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H_vMV-algebras, I

Mahmood Bakhshi

Abstract. The aim of this paper is to introduce the concept of H_vMV-algebras as a common generalization of MV-algebras and hyper MV-algebras. After giving some basic properties and related results, the concepts of H_vMV-subalgebras, H_vMV-ideals and weak H_vMV-ideals are introduced and some of their properties and the connections between them are obtained.

1. Introduction

In 1958, Chang [1], introduced the concept of an MV-algebra as an algebraic proof of completeness theorem for $\aleph_0$-valued Łukasiewicz propositional calculus, see also [2]. Many mathematicians have worked on MV-algebras and obtained significant results. Mundici [6] proved that MV-algebras and abelian ℓ-groups with strong unit are categorically equivalent.

The hyperstructure theory (called also multialgebras) was introduced in 1934 by Marty [5]. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Recently, Ghorbani et al. [4] applied the hyperstructures to MV-algebras and introduced the concept of hyper MV-algebras. Now hyperstructures have many applications to several sectors of both pure and applied sciences such as: geometry, hypergraphs, binary relations, lattices, fuzzy set and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence and probabilities.

H_v-structures were introduced by Vougiouklis in [7] as a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). The reader will find in [8] some basic definitions and theorems about H_v-structures. A survey of some basic definitions, results and applications one can find in [3] and [8].

In this paper, in order to obtain a suitable generalization of MV-algebras and hyper MV-algebras which may be equivalent (categorically) to a certain subclass of the class of H_v-groups, the concept of H_vMV-algebra is introduced and some related results are obtained. In particular, weak H_vMV-ideals generated by a subset are characterized.

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2. Preliminaries

In this section we present some basic definitions and results.

Definition 2.1. An MV-*algebra* is an algebra $(M; +, ^*, 0)$ of type (2,1,0) satisfying the following axioms:
(MV1) $+$ is associative,
(MV2) $+$ is commutative,
(MV3) $x + 0 = x$,
(MV4) $(x^*)^* = x$,
(MV5) $x + 0^* = 0^*$,
(MV6) $(x^* + y)^* + y = (y^* + x)^* + x$.

On any MV-algebra M we can defina a partial ordering $\leqslant$ by putting $x \leqslant y$ if and only if $x^* + y = 0^*$.

Definition 2.2. A *hyper* MV- *algebra* is a nonempty set H endowed with a binary hyperoperation '$\oplus$', a unary operation '*' and a constant '0' satisfying the following conditions: $\forall x, y, z \in M$,
(HMV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(HMV2) $x \oplus y = y \oplus x$,
(HMV3) $(x^*)^* = x$,
(HMV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(HMV5) $0^* \in x \oplus 0^*$,
(HMV6) $0^* \in x \oplus x^*$,
(HMV7) $x \ll y$ and $y \ll x$ imply $x = y$, where $x \ll y$ is defined as $0^* \in x^* \oplus y$.

For $A, B \subseteq H$, $A \ll B$ is defined as $a \ll b$ for some $a \in A$ and $b \in B$.

Proposition 2.3. *In any hyper* MV-*algebra H for all $x, y \in H$ we have*
1. $0 \ll x \ll 0^*$,
2. $x \ll x$,
3. $x \ll y$ *implies that* $y^* \ll x^*$,
4. $x \ll x \oplus y$,
5. $0 \oplus 0 = \{0\}$,
6. $x \in x \oplus 0$.

Definition 2.4. A nonempty subset I of hyper MV-algebra H is called a

- *hyper* MV-*ideal* if
  (I_0) $x \ll y$ and $y \in I$ imply $x \in I$,
  (I_1) $x \oplus y \subseteq I$ for all $x, y \in I$,
- *weak hyper* MV-*ideal* if (I_0) holds and
  (I_2) $x \oplus y \ll I$ for all $x, y \in I$.

Obviously, every hyper MV-ideal is a weak hyper MV-ideal.

3. H_vMV-algebras

Definition 3.1. An $\mathsf{H}_v\mathsf{MV}$-*algebra* is a nonempty set H endowed with a binary hyperoperation '$\oplus$', a unary operation '*' and a constant '0' satisfying the following conditions:

($\mathsf{H}_v\mathsf{MV}$1) $x \oplus (y \oplus z) \cap (x \oplus y) \oplus z \neq \emptyset$, (weak associativity)
($\mathsf{H}_v\mathsf{MV}$2) $x \oplus y \cap y \oplus x \neq \emptyset$, (weak commutativity)
($\mathsf{H}_v\mathsf{MV}$3) $(x^*)^* = x$,
($\mathsf{H}_v\mathsf{MV}$4) $(x^* \oplus y)^* \oplus y \cap (y^* \oplus x)^* \oplus x \neq \emptyset$,
($\mathsf{H}_v\mathsf{MV}$5) $0^* \in x \oplus 0^* \cap 0^* \oplus x$,
($\mathsf{H}_v\mathsf{MV}$6) $0^* \in x \oplus x^* \cap x^* \oplus x$,
($\mathsf{H}_v\mathsf{MV}$7) $x \in x \oplus 0 \cap 0 \oplus x$,
($\mathsf{H}_v\mathsf{MV}$8) $0^* \in x^* \oplus y \cap y \oplus x^*$ and $0^* \in y^* \oplus x \cap x \oplus y^*$ imply $x = y$.

Remark 3.2. On any $\mathsf{H}_v\mathsf{MV}$-algebra H, we can define a binary relation '$\preceq$' by

$$x \preceq y \ \Leftrightarrow \ 0^* \in x^* \oplus y \cap y \oplus x^*.$$

Hence, the condition ($\mathsf{H}_v\mathsf{MV}$8) can be redefined as follows:

$$x \preceq y \text{ and } y \preceq x \text{ imply } x = y.$$

Let A and B be nonempty subsets of H. By $A \preceq B$ we mean that there exist $a \in A$ and $b \in B$ such that $a \preceq b$. For $A \subseteq H$, we denote the set $\{a^* : a \in A\}$ by A^*, and 0^* by 1.

Obviously, every hyper MV-algebra is an $\mathsf{H}_v\mathsf{MV}$-algebra but the converse is not true. We say $\mathsf{H}_v\mathsf{MV}$-algebra H is *proper* if it is not a hyper MV-algebra.

Example 3.3. Let $H = \{0, a, 1\}$ and the operations $\oplus$ and * be defined as follows:

$\oplus$	0	a	1
0	{0}	{a}	{0,a,1}
a	{0,a}	{1}	{0,1}
1	{0,1}	{0,a,1}	{0,a,1}
$*$	1	a	0

Then $(H; \oplus, ^*, 0)$ is a proper $\mathsf{H}_v\mathsf{MV}$-algebra. □

Example 3.4. Similarly, $H = \{0, a, b, 1\}$ with the operations $\oplus$ and * defined by

$\oplus$	0	a	b	1
0	{0,a}	{0,a,b}	{0,a,b}	{0,a,b,1}
a	{0,a,b,1}	{0,b}	{0,1}	{a,b,1}
b	{a,b}	{0,a,b,1}	{0}	{0,a,b,1}
1	{0,a,1}	{0,a,b,1}	{1}	{0,a,b,1}
$*$	1	b	a	0

is a proper $\mathsf{H}_v\mathsf{MV}$-algebra. □

Proposition 3.5. *In any* $\mathsf{H}_v\mathsf{MV}$*-algebra* H *for* $x, y \in H$ *and* $A, B \subseteq H$ *the following hold:*

1. $x \preceq x$, $A \preceq A$,
2. $0 \preceq x \preceq 1$, $0 \preceq A \preceq 1$,
3. $x \preceq y$ *implies* $y^* \preceq x^*$,
4. $A \preceq B$ *implies* $B^* \preceq A^*$,
5. $A \preceq B$ *implies that* $0^* \in (A^* \oplus B) \cap (B \oplus A^*)$,
6. $(x^*)^* = x$ *and* $(A^*)^* = A$,
7. $0^* \in (A \oplus A^*) \cap (A^* \oplus A)$,
8. $A \cap B \neq \emptyset$ *implies that* $A \preceq B$,
9. $(A \cap B)^* = A^* \cap B^*$,
10. $(A \oplus B) \cap (B \oplus A) \neq \emptyset$,
11. $A \oplus (B \oplus C) \cap (A \oplus B) \oplus C \neq \emptyset$,
12. $(A^* \oplus B)^* \oplus B \cap (B^* \oplus A)^* \oplus A \neq \emptyset$. □

The following example shows that the relation $\preceq$ is not transitive.

Example 3.6. In the $\mathsf{H}_v\mathsf{MV}$-algebra $(H; \oplus, ^*, 0)$, where $H = \{0, a, b, c, 1\}$ and the operations are defined by

$\oplus$	0	a	b	c	1
0	{0}	{0,a}	{0,b}	{0,c}	{0,a,b,c,1}
a	{0,a}	{0,a}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}
b	{0,b}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c}	{0,a,b,c,1}
c	{0,c}	{0,a,b,c,1}	{0,a,b,c}	{0,a,b,c,1}	{0,a,b,c,1}
1	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}
$*$	1	b	a	c	0

we have $a \preceq b$ and $b \preceq c$ while $a \not\preceq c$, because $0^* \notin \{0, a, b, c\} = a^* \oplus c$. □

Now let $x \odot y = (x^* \oplus y^*)^*$.

Theorem 3.7. *In any* $\mathsf{H}_v\mathsf{MV}$*-algebra* H *for all* $x, y, z \in H$ *and all nonempty subsets* A *and* B *of* H *we have:*

(1) $x \odot (y \odot z) \cap (x \odot y) \odot z \neq \emptyset$,
(2) $x \odot y \cap y \odot x \neq \emptyset$,
(3) $0 \in x \odot 0 \cap 0 \odot x$,
(4) $0 \in x \odot x^* \cap x^* \odot x$,
(5) $x \in x \odot 1 \cap 1 \odot x$,
(6) $1 \in x \odot y^* \cap y^* \odot x$ *and* $1 \in y \odot x^* \cap x^* \odot y$ *imply* $x = y$,
(7) $(A \oplus B)^* = A^* \odot B^*$,
(8) $(A \odot B)^* = A^* \oplus B^*$,
(9) $x \in x \oplus x$ *if and only if* $x^* \in x^* \odot x^*$,
(10) $x \in x \odot x$ *if and only if* $x^* \in x^* \oplus x^*$.

Proof. It is enough to observe that for $x, y, z \in H$,

$$\begin{aligned} x \odot (y \odot z) &= \bigcup\{x \odot t : t \in (y^* \oplus z^*)^*\} \\ &= \bigcup\{(x^* \oplus t^*)^* : t \in (y^* \oplus z^*)^*\} \\ &= \bigcup\{(x^* \oplus t^*)^* : t^* \in y^* \oplus z^*\} \\ &= \bigcup\{a^* : a \in x^* \oplus t^* : t^* \in y^* \oplus z^*\} \\ &= \bigcup\{a^* : a \in x^* \oplus (y^* \oplus z^*)\} \end{aligned}$$

and similarly

$$(x \odot y) \odot z = \bigcup\{a^* : a \in (x^* \oplus y^*) \oplus z^*\}.$$

This proves (1).

The proofs of $(2) - (6)$ follow from ($\mathsf{H}_v\mathsf{MV2}$) and ($\mathsf{H}_v\mathsf{MV5}$)-($\mathsf{H}_v\mathsf{MV7}$). The proofs of $(7) - (10)$ follow from the definition. □

On H we also define two binary hyperoperations '$\vee$' and '$\wedge$' as

$$x \vee y = (x \odot y^*) \oplus y, \qquad x \wedge y = (x \oplus y^*) \odot y = (x^* \vee y^*)^*.$$

Theorem 3.8. *In any $\mathsf{H}_v\mathsf{MV}$-algebra H, the following hold:*

(1) $(x \wedge y)^* = x^* \vee y^*$, $(x \vee y)^* = x^* \wedge y^*$,
(2) $(x \vee y) \cap (y \vee x) \neq \emptyset$, $(x \wedge y) \cap (y \wedge x) \neq \emptyset$,
(3) $x \in (x \vee x) \cap (x \wedge x)$,
(4) $0 \in (x \wedge 0) \cap (0 \wedge x)$,
(5) $1 \in (x \vee 1) \cap (1 \vee x)$,
(6) $x \in (x \vee 0) \cap (0 \vee x)$,
(7) $x \in (x \wedge 1) \cap (1 \wedge x)$,
(8) $x \preceq y$ *implies* $y \in x \vee y$ *and* $x \in x \wedge y$,
(9) $x \in y \odot x$ *implies* $1 \in y \vee x^*$,
(10) $x \in y \oplus x$ *implies* $0 \in y \wedge x^*$,
(11) *If* $x \in x \oplus x$, *then* $0 \in x \wedge x^*$,
(12) *If* $x \in x \odot x$, *then* $1 \in x \vee x^*$.

Proof. (1). Let $x, y \in H$. Then,

$$x^* \vee y^* = (x^* \odot y) \oplus y^* = (x \oplus y^*)^* \oplus y^* = ((x \oplus y^*) \odot y)^* = (x \wedge y)^*.$$

Similarly, the second equality is proved.

(2). It follows from ($\mathsf{H}_v\mathsf{MV4}$).

(3). From $0 \in x \odot x^*$ it follows that $x \in 0 \oplus x \subseteq (x \odot x^*) \oplus x = x \vee x$. From $0^* \in x \oplus x^*$ it follows that

$$x = (x^*)^* \in (0 \oplus x^*)^* \subseteq ((x \oplus x^*)^* \oplus x^*)^* = (x \oplus x^*) \odot x = x \wedge x.$$

(4). From $1 = 0^* \in x \oplus 0^*$ it follows that $0 \in 1 \odot 0 \subseteq (x \oplus 0^*) \odot 0 = x \wedge 0$. Similarly, from $x^* \in 0 \oplus x^*$ it follows that $0 \in x^* \odot x \subseteq (0 \oplus x^*) \odot x = 0 \wedge x$. Thus, $0 \in (x \wedge 0) \cap (0 \wedge x)$.

(9). If $x \in y \odot x$, then $1 = 0^* \in x \oplus x^* \subseteq (y \odot x) \oplus x^* = y \vee x^*$.

(10). If $x \in y \oplus x$, then $0 \in x \odot x^* \subseteq (y \oplus x) \odot x^* = y \wedge x^*$.

The proofs of the other cases are easy. □

Proposition 3.9. *Let $x \in H$. Then*

(1) $0 \in x \wedge x^*$ *if and only if* $x \oplus x \preceq x$ *if and only if* $x^* \preceq x^* \odot x^*$,

(2) $1 \in x \vee x^*$ *if and only if* $x^* \oplus x^* \preceq x^*$ *if and only if* $x \preceq x \odot x$. □

4. Homomorphisms, subalgebras and $\mathsf{H}_v\mathsf{MV}$-ideals

In this section, homomorphisms, $\mathsf{H}_v\mathsf{MV}$-subalgebras, weak $\mathsf{H}_v\mathsf{MV}$-ideals and $\mathsf{H}_v\mathsf{MV}$-ideals are introduced and some their properties are obtained.

Definition 4.1. Let $(H; \oplus, ^*, 0_H)$ and $(K; \otimes, ^\star, 0_K)$ be $\mathsf{H}_v\mathsf{MV}$-algebras and let $f : H \longrightarrow K$ be a function satisfying the following conditions:

(1) $f(0_H) = 0_K$,

(2) $f(x^*) = f(x)^\star$,

(3) $f(x^*) \preceq f(x)^\star$,

(4) $f(x \oplus y) = f(x) \otimes f(y)$,

(5) $f(x \oplus y) \subseteq f(x) \otimes f(y)$.

f is called a *homomorphism* if it satisfies (1), (2) and (4), and it is called a *weak homomorphism* if it satisfies (1), (3) and (5). Clearly, $f(1) = 1$ if f is a homomorphism. Note that (1) is not a consequence of (2) and (4).

Example 4.2. The set $H = \{0, a, 1\}$ with the operations defined by the table

$\oplus$	0	a	1
0	{0}	{0,a}	{0,1}
a	{0,a}	{0,a,1}	{a,1}
1	{0,1}	{a,1}	{1}
$*$	1	a	0

is an $\mathsf{H}_v\mathsf{MV}$-algebra. The function $f : H \longrightarrow H$ such that $f(0) = 1$, $f(1) = 0$ and $f(a) = a$ satisfies (2) and (4) but not (1). □

Further, for simplicity, we will use the same symbols for operations in H and K.

Theorem 4.3. *Let $f : H \longrightarrow K$ be a homomorphism.*

(1) *f is one-to-one if and only if $kerf = \{0\}$.*

(2) *f is an isomorphism if and only if there exists a homomorphism f^{-1} from K onto H such that $ff^{-1} = 1_K$ and $f^{-1}f = 1_H$.*

Proof. We prove only (1). Assume that f is one-to-one and $x \in kerf$. Then, $f(x) = 0 = f(0)$ whence $x = 0$, i.e., $kerf = \{0\}$. Conversely, assume that $kerf = \{0\}$ and $f(x) = f(y)$, for $x, y \in H$. Then,

$$0^* \in f(x)^* \oplus f(y) \cap f(y) \oplus f(x)^* = f(x^* \oplus y) \cap f(y \oplus x^*)$$

whence $f(s) = 0^* = f(t)$, for some $t \in x^* \oplus y$ and $s \in y \oplus x^*$. Hence, $f(s^*) = f(t^*) = 0$, i.e., $s^*, t^* \in kerf = \{0\}$ and so $0^* = s \in y \oplus x^*$ and $0^* = t \in x^* \oplus y$ whence $x \preceq y$. Similarly, we can show that $y \preceq x$. Thus, $x = y$, i.e., f is one-to-one. □

Proposition 4.4. *A nonempty subset S of H is an $\mathsf{H}_v\mathsf{MV}$-subalgebra of H if and only if $0 \in S$ and $x^* \oplus y \subseteq S$ for all $x, y \in S$.* □

Definition 4.5. A nonempty subset I of H such that $x \preceq y$ and $y \in I$ imply $x \in I$ is called

an $\mathsf{H}_v\mathsf{MV}$-*ideal* if $x \oplus y \subseteq I$, for all $x, y \in I$, and

a *weak* $\mathsf{H}_v\mathsf{MV}$-*ideal* if $x \oplus y \preceq I$, for all $x, y \in I$.

From Proposition 3.5 (8) it follows that every $\mathsf{H}_v\mathsf{MV}$-ideal is a weak $\mathsf{H}_v\mathsf{MV}$-ideal.

Theorem 4.6. *A nonempty subset I of H is a weak $\mathsf{H}_v\mathsf{MV}$-ideal if and only if $x \preceq y$ and $y \in I$ imply $x \in I$ and for all $x, y \in I$ we have $(x \oplus y) \cap I \neq \emptyset$.* □

Theorem 4.7. *If I is an $\mathsf{H}_v\mathsf{MV}$-ideal of an $\mathsf{H}_v\mathsf{MV}$-algebra H in which $x \preceq x \vee y$ holds for all $x, y \in H$, then $0 \in I$, and $a \odot b^* \subseteq I$ together with $b \in I$ imply $a \in I$.*

Proof. If I is an $\mathsf{H}_v\mathsf{MV}$-idealm then obviously, $0 \in I$. Now, let $a \odot b^* \subseteq I$ and $b \in I$. Then, $a \preceq a \vee b = (a \odot b^*) \oplus b \subseteq I$, whence $a \in I$. □

Definition 4.8. A nonempty subset A of H is called $S_\odot$-*reflexive* if $x \odot y \cap A \neq \emptyset$ implies that $x \odot y \subseteq A$. Similarly, A is called $S_\oplus$-reflexive if $x \oplus y \cap A \neq \emptyset$ implies that $x \oplus y \subseteq A$.

Theorem 4.9. *If in an $\mathsf{H}_v\mathsf{MV}$-algebra H for all $x, y \in H$ we have $x \wedge y \preceq x \preceq x \vee y$, then each its $S_\odot$-reflexive and $S_\oplus$-reflexive subset is an $\mathsf{H}_v\mathsf{MV}$-ideal of H.*

Proof. Let $x, y \in H$ be such that $x \preceq y$ and $y \in I$. Then, $0^* \in x^* \oplus y$ and so $0 \in x \odot y^*$, whence $(x \odot y^*) \cap I \neq \emptyset$. Since, I is $S_\odot$-reflexive, $x \odot y^* \subseteq I$ and so $x \in I$. Thus, $x \preceq y$ and $y \in I$ imply $x \in I$. Now, let $x, y \in I$. Then, $(x \oplus y) \odot y^* = x \wedge y^* \preceq x$ and hence, $c \preceq x \in I$, where $c \in x \wedge y^*$. This implies that $c \in I$ and so $(x \oplus y) \odot y^* \cap I \neq \emptyset$. Hence, there exists $a \in x \oplus y$ such that $a \odot y^* \cap I \neq \emptyset$ combining $y \in I$ we get $a \in I$, i.e., $x \oplus y \cap I \neq \emptyset$, whence $x \oplus y \subseteq I$. Thus, I is an $\mathsf{H}_v\mathsf{MV}$-ideal of H. □

Corollary 4.10. *In a hyper* MV*-algebra, every* $S_\odot$*-reflexive and* $S_\oplus$*-reflexive subset* I *that* $x \preceq y$ *and* $y \in I$ *imply* $x \in I$ *is a hyper* MV*-ideal.* □

Theorem 4.11. *Let* $f : H \longrightarrow K$ *be a homomorphism. Then*

(1) $kerf$ *is a weak* $\mathsf{H}_v\mathsf{MV}$*-ideal of* H*.*

(2) *If* I *is an* $\mathsf{H}_v\mathsf{MV}$*-ideal of* K*,* $f^{-1}(I)$ *is an* $\mathsf{H}_v\mathsf{MV}$*-ideal of* H*.*

(3) *Assume that* $x \preceq x \vee y$ *holds for all* $x, y \in H$*. If* f *is onto and* I *is an* $S_\odot$*–reflexive* $\mathsf{H}_v\mathsf{MV}$*-ideal of* H *containing* $ker\ f$*, then* $f(I)$ *is an* $\mathsf{H}_v\mathsf{MV}$*-ideal of* K*.*

Proof. (1). Let $x, y \in H$ be such that $x \preceq y$ and $y \in kerf$. Then, $0^* \in (x^* \oplus y) \cap (y \oplus x^*)$ and $f(y) = 0$. Thus

$$0^* = f(0^*) \in f(x^* \oplus y) \cap f(y \oplus x^*) = f(x)^* \oplus 0 \cap 0 \oplus f(x)^*,$$

which implies that $f(x) \preceq 0$. Hence, $f(x) = 0$, i.e., $x \in kerf$.

Now, let $x, y \in kerf$. Then, $0 \in 0 \oplus 0 = f(x) \oplus f(y) = f(x \oplus y)$ and so $f(t) = 0$, for some $t \in x \oplus y$. This implies that $(x \oplus y) \cap kerf \neq \emptyset$ and so by Theorem 4.6, $kerf$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.

(2) It is easy.

(3) Assume that f is onto and I is an $\mathsf{H}_v\mathsf{MV}$-ideal of H. Let $x \preceq y$ and $y \in f(I)$. Then, $0^* \in x^* \oplus y \cap y \oplus x^*$ and $y = f(b)$, for some $b \in I$. Since, f is onto, there exists $a \in H$ such that $f(a) = x$. Hence,

$$0^* \in f(a^*) \oplus f(b) \cap f(b) \oplus f(a^*) = f(a^* \oplus b) \cap f(b \oplus a^*),$$

whence $f(u) = 0^* = f(v)$, for some $u \in a^* \oplus b$ and $v \in b \oplus a^*$. This implies that $u^*, v^* \in kerf \subseteq I$, i.e., $a \odot b^* \cap I \neq \emptyset$, whence $a \odot b^* \subseteq I$. Since, $b \in I$, so $a \in I$ and hence, $x = f(a) \in f(I)$.

Let now $x, y \in f(I)$. Then, there exist $a, b \in I$ such that $f(a) = x$ and $f(b) = y$. From $a \oplus b \subseteq I$ it follows that $x \oplus y \subseteq f(I)$, proving $f(I)$ is an $\mathsf{H}_v\mathsf{MV}$-ideal of K. □

Definition 4.12. Let A be a nonempty subset of H. The smallest (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H containing A is called the (weak)$\mathsf{H}_v\mathsf{MV}$-*ideal generated* by A and is denoted by $\langle A \rangle$ (by $\langle A \rangle_w$ respectively).

It is clear that

$$\langle A\rangle \supseteq \{x \in H : x \preceq (\cdots((a_1 \oplus a_2) \oplus \cdots) \oplus a_n, \text{ for some } n \in \mathbb{N}, a_1, \ldots, a_n \in A\}.$$

Theorem 4.13. *Assume that $|x \oplus y| < \infty$, for all $x, y \in H$, $\preceq$ is transitive and monotone, and $x \oplus y \in R(H) = \{a \in H : |z \oplus a| = 1\ \forall z \in H\}$ for all $x, y \in R(H)$. Then*

$$\langle A\rangle_w = \{x \in H : x \preceq (\cdots((a_1 \oplus a_2) \oplus \cdots) \oplus a_n, \text{ for some } n \in \mathbb{N}, a_1, \ldots, a_n \in A\}$$

for any nonempty subset A of H contained in $R(H)$.

Proof. Assume that

$$B = \{x \in H : x \preceq (\cdots((a_1 \oplus a_2) \oplus \cdots) \oplus a_n, \text{ for some } n \in \mathbb{N}, a_1, \ldots, a_n \in A\}.$$

Obviously, $A \subseteq B$. Now, let $x, y \in H$ be such that $x \preceq y$ and $y \in B$. Since, $|x \oplus y| < \infty$, so $y \preceq (\cdots((a_1 \oplus a_2) \oplus \cdots) \oplus a_n$ for some $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A$. This implies that $0^* \in y^* \oplus ((\cdots((a_1 \oplus a_2) \oplus \cdots) \oplus a_n)$. On the other hand, $x \preceq y$ implies that $y^* \preceq x^*$, whence

$$0^* \in \{0^*\} = y^* \oplus (\cdots((a_1 \oplus a_2) \oplus \cdots) \oplus a_n) \preceq x^* \oplus (\cdots((a_1 \oplus a_2) \oplus \cdots) \oplus a_n),$$

which gives $0^* \in x^* \oplus (\cdots((a_1 \oplus a_2) \oplus \cdots) \oplus a_n)$, i.e., $x \preceq (\cdots((a_1 \oplus a_2) \oplus \cdots) \oplus a_n)$. Thus, $x \in B$.

Now, let $x, y \in B$. Then,

$$x \preceq (\cdots(a_1 \oplus a_2) \oplus \cdots) \oplus a_n \quad \text{and} \quad y \preceq (\cdots(b_1 \oplus b_2) \oplus \cdots) \oplus b_m$$

for some $n, m \in \mathbb{N}$, $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$. Since, $\preceq$ is monotone,

$$\begin{aligned} x \oplus y &\preceq x \oplus ((\cdots(b_1 \oplus b_2) \oplus \cdots) \oplus b_m) \\ &\preceq ((\cdots(a_1 \oplus a_2) \oplus \cdots) \oplus a_n) \oplus ((\cdots(b_1 \oplus b_2) \oplus \cdots) \oplus b_m) \end{aligned}$$

and hence there exists $u \in x \oplus y$ such that

$$\begin{aligned} u &\preceq x \oplus ((\cdots(a_1 \oplus a_2) \oplus \cdots) \oplus a_n) \\ &\preceq ((\cdots(a_1 \oplus a_2) \oplus \cdots) \oplus a_n) \oplus ((\cdots(b_1 \oplus b_2) \oplus \cdots) \oplus b_m) \\ &= (\cdots(((\cdots(a_1 \oplus a_2) \oplus \cdots) \oplus a_n) \oplus b_1) \oplus \cdots) \oplus b_m \end{aligned}$$

because $\preceq$ is transitive. The equality holds for $A \cap B \neq \emptyset$, and $|A| = 1 = |B|$ imply $A = B$. Thus $u \in B$ and so $x \oplus y \preceq B$. Therefore, B is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. Obviously, B is the least weak $\mathsf{H}_v\mathsf{MV}$-ideal of H containing A. □

Let $\mathsf{H}_v\mathsf{MVI}$ ($\mathsf{WH}_v\mathsf{MVI}$) denotes the set of all $\mathsf{H}_v\mathsf{MV}$-ideals (weak $\mathsf{H}_v\mathsf{MV}$-ideals) of H. Then, $\mathsf{H}_v\mathsf{MVI}$ ($\mathsf{WH}_v\mathsf{MVI}$) together with the set inclusion, as a partial ordering, is a poset in which for all $A_i \subseteq \mathsf{H}_v\mathsf{MVI}$, $\bigwedge A_i = \bigcap A_i$ and $\bigvee A_i = \langle A_i\rangle$. So, we have

Theorem 4.14. ($\mathsf{H}_v\mathsf{MVI}$,$\subseteq$) *is a complete lattice, and if* $\mathsf{WH}_v\mathsf{MVI}$ *is closed with respect to the intersection,* $\mathsf{H}_v\mathsf{MVI}$ *is a complete sublattice of the complete lattice* ($\mathsf{WH}_v\mathsf{MVI}$,$\subseteq$). □

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Department of Mathematics, Univesity of Bojnord, Bojnord, Iran
E-mail: bakhshi@ub.ac.ir