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# Parastrophically equivalent identities characterizing quasigroups isotopic to abelian groups

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**Abstract.** We study parastrophic equivalence of identities in primitive quasigroups and parastrophically equivalent balanced and near-balanced identities characterizing quasigroups isotopic to groups (to abelian groups). Some identities in quasigroups with 0-ary operation characterizing quasigroups isotopic to abelian groups are given.

## 1. Introduction

Quasigroups isotopic to groups, to abelian groups consist important classes of quasigroups. They arise under investigation of different classes and systems of quasigroups and are used in distinct applications. Medial quasigroups, linear quasigroups and $T$-quasigroups are the most known classes of these quasigroups. Quasigroups isotopic to groups (to abelian groups), their subclasses and identities reducing to them were investigated by many authors. Recall some of the known results.

In [1] and [3] these quasigroups arose under the research of balanced identities, including arbitrary number of variables, and under the study of four quasigroups connected by the law of general associativity. As it was proved, these four quasigroups are isotopic to the same group.

In [3], V. D. Belousov found a balanced identity of five (of four) variables in a primitive quasigroup $(Q, \cdot, \backslash, /)$ that characterizes quasigroups isotopic to groups (to abelian groups).

As M. M. Glukhov informed, he proved that among of the identities characterizing the variety of quasigroups isotopic to abelian groups there not exist balanced identities of three variables and listed six balanced identities of four variables obtained by different authors (see section 5).

In [6], [7] and [9], some identities with permutations of three variables that characterize quasigroups isotopic to groups (to abelian groups) were considered. In

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[7], the type of these identities was defined and some identities with permutations characterizing quasigroups isotopic to abelian groups were given.

In the article we continue this research, consider parastrophic equivalence of identities in primitive quasigroups and parastrophically equivalent balanced and near-balanced identities characterizing quasigroups isotopic to groups (to abelian groups). Some unbalanced identities of Theorem 1.2.1a of [7] are simplified and some identities in quasigroups with 0-ary operation characterizing quasigroups isotopic to abelian groups are given.

## 2. Preliminaries

Let $(Q,\cdot)$ be a quasigroup. The operation $(\backslash)$, or $(\cdot)^{-1}$ ($(/)$ or ${}^{-1}(\cdot)$):

$$x\backslash y = z \Leftrightarrow xz = y \quad (x/y = z \Leftrightarrow zy = x)$$

is called *the right (left) inverse operation for the operation* $(\cdot)$, and the quasigroup $(Q,\backslash)$ ($(Q,/)$) is called *the right (left) inverse quasigroup* for the quasigroup $(Q,\cdot)$. In addition,

$$x\backslash y = L_x^{-1}y, \quad x/y = R_y^{-1}x,$$

where $L_x y = x\cdot y = R_y x$.

If a quasigroup operation is denoted by $A$, then its right (left) inverse operation is denoted by $A^{-1}$ (respectively, by ${}^{-1}A$).

*The primitive quasigroup* [4] $(Q,\cdot,\backslash,/)$, e.i., the algebra with three operations satisfying to the following four identities:

$$xy/y = x, \ (x/y)y = x, \ y(y\backslash x) = x, \ y\backslash yx = x,$$

corresponds to every quasigroup $(Q,\cdot)$.

We also shall use *primitive quasigroups* $(Q,\cdot,\backslash,/,a)$ *with one 0-operation* $c_a$: $\{\emptyset\} \rightarrow a$, where $a$ is some fixed element of the set $Q$.

It is known that every quasigroup operation $A$ has the following six parastrophic operations (conjugates or parastrophes):

$$A = (\cdot),\ A^{-1} = (\backslash), {}^{-1}A = (/),\ {}^{-1}(A^{-1}) = (\otimes_1),\ ({}^{-1}A)^{-1} = (\otimes_2),\ A^* = (\cdot)^*,$$

where $A^*(x,y) = A(y,x)$ [4], moreover, $x \otimes_1 y = y/x, \ x \otimes_2 y = y\backslash x$.

If a quasigroup $(Q,\cdot)$ is isotopic to a group, then every its parastrophe is also isotopic to the same group (see, for example, [7] and [12]).

## 3. Equivalent identities in primitive quasigroups

For the first time the concept of conjugate or parastrophic identities (in a quasigroup $(Q,\cdot)$) was introduced by A. Sade in [11]. He also gave a number of rules for simplifying of identities in $(Q,\cdot)$ that involves more than one of parastrophic operations. V.D. Belousov in [2] listed the parastrophic identities in a quasigroup $(Q,\cdot)$ for a number of the well-known identities. Part of these identities was earlier given by S.K. Stein in [13]. In [5], V.D. Belousov considered parastrophic equivalence of the minimal identities in a quasigroup $(Q,\cdot)$ connected with orthogonality. Below we shall consider parastrophic equivalence of identities in primitive quasigroups.

Let a primitive quasigroup $(Q,\cdot,\backslash,/)$ satisfy an identity. Changing the operation $(\cdot)$ for some its parastrophe $\sigma$ in this identity, we shall obtain an identity, which holds in a primitive quasigroup with another triple of operations. For example, using the change $(\cdot)\to(/)$, we shall get an identity in the quasigroup $(Q,/,\otimes_2,\cdot)$, since in this case $(\backslash)\to(/)^{-1}=(\otimes_2)$, $(/)\to^{-1}(/)=(\cdot)$, $\otimes_1\to(*)$, $(\otimes_2)\to(\backslash)$, $(*)\to(\otimes_1)$ (see Tab. 1 below). But

$$x\otimes_1 y=y/x,\ \ x\otimes_2 y=y\backslash x,\ \ x*y=y\cdot x. \tag{1}$$

After use of these equalities we get another identity in the quasigroup $(Q,\cdot,\backslash,/)$.

Consider the transformations of identities in a primitive quasigroup $(Q,\cdot,\backslash,/)$ that are connected with the change of the operation $(\cdot)$ to its parastrophes. The following Table 1 shows how parastrophes are changed in an identity under the change of the operation $(\cdot)$ for some its parastrophe.

| $(\cdot)$ | $(\cdot)$ | $(\backslash)$ | $(/)$ | $(\otimes_1)$ | $(\otimes_2)$ | $(*)$ |
|---|---|---|---|---|---|---|
| $(\cdot)$ | $(\cdot)$ | $(\backslash)$ | $(/)$ | $(\otimes_1)$ | $(\otimes_2)$ | $(*)$ |
| $(\backslash)$ | $(\backslash)$ | $(\cdot)$ | $(\otimes_1)$ | $(/)$ | $(*)$ | $(\otimes_2)$ |
| $(/)$ | $(/)$ | $(\otimes_2)$ | $(\cdot)$ | $(*)$ | $(\backslash)$ | $(\otimes_1)$ |
| $(\otimes_1)$ | $(\otimes_1)$ | $(*)$ | $(\backslash)$ | $(\otimes_2)$ | $(\cdot)$ | $(/)$ |
| $(\otimes_2)$ | $(\otimes_2)$ | $(/)$ | $(*)$ | $(\cdot)$ | $(\otimes_1)$ | $(\backslash)$ |
| $(*)$ | $(*)$ | $(\otimes_1)$ | $(\otimes_2)$ | $(\backslash)$ | $(/)$ | $(\cdot)$ |

Table 1

Indeed, if $(\cdot)\to(\backslash)$, then we have the following change of parastrophes: $(\backslash)\to(\backslash)^{-1}=(\cdot)$, $(/)\to^{-1}(\backslash)=(\otimes_1)$, $(\otimes_1)\to(\backslash)^{\otimes_1}=(/)$, $(\otimes_2)\to(\backslash)^{\otimes_2}=(*)$, $(*)\to(\backslash)^*=(\otimes_2)$. Thus we get an identity in the quasigroup $(Q,\backslash,\cdot,\otimes_1)$. This result is reflected in the second row of Table 1. The remaining rows are fulled analogously.

Below we shall consider the parastrophic equivalence of identities in the following sense.

**Definition 1.** An identity $\beta$ in a quasigroup $(Q,\cdot,\backslash,/)$ is called *parastrophically equivalent* to an identity $\alpha$ if $\beta$ can be obtained from $\alpha$ under the change of the operation $(\cdot)$ to one of six its parastrophic operations with the successive passage, if it necessary, to the signature $(\cdot,\backslash,/)$ in the obtained identity by means of the equalities (1).

Note that according to this definition in each identity $\alpha$ we use only the operation $(\cdot)$ even if this operation is absent in this identity (see, for example, the identities 8) and 9) in section 5).

It is easy to prove that the relation of this definition is a relation of equivalence (i. e., symmetric, reflexive and transitive), taking into account that the parastrophic transformations of a quasigroup $(Q, \cdot)$ form the symmetrical group $S_3$ (see, for example, [8], where the multiplication table of the parastrophic transformations of a quasigroup $(Q, \cdot)$ is given).

Two identities in a quasigroup $(Q, \cdot, \backslash, /)$ is called *mutually symmetric* if one identity is obtained from another one under the passage from the operation $(\cdot)$ to the parastrophe $(*)$. An identity is called *symmetric* if it coincides with the mutually symmetric identity.

Note that if an identity in a quasigroup $(Q, \cdot, \backslash, /)$ characterizes some property of the quasigroup $(Q, \cdot)$, then an identity, which is parastrophically equivalent to this identity, not necessarily characterizes this property of the quasigroup $(Q, \cdot)$, although characterizes it in the corresponding parastrophic quasigroup. However, such situation is sometimes possible.

## 4. Identities for quasigroups isotopic to groups

Let some identity in a primitive quasigroup $(Q, \cdot, \backslash, /)$ characterize quasigroup $(Q, \cdot)$ isotopic to a group (to an abelian group). Then any identity parastrophically equivalent to this identity also characterizes the property of this quasigroup to be isotopic to this group (to this abelian group).

Indeed, it is known that if a quasigroup $(Q, \cdot)$ is isotopic to a group (to an abelian group) $(Q, +)$, then any its parastrophe is isotopic to this group (see [12] and Lemma 1.1.1 [7]). Write some identity $\alpha$ in a quasigroup $(Q, \cdot, \backslash, /)$ that characterizes the property of quasigroup $(Q, \cdot)$ to be isotopic to a group (to an abelian group) for some parastrophe $(Q, \sigma)$ of $(Q, \cdot)$ and use the equalities (1). Then we shall get an identity $\beta$ in the same signature, which is necessary and sufficient for isotopy of the quasigroup $(Q, \sigma)$ (and of the quasigroup $(Q, \cdot)$, by Lemma 1.1.1 [7]) to the same group (to the same abelian group).

Thus from one identity we can obtain the class of parastrophically equivalent identities in a primitive quasigroup every of which also characterizes the variety of quasigroups isotopic to groups (to abelian groups).

Recall that an identity $w_1 = w_2$ in a quasigroup $(Q, \cdot, \backslash, /)$ is called *balanced* [3], if every variable appears from the left and from the right exactly one time. Let $[w_1] = [w_2] = n$ for an identity $w_1 = w_2$, where $[w]$ is the number of appearances of variables in a word $w$. Then the number $n$ is called the length of this identity.

Below we shall consider near-balanced identities in the following sense.

**Definition 2.** An identity $w_1 = w_2$ of length $n + 1$ of $n \geq 2$ variables in a quasigroup $(Q, \cdot, \backslash, /)$ is called *near-balanced* if every of $n - 1$ variables appears exactly one time, but the single variable appears exactly two times on each side of the identity.

From the known balanced identity of Belousov (2) characterizing quasigroups isotopic to groups we can obtain a class of parastrophically equivalent balanced identities (or a parastrophical-equivalent class) characterizing these quasigroups.

**Theorem 1.** *The following balanced identities of five variables:*

$$((x(y\backslash z))/u)v = x(y\backslash((z/u)v)), \tag{2}$$

$$(u/(x\backslash yz)\backslash v = x\backslash(y((u/z)\backslash v)), \tag{3}$$

$$((x/(z\backslash y))u)/v = x/((zu/v)\backslash y) \tag{4}$$

*form a parastrophical-equivalent class of identities characterizing quasigroups isotopic to groups.*

*Proof.* Transform the Belousov identity (2), changing the operation $(\cdot)$ for parastrophes and using equalities (1):

$(\cdot) \to (\backslash)$: $((x\backslash(yz)) \otimes_1 u)\backslash v = x\backslash(y((z \otimes_1 u)\backslash v))$ or (3):

$$(u/(x\backslash yz)\backslash v = x\backslash(y((u/z)\backslash v)).$$

$(\cdot) \to (/)$: $((x/(y \otimes_2 z))u)/v = x/(y \otimes_2 (zu/v))$ or (4):

$$((x/(z\backslash y))u)/v = x/((zu/v)\backslash y).$$

$(\cdot) \to (\otimes_1)$: $((x \otimes_1 (y * z))\backslash u) \otimes_1 v = x \otimes_1 (y * ((z\backslash u) \otimes_1 v))$ or

$$v/((zy/x)\backslash u) = ((v/(z\backslash u))y)/x.$$

But it is the identity (4) after transpositions $(x, v)$, $(y, u)$ of variables.

$(\cdot) \to (\otimes_2)$: $((x \otimes_2 (y/z)) * u) \otimes_2 v = x \otimes_2 (y/((z * u) \otimes_2 v))$ or

$$v\backslash(u((y/z)\backslash x)) = (y/(v\backslash uz))\backslash x.$$

It is the identity (2) after the transpositions $(x, v)$, $(y, u)$.

$(\cdot) \to (*)$: $((x * (y \otimes_1 z)) \otimes_2 u) * v = x * (y \otimes_1 ((z \otimes_2 u) * v))$ or

$$v(u\backslash((z/y)x)) = ((v(u\backslash z))/y)x.$$

It is (2) after the transpositions as above.

Hence, from the identity (2) we obtain a parastrophical-equivalent class of balanced identities of length five. This class contains three identities. Note that the identity (2) is symmetric, the identities (3) and (4) are mutually symmetric (use the transpositions $(x, v)$, $(y, u)$ of variables). □

**Theorem 1a.** *The following near-balanced identities of four variables:*

$$((x(u\backslash z))/u)v = x(u\backslash((z/u)v)), \tag{2a}$$

$$(u/(x\backslash uz))\backslash v = x\backslash(u((u/z)\backslash v)), \tag{3a}$$

$$((x/(z\backslash u))u)/v = x/((zu/v)\backslash u) \tag{4a}$$

*form a parastrophical-equivalent class of identities characterizing the quasigroups isotopic to groups.*

*Proof.* These three identities, each of which characterizes quasigroups isotopic to groups, were given in Corollary 1.1.2 of [7]. The identity $(2a)$ was obtained from the balanced Belousov identity of five variables by F. N. Sokhats'kyi in [12] (the identity (38)). The identity ($3a$) (the identity ($4a$)) is obtained from ($2a$) by the change of the operation ($\cdot$) for the operation ($\backslash$) (for the operation ($/$)). Indeed, if in ($2a$) $(\cdot) \to (\backslash)$, then $((x\backslash(uz)) \otimes_1 u)\backslash v = x\backslash(u \cdot ((z \otimes_1 u)\backslash v))$ or

$$(u/(x\backslash(uz))\backslash v = x\backslash(u \cdot ((u/z)\backslash v)). \text{ It is } (3a).$$

If in ($2a$) $(\cdot) \to (/)$, then $((x/(u\otimes_2 z))\cdot u)/v = x/(u\otimes_2((z\cdot u)/v))$ or $((x/(z\backslash u))\cdot u)/v = x/((z \cdot u)/v)\backslash u)$. It is ($4a$).

Consider the identities which can be obtained from ($2a$) if to change the operation ($\cdot$) for the operation ($\otimes_1$) (for ($\otimes_2$) and for ($*$) respectively) and to use Table 1 and the equalities (1):

$(\cdot) \to (\otimes_1)$: $((x \otimes_1 (u * z))\backslash u) \otimes_1 v = x \otimes_1 (u * ((z\backslash u) \otimes_1 v))$ or

$$v/((zu/x)\backslash u) = ((v/(z\backslash u))u)/x.$$

But it is ($4a$) after the change of the positions of variables $x, v$.

$(\cdot) \to (\otimes_2)$: $((x \otimes_2 (u/z)) * u) \otimes_2 v = x \otimes_2 (u/((z * u) \otimes_2 v))$ or

$$v\backslash(u((u/z)\backslash x)) = (u/(v\backslash uz))\backslash x.$$

It is ($3a$) if to change the positions of variables $x, v$.

$(\cdot) \to (*)$: $((x * (u \otimes_1 z)) \otimes_2 u) * v = x * (u \otimes_1 ((z \otimes_2 u) * v))$ or

$$v(u\backslash((z/u)x)) = ((v(u\backslash z))/u)x. \text{ It is } (2a).$$

Note that the identity ($2a$) is symmetric, and the identities ($3a$) and ($4a$) are mutually symmetric. □

## 5. Quasigroups isotopic to abelian groups

M. M. Gluchov informed the author about some his unpublished results. In particular, he proved that among of the identities characterizing the variety of quasigroups isotopic to abelian groups there not exist balanced identities of three variables and listed the following six balanced identities of length four, every of which characterizes the quasigroups isotopic to abelian groups:

1) $x\backslash(y(u\backslash v)) = u\backslash(y(x\backslash v))$; 2) $(x/y)(u\backslash v) = (v/y)(u\backslash x)$;
3) $((xy)/u)v = ((xv)/u)y$; 4) $xu/(v\backslash y) = vu/(x\backslash y)$;
5) $x(y\backslash(uv)) = u(y\backslash(xv))$; 6) $((u/v)x)/y = ((u/y)x)/v$.

The identities 1), 2) were established by V. D. Belousov [3], the identities 3), 4) and 5) were given by A. Drapal [10], A. Tabarov found the identity 6).

Using the change of the quasigroup operation ($\cdot$) for distinct parastrophic operations in six identities pointed out above, we found yet three balanced identities of length four, every of which characterizes quasigroups isotopic to abelian groups:

7) $(u/v)\backslash yx = (u/x)\backslash yv$; 8) $(y/(v\backslash u))\backslash x = (y/(v\backslash x))\backslash u$;

$$9)\quad x/((u/v)\backslash y) = u/((x/v)\backslash y).$$

The identity 7) is obtained from 2) under the change $(\cdot) \to (\backslash)$, and when $(\cdot) \to (\otimes_2)$, also from 4) if $(\cdot) \to (\otimes_1)$, and when $(\cdot) \to (*)$.

The identity 8) is obtained from 1) under $(\cdot) \to (/)$, from 3) if $(\cdot) \to (\backslash)$, from 5) when $(\cdot) \to (\otimes_2)$ and from 6) if $(\cdot) \to (\otimes_1)$.

The identity 9) follows from 1) under $(\cdot) \to (\otimes_2)$, from 3) if $(\cdot) \to (\otimes_1)$, from 5) when $(\cdot) \to (/)$ and from 6) if $(\cdot) \to (\backslash)$.

**Theorem 2.** *All nine balanced identities of length four, pointed out above and characterizing quasigroups isotopic to abelian groups, form two parastrophical-equivalent classes:*

$$\{1), 3), 5), 6), 8), 9)\} \textit{ and } \{2), 4), 7)\}.$$

*Proof.* Transform the identity 1): $x\backslash(y(u\backslash v)) = u\backslash(y(x\backslash v))$ using the passage from the operation $(\cdot)$ to the remaining five its parastrophes, Table 1 and the equalities (1):

$(\cdot) \to (\backslash)$: $x(y\backslash(uv)) = u(y\backslash(xv))$. It is the identity 5).

$(\cdot) \to (/)$: $x \otimes_2 (y/(u \otimes_2 v)) = u \otimes_2 (y/(x \otimes_2 v))$ or $(y/(v\backslash u))\backslash x = (y/(v\backslash x))\backslash u$. It is the identity 8).

$(\cdot) \to (\otimes_1)$: $x * (y \otimes_1 vu) = u * (y \otimes_1 vx)$ or $(vu/y)x = (vx/y)u$. It is 3).

$(\cdot) \to (\otimes_2)$: $x/(y \otimes_2 (u/v)) = u/(y \otimes_2 (x/v))$ or $x/((u/v)\backslash y) = u/((x/v)\backslash y)$. It is 9).

$(\cdot) \to (*)$: $x \otimes_1 ((u \otimes_1 v)y) = u \otimes_1 ((x \otimes_1 v)y)$ or $((v/u)y)/x = ((v/x)y)/u$. It is 6).

Thus for the identity 1) we have six parastrophically equivalent identities: 1), 3), 5), 6), 8) and 9). Moreover, the following pairs of identities are mutually symmetric: 1) and 6); 3) and 5); 8) and 9).

Consider the identity 2): $(x/y)(u\backslash v) = (v/y)(u\backslash x)$.

$(\cdot) \to (\backslash)$: $(x \otimes_1 y)\backslash uv = (v \otimes_1 y)\backslash ux$ or $(y/x)\backslash uv = (y/v)\backslash ux$. It is the identity 7).

$(\cdot) \to (/)$: $xy/(u \otimes_2 v) = vy/(u \otimes_2 x)$ or $xy/(v\backslash u) = vy/(x\backslash u)$. It is 4).

$(\cdot) \to (\otimes_1)$: $(x\backslash y) \otimes_1 vu = (v\backslash y) \otimes_1 xu$ or $vu/(x\backslash y) = xu/(v\backslash y)$. It is the identity 4).

$(\cdot) \to (\otimes_2)$: $yx \otimes_2 (u/v) = yv \otimes_2 (u/x)$ or $(u/v)\backslash yx = (u/x)\backslash yv$. It is the identity 7).

$(\cdot) \to (*)$: $(x \otimes_2 y) * (u \otimes_1 v) = (v \otimes_2 y) * (u \otimes_1 x)$, or $(v/u)(y\backslash x) = (x/u)(y\backslash v)$. It is 2).

Thus we have the class of three parastrophically equivalent identities: 2), 4) and 7). The pair of the identities 4) and 7) is mutually symmetric and the identity 2) is symmetric. □

**Corollary 1.** *There exist at least two parastrophically equivalent classes of balanced identities of length four characterizing the variety of quasigroups isotopic to abelian groups.* □

## 6. Identities with a 0-ary operation

In [6], [7] and [9] were considered identities with permutations (or, simply, identities) in a quasigroup $(Q,\cdot)$:

$$\alpha_1(\alpha_2(x \oplus_1 y) \oplus_2 z) = \alpha_3 x \oplus_3 \alpha_4(\alpha_5 y \oplus_4 \alpha_6 z), \tag{5}$$

where $x, y, z$ are variables, $\alpha_i$, $i = 1, 2, ..., 6$ ($i \in \overline{1,6}$), is a permutation of the set $Q$, $(\oplus_k)$, $k = 1, 2, 3, 4$, is a parastrophic operation for the quasigroup operation $(\cdot)$. These identities form the special case of the generalized associativity identity [1]. A particular case of this identity (when $\alpha_1$ is the identity permutation) is the identity with permutations

$$\alpha_2(x \oplus_1 y) \oplus_2 z = \alpha_3 x \oplus_3 \alpha_4(\alpha_5 y \oplus_4 \alpha_6 z). \tag{6}$$

The ordered collection $(\oplus_1, \oplus_2, \oplus_3, \oplus_4)$ of parastrophic operations in an identity (5) is called the *type* of this identity. Note that three variables in (5) (in (6)) are ordered uniformly from the left and from the right.

In [7], it was proved that if a quasigroup $(Q,\cdot)$ satisfies an identity with permutations of the form (5), then the quasigroup $(Q,\cdot)$ is isotopic to a group. A quasigroup $(Q,\cdot)$ is isotopic to a group if and only if it satisfies the following identity with permutations of the type $(\circ, \circ, \circ, \circ)$:

$$R_a^{-1}(x \circ L_a^{-1} y) \circ z = x \circ L_a^{-1}(R_a^{-1} y \circ z)$$

for a fixed element $a \in Q$, where $(\circ)$ is a parastrophe of the operation $(\cdot)$, $R_a x = x \circ a$, $L_a x = a \circ x$ (Theorems 1.1.1 and 1.1.1a of [7]).

For quasigroups isotopic to abelian groups it was proved the following

**Theorem 1.2.1.** [7] *Let the identity* (6) *of the type* $(\oplus_1, \circ, \circ^*, \oplus_4)$ *with some parastrophes* $(\circ)$, $(\oplus_1), (\oplus_4)$ *and with some permutations* $\alpha_i$, $i \in \overline{2,6}$, *hold in a quasigroup* $(Q,\cdot)$. *Then the quasigroup* $(Q,\cdot)$ *is isotopic to an abelian group.*

*For any type* $(\oplus_1, \oplus_2, \oplus_3, \oplus_4)$ *different from* $(\cdot,\cdot,\cdot,\cdot)$ *and* $(*,*,*,*)$, *where* $(\oplus_i) = (\cdot)$ *or* $(\oplus_i) = (\cdot)^*$, $i \in \overline{1,4}$, *there exists an identity of the form* (6) *that characterizes quasigroups* $(Q,\cdot)$ *isotopic to abelian groups.*

Below we show that in each of nine identities of section 5 characterizing quasigroups isotopic to abelian groups one of variables may be fixed. As a result, we obtain some identities of three variables in a quasigroup with a 0-ary operation. Both identities with permutations characterizing quasigroups isotopic to groups and identities with permutations characterizing quasigroups isotopic to abelian groups, can have different types.

**Theorem 3.** *Each of the following identities in a quasigroup $(Q,\cdot,\backslash,/,a)$ with a 0-operation characterizes quasigroups $(Q,\cdot)$ isotopic to abelian groups:*

$$x\backslash(a(u\backslash v)) = u\backslash(a(x\backslash v)),\ x\backslash(y(u\backslash a)) = u\backslash(y(x\backslash a)); \qquad 1a)$$

$$(x/y)(a\backslash v) = (v/y)(a\backslash x),\ (x/a)(u\backslash v) = (v/a)(u\backslash x); \qquad 2a)$$

$$((xy)/a)v = ((xv)/a)y,\ ((ay)/u)v = ((av)/u)y; \qquad 3a)$$

$$xa/(v\backslash y) = va/(x\backslash y),\ xu/(v\backslash a) = vu/(x\backslash a); \qquad 4a)$$

$$x(a\backslash(uv)) = u(a\backslash(xv)),\ x(y\backslash(ua)) = u(y\backslash(xa)); \qquad 5a)$$

$$((u/v)a)/y = ((u/y)a)/v,\ ((a/v)x)/y = ((a/y)x)/v; \qquad 6a)$$

$$(u/v)\backslash ax = (u/x)\backslash av,\ (a/v)\backslash yx = (a/x)\backslash yv; \qquad 7a)$$

$$(y/(a\backslash u))\backslash x = (y/(a\backslash x))\backslash u,\ (a/(v\backslash u))\backslash x = (a/(v\backslash x))\backslash u; \qquad 8a)$$

$$x/((u/v)\backslash a) = u/((x/v)\backslash a),\ x/((u/a)\backslash y) = u/((x/a)\backslash y). \qquad 9a)$$

*Proof.* We shall obtain every pair of the identities 1a) – 9a) from the identities 1)– 9) respectively using the sufficient conditions of Theorem 1.2.1 [7] and the equalities (1). In all cases the necessity of the obtained identities follows from the identities 1)– 9) respectively.

$1a$). The identity $x\backslash(a(u\backslash v)) = u\backslash(a(x\backslash v))$ or $x\backslash L_a(u\backslash v) = u\backslash L_a(x\backslash v)$, where $L_a x = a\cdot x$, is the identity 1) for $y = a$ and can be written as

$$L_a(u\backslash v)\otimes_2 x = u\backslash L_a(v\otimes_2 x),$$

where $(\otimes_2) = (\backslash)^*$. This identity has the form (6) and the type $(\backslash,\otimes_2,\backslash,\otimes_2)$. Therefore, by Theorem 1.2.1 of [7], the quasigroup $(Q,\cdot)$ is isotopic to an abelian group even if in 1) the variable $y$ is fixed.

Putting in 1) $v = a$, we obtain the second identity of $1a$):

$$x\backslash(y(u\backslash a)) = u\backslash(y(x\backslash a)),\ x\backslash(y(a\otimes_2 u)) = u\backslash(y(a\otimes_2 x)$$

or $(y\cdot L_a^{\otimes_2}u)\otimes_2 x = u\backslash(y\cdot L_a^{\otimes_2}x)$, where $L_a^{\otimes_2}x = a\otimes_2 x$. Transforming the last identity, we get the identity

$$(u * y)\otimes_2 x = (L_a^{\otimes_2})^{-1}u\backslash(y\cdot L_a^{\otimes_2}x) \text{ of the type } (*,\otimes_2,\backslash,\cdot).$$

Note that the identity obtained from 1) can not be reduced to the required form of Theorem 1.2.1 of [7] if to fix one of the rest two variables.

Thus from the identity 1) in a primitive quasigroup $(Q,\cdot,\backslash,/)$ we obtain two identities in the quasigroup $(Q,\cdot,\backslash,/,a)$ for an element $a\in Q$.

$2a$). Put $u = a$ in 2): $(x/y)(a\backslash v) = (v/y)(a\backslash x)$, $(x/y)\cdot L_a^{-1}v = (v/y)\cdot L_a^{-1}x$ whence we have the identity

$$(x/y)\cdot v = L_a^{-1}x * (y\otimes_1 L_a v) \text{ of the type } (/,\cdot,*,\otimes_1).$$

If $y = a$ in 2), we obtain the second identity of 2$a$): $(x/a)(u\backslash v) = (v/a)(u\backslash x)$, $R_a^{-1}x \cdot (u\backslash v) = R_a^{-1}v \cdot (u\backslash x)$ $R_a x = xa$, and the identity

$$(v \otimes_2 u) * x = R_a^{-1}v \cdot (u\backslash R_a x) \text{ of the type } (\otimes_2, *, \cdot, \backslash).$$

3$a$). For $u = a$ in 3) we have $((xy)/a)v = ((xv)/a)y$, $R_a^{-1}(xy)\cdot v = R_a^{-1}(xv)\cdot y$, and the identity

$$R_a^{-1}(y * x) \cdot v = y * R_a^{-1}(xv) \text{ of the type } (*, \cdot, *, \cdot).$$

Let $x = a$ in 3): $((ay)/u)v = ((av)/u)y$, $(L_a y/u)v = (L_a v/u)y$, Hence, the following identity of the type $(/, \cdot, *, \otimes_1)$ holds

$$(y/u) \cdot v = L_a^{-1}y * (u \otimes_1 L_a v).$$

4$a$). Put in 4) $u = a$: $xa/(v\backslash y) = va/(x\backslash y)$, $R_a x/(v\backslash y) = R_a v/(x\backslash y)$ whence for $(\otimes_1) = (/)^*$ it follows the identity

$$(v\backslash y) \otimes_1 x = R_a v/(y \otimes_2 R_a^{-1}x) \text{ of the type } (\backslash, \otimes_1, /, \otimes_2).$$

If $y = a$, then $xu/(v\backslash a) = vu/(x\backslash a)$, $xu/L_a^{\otimes_2}v = vu/L_a^{\otimes_2}x$ whence we obtain the identity

$$xu/v = L_a^{\otimes_2}x \otimes_1 (u * (L_a^{\otimes_2})^{-1}v) \text{ of the type } (\cdot, /, \otimes_1, *).$$

5$a$). Let $y = a$ in 5): $x(a\backslash(uv)) = u(a\backslash(xv))$, $x \cdot L_a^{-1}(uv) = u \cdot L_a^{-1}(xv)$, and we have the identity

$$L_a^{-1}(uv) * x = u \cdot L_a^{-1}(v * x) \text{ of the type } (\cdot, *, \cdot, *).$$

For $v = a$ in 5) we get $x(y\backslash(ua)) = u(y\backslash(xa))$, $x(y\backslash R_a u) = u \cdot (y\backslash R_a x)$, $(y\backslash R_a u) * x = u(y\backslash R_a x)$, and the identity

$$(u \otimes_2 y) * x = R_a^{-1}u \cdot (y\backslash R_a x) \text{ of the type } (\otimes_2, *, \cdot, \backslash).$$

6$a$). We obtain the identities $((u/v)a)/y = ((u/y)a)/v$, $R_a(u/v)/y = R_a(u/y)/v$ if in 6) $x = a$. From the last identity it follows the identity

$$R_a(v \otimes_1 u)/y = v \otimes_1 R_a(u/y) \text{ of the type } (\otimes_1, /, \otimes_1, /).$$

If $u = a$, then $((a/v)x)/y = ((a/y)x)/v$, $(R_a^{\otimes_1}v \cdot x)/y = (R_a^{\otimes_1}y \cdot x)/v$, where $R_a^{\otimes_1}x = x \otimes_1 a$. Hence, we get the following identity:

$$(R_a^{\otimes_1}v \cdot x)/y = v \otimes_1 (x * R_a^{\otimes_1}y) \text{ of the type } (\cdot, /, \otimes_1, *).$$

7$a$). Put $y = a$ in 7): $(u/v)\backslash ax = (u/x)\backslash av$, $(u/v)\backslash L_a x = (u/x)\backslash L_a v$,

$$(v \otimes_1 u)\backslash x = L_a v \otimes_2 (u/L_a^{-1}x).$$

The type of this identity is $(\otimes_1, \backslash, \otimes_2, /)$.

If $u = a$, we get $(a/v)\backslash yx = (a/x)\backslash yv$, $(v \otimes_1 a)\backslash(yx) = (x \otimes_1 a)\backslash(yv)$, $yx \otimes_2 R_a^{\otimes_1} v = R_a^{\otimes_1} x\backslash yv$, and the identity

$$(x * y) \otimes_2 v = R_a^{\otimes_1} x\backslash(y \cdot (R_a^{\otimes_1})^{-1} v) \text{ of the type } (*, \otimes_2, \backslash, \cdot).$$

8$a$). Let $v = a$ in 8): $(y/(a\backslash u))\backslash x = (y/(a\backslash x))\backslash u$, $(y/L_a^{-1}u)\backslash x = (y/L_a^{-1}x)\backslash u$ whence we obtain the identity

$$(u \otimes_1 y)\backslash x = L_a u \otimes_2 (y/L_a^{-1}x) \text{ of the type } (\otimes_1, \backslash, \otimes_2, /).$$

Put $y = a$: $(a/(v\backslash u))\backslash x = (a/(v\backslash x))\backslash u$, $(a/(u \otimes_2 v)\backslash x = u \otimes_2 (a/(v\backslash x))$. Let $L_a^{(/)} x = a/x$, then from the last identity we have the identity

$$L_a^{(/)}(u \otimes_2 v)\backslash x = u \otimes_2 L_a^{(/)}(v\backslash x) \text{ of the type } (\otimes_2, \backslash, \otimes_2, \backslash).$$

9$a$). Putting $y = a$ in 9), we get $x/((u/v)\backslash a) = u/((x/v)\backslash a)$, $(a \otimes_2 (u/v)) \otimes_1 x = u/(a \otimes_2 (x/v))$, and the following identity:

$$L_a^{\otimes_2}(u/v) \otimes_1 x = u/L_a^{\otimes_2}(v \otimes_1 x) \text{ of the type } (/, \otimes_1, /, \otimes_1).$$

If $v = a$, $x/((u/a)\backslash y) = u/((x/a)\backslash y)$, $((u/a)\backslash y) \otimes_1 x = u/((y \otimes_2 (x/a))$. Let $R_a^{(/)} x = x/a$, then we have the identity

$$(R_a^{(/)} u\backslash y) \otimes_1 x = u/(y \otimes_2 R_a^{(/)} x) \text{ of the type } (\backslash, \otimes_1, /, \otimes_2).$$ □

## 7. Near-balanced identities for quasigroups

In [7], the identities with permutations of different types were considered. As a corollary, in a quasigroup $(Q, \cdot, \backslash, /)$ the following identities characterizing quasigroups isotopic to abelian groups were obtained (Theorem 1.2.1a, the identities (1.2.9) – (1.2.15) of [7]):

$$((x \cdot y)/u) \cdot (u\backslash z) = x \cdot u\backslash((z/u) \cdot y), \tag{7}$$

$$(((y/u) \cdot (u\backslash x))/u) \cdot (u\backslash z) = (x/u) \cdot (u\backslash((z/u) \cdot (u\backslash y))), \tag{8}$$

$$z \cdot (u\backslash((y/u) \cdot x)) = ((z \cdot x)/u) \cdot (u\backslash y), \tag{9}$$

$$((y \cdot x)/u) \cdot z = ((y \cdot z)/u) \cdot x, \tag{10}$$

$$z \cdot u\backslash((y/u) \cdot (u\backslash x)) = (x/u) \cdot u\backslash(z \cdot (u\backslash y)), \tag{11}$$

$$((x/u) \cdot y)/u \cdot (u\backslash z) = ((z/u) \cdot y)/u \cdot (u\backslash x), \tag{12}$$

$$((x/u) \cdot y)/u \cdot (u\backslash z) = ((z/u) \cdot (u\backslash x))/u \cdot y \tag{13}$$

Consider these identities more carefully. First note that the identities (7) and (9) coincide although they correspond to different types of identities with permutations (unfortunately, that was not noticed in [7]).

The identity (10) is the balanced identity 3) of the Glukhov list, the rest identities of four variables are unbalanced. Show that identities (11), (12) and (13) can be simplified and reduced to the known balanced identities.

**Proposition 1.** *The identity* (10) *is the identity* 3)*, the identity* (11) *is reduced to the identity* 5)*, the identities* (12) *and* (13) *are reduced to* 3)*.*

*Proof.* (10) is the balanced identity 3) of the Glukhov list. The identity (11) means that $z\cdot L_u^{-1}(R_u^{-1}y\cdot L_u^{-1}x)=R_u^{-1}x\cdot L_u^{-1}(z\cdot L_u^{-1}y)$ or $z\cdot L_u^{-1}(\alpha_u y\cdot x)=\alpha_u x\cdot L_u^{-1}(z\cdot y)$, where $\alpha_u = R_u^{-1}L_u$. From where for $z=u$ it follows that $\alpha_u y\cdot x=\alpha_u x\cdot y$. Therefore, $z\cdot L_u^{-1}(\alpha_u x\cdot y)=\alpha_u x\cdot L_u^{-1}(z\cdot y)$ or $z\cdot(u\backslash xy)=x\cdot(u\backslash zy)$. It is the balanced identity 5) of the Glukhov list.

(12) is $R_u^{-1}(R_u^{-1}x\cdot y)\cdot L_u^{-1}z=R_u^{-1}(R_u^{-1}z\cdot y)\cdot L_u^{-1}x$ or $R_u^{-1}(\alpha_u x\cdot y)\cdot z=R_u^{-1}(\alpha_u z\cdot y)\cdot x$, from where for $y=u$ we get $\alpha_u x\cdot z=\alpha_u z\cdot x$. Hence, $R_u^{-1}(yx)\cdot z=R_u^{-1}(y\cdot z)\cdot x$ or $(yx/u)\cdot z=(yz/u)\cdot x$. It is the identity 3).

(13) is $R_u^{-1}(R_u^{-1}x\cdot y)\cdot L_u^{-1}z=R_u^{-1}(R_u^{-1}z\cdot L_u^{-1}x)\cdot y$ or $R_u^{-1}(R_u^{-1}L_u x\cdot y)\cdot z=R_u^{-1}(R_u^{-1}L_u z\cdot x)\cdot y$.

If $y=u$, we have the equality $\alpha_u x\cdot z=\alpha_u z\cdot x$ and the identity 3) of Glukhov's list: $((xy)/u)\cdot z=((xz)/u)\cdot y$. □

**Theorem 4.** *The following near-balanced identities of length five in a quasigroup* $(Q,\cdot,\backslash,/)$*:*

$$(xy/u)\cdot(u\backslash z)=x\cdot(u\backslash((z/u)\cdot y)), \tag{14}$$

$$(u/(x\backslash y))\backslash(uz)=x\backslash(u\cdot((u/z)\backslash y)), \tag{15}$$

$$((x/y)\cdot u)/(z\backslash u)=x/((zu/y)\backslash u), \tag{16}$$

$$zu/((y/x)\backslash u)=((y/(z\backslash u))\cdot u)/x, \tag{17}$$

$$(u/z)\backslash(u\cdot(y\backslash x))=(u/(y\backslash uz))\backslash x, \tag{18}$$

$$(z/u)\cdot(u\backslash yx)=((y\cdot(u\backslash z))/u)\cdot x \tag{19}$$

*form a parastrophical-equivalent class of identities characterizing quasigroups isotopic to abelian groups.*

*Proof.* The identity (14) is (7) and, by Theorem 1.2.1a [7], characterizes quasigroups isotopic to abelian groups. At first we shall give the short proof from [7] of this fact.

Let a quasigroup $(Q,\cdot)$ be isotopic to an abelian group. Then, by Albert's theorem (see [4]), we conclude that the loop $(Q,+)$: $x+y=R_a^{-1}x\cdot L_a^{-1}y$, which is principally isotopic to this quasigroup, is an abelian group for any element $a\in Q$.

Hence, the identity $(x+y)+z = x+(z+y)$ is fulfilled. Pass in this identity to the operation $(\cdot)$:

$$R_a^{-1}(R_a^{-1}x \cdot L_a^{-1}y) \cdot L_a^{-1}z = R_a^{-1}x \cdot L_a^{-1}(R_a^{-1}z \cdot L_a^{-1}y). \tag{20}$$

From this identity with permutations after the respective change of variables we have the following identity:

$$R_a^{-1}(x \cdot y) \cdot L_a^{-1}z = x \cdot L_a^{-1}(R_a^{-1}z \cdot y). \tag{21}$$

This identity is true for any element $a \in Q$. Thus we have the identity (7).

Conversely, if the identity (7) holds, then the identity (21) and the identity $R_a^{-1}(x \cdot y) \cdot z = x \cdot L_a^{-1}(y * R_a^{-1}L_a z)$ hold. But the last identity is an identity of the form (5) and, by Theorem 1.1.1 [7], the quasigroup $(Q, \cdot)$ is isotopic to a group. After the inverse transformation of the last identity we obtain the identity (20), which means that in the group $(Q,+)$: $x+y = R_a^{-1}x \cdot L_a^{-1}y$ the identity $(x+y)+z = x+(z+y)$ holds. From where it follows that $(Q,+)$ is an abelian group (put $x=0$, where 0 is the unit of the group).

The rest identities of the theorem we shall obtain from the identity (14) passing on in this identity from the operation $(\cdot)$ to the parastrophes $(\backslash)$, $(/)$, $(\otimes_1)$, $(\otimes_2)$ and $(*)$ respectively and using Tale 1 and the equalities (1).

For example, in (14) change the operation $(\cdot)$ for the parastrophe $(\backslash)$:

$$((x\backslash y) \otimes_1 u)\backslash(uz) = x\backslash(u \cdot ((z \otimes_1 u)\backslash y)) \text{ or } (u/(x\backslash y))\backslash(uz) = x\backslash(u \cdot ((u/z)\backslash y)).$$

It is the identity (15).

The remaining identities are checked analogously.

Note that each of the pairs of identities (19) and (14), (15) and (17), (18) and (16) is mutually symmetric. □

**Proposition 2.** *The identity* (8) *is reduced to the identity* (19).

*Proof.* Writing the identity (8):

$$(((y/u) \cdot (u\backslash x))/u) \cdot (u\backslash z) = (x/u) \cdot (u\backslash((z/u)) \cdot (u\backslash y))$$

with the help of translations, we obtain the identities

$$R_u^{-1}(R_u^{-1}y \cdot L_u^{-1}x) \cdot L_u^{-1}z = R_u^{-1}x \cdot L_u^{-1}(R_u^{-1}z \cdot L_u^{-1}y),$$
$$R_u^{-1}(R_u^{-1}L_u y \cdot x) \cdot z = R_u^{-1}L_u x \cdot L_u^{-1}(R_u^{-1}L_u z \cdot y).$$

Let $R_u^{-1}L_u = \alpha_u$, $z = u$, then $\alpha_u y \cdot x = \alpha_u x \cdot y$ and $R_u^{-1}(y \cdot L_u^{-1}x) \cdot z = R_u^{-1}x \cdot L_u^{-1}(y \cdot z)$ or $((y \cdot (u\backslash x))/u) \cdot z = (x/u) \cdot (u\backslash yz)$. But it is the identity (19) if to interchange the positions of the variables $x$ and $z$. □

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