \odot y \ll t$, whence $a \leqslant t$, for some $a \in y \odot y$. Since $y \to a \leqslant y \to t$, then by Lemma 4.3, we obtain $\emptyset \neq D \cap (y \to t) \subseteq y \to (u \to x) \leqslant u \to (y \to x)$. Since D is a weak deductive system of L by Theorem 3.8, $u \to (y \to x) \cap D \neq \emptyset$. Now, $u \in D$ implies $(y \to x) \cap D \neq \emptyset$. Therefore, D is an $IWDS$ of L. □

Theorem 4.9. *Let F and G be two weak deductive system of L such that $F \subseteq G$. If F is an $IWDS$ of L, then G is an $IWDS$ of L, too.*

Proof. It follows from Theorem 4.8. □

Corollary 4.10. *Any weak deductive systems of L is an $IWDS$ of L if and only if $\{1\}$ is an $IWDS$ of L, or equivalently, if and only if $x \leqslant u$, for any $u \in x \odot x$.*

Proof. (i). Let $(x \to y) \cap \{1\} \neq \emptyset$ and $x \in \{1\}$. Then $1 \ll 1 \to y$ and so $1 \odot 1 \ll y$. Since $1 \odot u = \{u\}$, for all $u \in L$, we get $1 \ll y$ and so $y = 1$. Hence $\{1\}$ is a weak deductive system of L, Now, by using of Theorem 4.9, we get $\{1\}$ is an $IWDS$ if and only if any weak deductive system of L is an $IWDS$ of L.

(ii). By Proposition 2.3(2), we have $x \leqslant y$ if and only if $1 \in x \to y$. Suppose that $\{1\}$ is an $IWDS$ of L. Then by Theorem 4.8, $1 \in x \to u$, for any $u \in x \odot x$ and so $x \leqslant u$, for any $u \in x^2$. Conversely, suppose that $x \leqslant u$, for all $u \in x^2$ and $(a \to (a \to b)) \cap \{1\} \neq \emptyset$, for some $a, b \in L$. Then $1 \in a \to (a \to b)$ and so by Proposition 2.3(4), $\{a\} = 1 \odot a \ll a \to b$. Hence $a \odot a \ll b$. By assumption we get $a \leqslant b$ and so $1 \in a \to b$. Therefore, $(a \to b) \cap \{1\} \neq \emptyset$ and so $\{1\}$ is an $IWDS$ of L. □

We note that, if $\{1\}$ is an $IWDS$ of L, then Corollary 4.10 and Proposition 2.3(8), imply $x \in x \odot x$, for all $x \in L$.

Theorem 4.11. *Let D be a weak deductive system of L. Then D is a maximal and implicative weak deductive system of L if and only if $x \to y \cap D \neq \emptyset$ and $y \to x \cap D \neq \emptyset$, for all $x, y \in L - D$.*

Proof. Suppose that D is a maximal and implicative weak deductive system and $x, y \in L - D$. By Proposition 2.3(3) and (8), we get that $x \in D_x$, $y \in D_y$, $D \subseteq D_x \subseteq L$ and $D \subseteq D_y \subseteq L$. Moreover, Theorem 4.5(ii) implies D_x and D_y are weak deductive systems of L. Hence by assumption $D_x = L = D_y$ and so $y \in D_x$, $x \in D_y$. Therefore, $x \to y \cap D \neq \emptyset$ and $y \to x \cap D \neq \emptyset$. Conversely, let D be a weak deductive system such that $x \to y \cap D \neq \emptyset$ and $y \to x \cap D \neq \emptyset$, for all $x, y \in L - D$. If there exists $a \in L$ such that D_a is not weak deductive systems of L, then there are $x, y \in L$ such that $x \to y \cap D_a \neq \emptyset$, $x \in D_a$ and $y \notin D_a$. Hence $a \to x \cap D \neq \emptyset$ and $a \to u \cap D \neq \emptyset$, for some $u \in x \to y$. But $a \to y \cap D = \emptyset$. From Proposition 2.3(8) and Theorem 3.8, we get that $y \notin D$. Hence by assumption $a \in D$. Since $a \to x \cap D \neq \emptyset$ and $a \to x \cap D \neq \emptyset$, then we get $x \in D$ and $u \in D$. It follows that $x \to y \cap D \neq \emptyset$. That is $y \in D$, which is contradiction. Hence D_a is a weak deductive system of L, for any $a \in L$. By Theorem 4.5(ii), D is an implicative deductive system. Now, we show that, D_a is the least weak deductive system of L containing $D \cup \{a\}$, for any $a \in L - D$. Let $a \in L - D$ and D' be a weak deductive system of L containing $D \cup \{a\}$ and u be an arbitrary element of D_a. Then $a \to u \cap D \neq \emptyset$ and so $a \to u \cap D' \neq \emptyset$. Since $a \in D'$, then $u \in D'$. Hence $D_a \subseteq D'$. That is D_a is the least weak deductive system of L containing $D \cup \{a\}$. Assume that $D \subsetneq E \subseteq L$, for some weak deductive system E of L. Then there exists $a \in E - D$. It follows that $D_a \subseteq E$. Since $a \in L - D$, by assumption of Proposition 2.3(8) and Theorem 3.8, we get $D_a = L$ and so $E = L$. Therefore, D is a maximal weak deductive system of L. □

5. Relation between hyper MV-algebras and hyper residuated lattices

Definition 5.1. [8] A *hyper MV-algebra* is a non-empty set M endowed with a binary hyper operation $\oplus$, a unary operation $*$ and a constant 0 satisfying the following conditions: for all $x, y, z \in M$

($hMV1$) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
($hMV2$) $x \oplus y = y \oplus x$,
($hMV3$) $(x^*)^* = x$,
($hMV4$) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
($hMV5$) $0^* \in x \oplus 0^*$,
($hMV6$) $0^* \in x \oplus x^*$,
($hMV7$) if $x \ll y$ and $y \ll x$, then $x = y$,

where $x \ll y$ is defined by $0^* \in x^* \oplus y$.

For every $A, B \subseteq M$, we define $A \ll B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \ll b$ and $A \oplus B = \cup\{a \oplus b \mid a \in A, b \in B\}$. Also, we define $0^* = 1$ and $A^* = \{a^* \mid a \in A\}$.

Lemma 5.2. *Let* $(L, \vee, \wedge, \odot, \to, 0, 1)$ *be a hyper residuated lattice and* $\neg\neg x = \{x\}$, *for all* $x \in L$. *Then* $|\neg x| = 1$, *for all* $x \in L$.

Proof. Let $x \in L$ and $a, b \in \neg x$. Then $\neg a \subseteq \neg\neg x = \{x\}$ and so $\neg a = \{x\}$. Similarly, $\neg b = \{x\}$. It follows that $\neg a = \neg b$ and so $\{a\} = \neg\neg a = \neg\neg b = \{b\}$. Hence $a = b$. Therefore, $|\neg x| = 1$. □

Theorem 5.3. *Let* $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *be a hyper residuated lattice satisfying the following conditions:*

(i) $\neg\neg x = \{x\}$, *for all* $x \in L$,

(ii) $\neg(x \odot \neg y) \odot \neg y = \neg(y \odot \neg x) \odot \neg x)$, *for all* $x, y \in L$.

Let $x+y = \neg(\neg x \odot \neg y)$. *Then* $(M, +, \neg, 0)$ *is a hyper MV-algebra (since* $|\neg x| = 1$, *we use* $\neg x$ *to denote the only element of* $\neg x$*).*

Proof. Let $x, y, z \in L$.

(1). Since $(L, \odot, 1)$ is a commutative semihypergroup, then we have

$$(x+y) = \neg(\neg x \odot \neg y) = \neg(\neg y \odot \neg x) = (y+x).$$

$$\begin{aligned}(2).\quad (x+y)+z = \cup\{a+z|a \in x+y\} &= \cup\{\neg(\neg a \odot \neg z) \mid a \in x+y\}\\ &= \cup\{\neg(\neg a \odot \neg z) \mid a \in \neg(\neg x \odot \neg y)\}\\ &= \cup\{\neg(\neg\neg b \odot \neg z) \mid b \in (\neg x \odot \neg y)\}\\ &= \cup\{\neg(b \odot \neg z) \mid b \in (\neg x \odot \neg y)\}\\ &= \neg((\neg x \odot \neg y) \odot \neg z)\end{aligned}$$

By the similar way, we can show that $\neg((\neg x \odot \neg y) \odot \neg z) = x + (y+z)$. Therefore, $x + (y+z) = (x+y)+z$.

(3). By Proposition 2.3(5), we get $x + 1 = x + \neg 0 = \neg(\neg x \odot \neg\neg 0) \supseteq \neg 0 = 1$.

(4). By Proposition 2.3(6), $(x \odot \neg x \ll 0$, so $0 \in x \odot \neg x$. Hence

$$(x + \neg x) = \neg(\neg x \odot \neg\neg x) = \neg(\neg x \odot x) \supseteq \neg 0 = 1.$$

(5). Let $1 \in (\neg x + y) \cap (x + \neg y)$. Then $1 \in \neg(\neg\neg x \odot \neg y) \cap \neg(\neg x \odot \neg\neg y)$ and so $0 \in (x \odot \neg y) \cap (\neg x \odot y)$. It follows that $x \odot \neg y \ll 0$ and $\neg x \odot y \ll 0$. Hence $x \ll \neg\neg y$ and $y \ll \neg\neg x$ and so $x = y$.

$$\begin{aligned}(6).\quad \neg(\neg x + y) + y = (\neg\neg(x \odot \neg y)) + y &= (x \odot \neg y) + y\\ &= \neg(\neg(x \odot \neg y) \odot \neg y)\\ &= \neg(\neg(y \odot \neg x) \odot \neg x), \text{ by assumption}\\ &= \neg(\neg y + x) + x.\end{aligned}$$

From (i) and (1) – (6), it follows that, $(M, +, \neg, 0)$ is a hyper MV-algebra. □

Example 5.4. Let $(P', \vee, \wedge, \odot, \rightsquigarrow, 0, 1)$ be a hyper residuated lattice in Example 4.6. Then P' satisfies the conditions of Theorem 5.3. □

Open problem: *Under what conditions we can obtain a hyper residuated lattice from a hyper MV-algebra?*

6. Conclusions and future works

In this paper, we introduce the concept of hyper residuated lattice which is a generalization of the concept of residuated lattice, and we give some properties and related results. The category of hyper residuated lattices, quotient structure, filter theory, lattice structures of filters and hyper residuated lattices could be topics for future researchs.

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