

Free products of dimonoids

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Abstract. We construct a free product of dimonoids which generalizes a free dimonoid presented by J.-L. Loday and describe its structure.

1. Introduction

A *dimonoid* is a nonempty set D equipped with two binary associative operations \circ and $*$ satisfying the axioms $(x \circ y) \circ z = x \circ (y * z)$, $(x * y) \circ z = x * (y \circ z)$, $(x \circ y) * z = x * (y * z)$, while a *dialgebra* is just a linear analog of a dimonoid. These notions were introduced by J.-L. Loday [1] for solving of problems in Leibniz algebras and investigated by many authors (see, e.g., [2]).

The construction of a free dimonoid generated by a given set was presented by J.-L. Loday [1] and applied to the study of free dialgebras and a cohomology of dialgebras. Structural properties of free dimonoids have been investigated in [3].

In this paper we present a construction of a free product of arbitrary dimonoids which generalizes a free dimonoid and describe its structure. The obtained results extend the corresponding results from [1] and [3].

2. The main result

As usual, we denote the set of all positive integers by \mathbb{N} .

Let $Fr[S_i]_{i \in I}$ be the free product of arbitrary semigroups S_i , $i \in I$. For every $w \in Fr[S_i]_{i \in I}$ denote the first (respectively, last) letter of w by $w^{(0)}$ (respectively, $w^{(1)}$) and the length of w by l_w . Consider the set

$$G(S_i)_{i \in I} = \{(w, m) \in Fr[S_i]_{i \in I} \times \mathbb{N} \mid l_w \geq m\}.$$

For all $(w, m) \in G(S_i)_{i \in I}$ and $u \in Fr[S_i]_{i \in I}$ assume

$$f_{(w,m)}^u = \begin{cases} l_u + m, & l_{u^{(1)}w^{(0)}} = 2, \\ l_u + m - 1, & l_{u^{(1)}w^{(0)}} = 1. \end{cases} \quad (1)$$

We need the following two lemmas.

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Lemma 2.1. Let $(u, s), (\omega, s) \in G(S_i)_{i \in I}$ and $w \in Fr[S_i]_{i \in I}$. If $u^{(0)}, \omega^{(0)} \in S_i$ for some $i \in I$, then $f_{(u,s)}^w = f_{(\omega,s)}^w$. \square

Lemma 2.2. For all $w_1, w_2, w_3 \in Fr[S_i]_{i \in I}$ and $(w_3, m_3) \in G(S_i)_{i \in I}$,

$$f_{(w_2 w_3, f_{(w_3, m_3)}^{w_2})}^{w_1} = f_{(w_3, m_3)}^{w_1 w_2}.$$

Proof. As $(w_2 w_3)^{(0)} \in S_i \Leftrightarrow w_2^{(0)} \in S_i$ for some $i \in I$ and $(w_1 w_2)^{(1)} \in S_j \Leftrightarrow w_2^{(1)} \in S_j$ for some $j \in I$, then

$$l_{w_1^{(1)}(w_2 w_3)^{(0)}} = l_{w_1^{(1)} w_2^{(0)}}, \quad (2)$$

$$l_{(w_1 w_2)^{(1)} w_3^{(0)}} = l_{w_2^{(1)} w_3^{(0)}}. \quad (3)$$

It is not difficult to see that

$$l_{w_1 w_2} = \begin{cases} l_{w_1} + l_{w_2}, & l_{w_1^{(1)} w_2^{(0)}} = 2, \\ l_{w_1} + l_{w_2} - 1, & l_{w_1^{(1)} w_2^{(0)}} = 1. \end{cases} \quad (4)$$

By (1) – (3) we have

$$f_{(w_3, m_3)}^{w_2} = \begin{cases} l_{w_2} + m_3, & l_{w_2^{(1)} w_3^{(0)}} = 2, \\ l_{w_2} + m_3 - 1, & l_{w_2^{(1)} w_3^{(0)}} = 1, \end{cases} \quad (5)$$

$$f_{(w_2 w_3, f_{(w_3, m_3)}^{w_2})}^{w_1} = \begin{cases} l_{w_1} + f_{(w_3, m_3)}^{w_2}, & l_{w_1^{(1)} w_2^{(0)}} = 2, \\ l_{w_1} + f_{(w_3, m_3)}^{w_2} - 1, & l_{w_1^{(1)} w_2^{(0)}} = 1, \end{cases} \quad (6)$$

$$f_{(w_3, m_3)}^{w_1 w_2} = \begin{cases} l_{w_1 w_2} + m_3, & l_{w_2^{(1)} w_3^{(0)}} = 2, \\ l_{w_1 w_2} + m_3 - 1, & l_{w_2^{(1)} w_3^{(0)}} = 1. \end{cases} \quad (7)$$

Further, using (4) – (7), consider the following four cases.

Case 1. $l_{w_1^{(1)} w_2^{(0)}} = l_{w_2^{(1)} w_3^{(0)}} = 2$. Then $f_{(w_3, m_3)}^{w_2} = l_{w_2} + m_3$ and

$$f_{(w_2 w_3, f_{(w_3, m_3)}^{w_2})}^{w_1} = l_{w_1} + f_{(w_3, m_3)}^{w_2} = l_{w_1} + l_{w_2} + m_3 = l_{w_1 w_2} + m_3 = f_{(w_3, m_3)}^{w_1 w_2}.$$

Case 2. $l_{w_1^{(1)} w_2^{(0)}} = 2$ and $l_{w_2^{(1)} w_3^{(0)}} = 1$. Then $f_{(w_3, m_3)}^{w_2} = l_{w_2} + m_3 - 1$ and

$$f_{(w_2 w_3, f_{(w_3, m_3)}^{w_2})}^{w_1} = l_{w_1} + f_{(w_3, m_3)}^{w_2} = l_{w_1} + l_{w_2} + m_3 - 1 = l_{w_1 w_2} + m_3 - 1 = f_{(w_3, m_3)}^{w_1 w_2}.$$

Case 3. $l_{w_1^{(1)} w_2^{(0)}} = 1$ and $l_{w_2^{(1)} w_3^{(0)}} = 2$. Then $f_{(w_3, m_3)}^{w_2} = l_{w_2} + m_3$ and

$$f_{(w_2 w_3, f_{(w_3, m_3)}^{w_2})}^{w_1} = l_{w_1} + f_{(w_3, m_3)}^{w_2} - 1 = l_{w_1} + l_{w_2} + m_3 - 1 = l_{w_1 w_2} + m_3 = f_{(w_3, m_3)}^{w_1 w_2}.$$

Case 4. $l_{w_1^{(1)} w_2^{(0)}} = l_{w_2^{(1)} w_3^{(0)}} = 1$. Then $f_{(w_3, m_3)}^{w_2} = l_{w_2} + m_3 - 1$ and

$$f_{(w_2 w_3, f_{(w_3, m_3)}^{w_2})}^{w_1} = l_{w_1} + f_{(w_3, m_3)}^{w_2} - 1 = l_{w_1} + l_{w_2} + m_3 - 2 = l_{w_1 w_2} + m_3 - 1 = f_{(w_3, m_3)}^{w_1 w_2}.$$

Thus, $f_{(w_2 w_3, f_{(w_3, m_3)}^{w_2})}^{w_1} = f_{(w_3, m_3)}^{w_1 w_2}$. \square

For a given relation ρ on a dimonoid $(D, \circ, *)$, the congruence generated by ρ is the least congruence on $(D, \circ, *)$ containing ρ . It will be denoted by ρ^* and can be characterized as the intersection of all congruences on $(D, \circ, *)$ containing ρ .

Let $\{(D_i, \circ_i, *_i)\}_{i \in I}$ be a family of arbitrary pairwise disjoint dimonoids. Operations on $Fr[(D_i, \circ_i)]_{i \in I}$ and $Fr[(D_i, *_i)]_{i \in I}$ will be denoted by \circ and $*$ respectively. For every $i \in I$ consider a relation $\theta_i = \{(a *_i b, a \circ_i b) \mid a, b \in D_i\}$ on a dimonoid $(D_i, \circ_i, *_i)$. It is clear that operations of $(D_i, \circ_i, *_i)/\theta_i^*$ coincide and it is a semigroup.

Let $\omega_1 = (x_1 x_2 \dots x_s, t)$, $\omega_2 = (y_1 y_2 \dots y_p, r) \in G((D_i, \circ_i))_{i \in I}$, where elements $x_1, \dots, x_s, y_1, \dots, y_p \in \bigcup_{i \in I} D_i$. Define a relation \sim on $G((D_i, \circ_i))_{i \in I}$ by putting

$$\omega_1 \sim \omega_2 \Leftrightarrow \begin{cases} s = p, t = r \text{ and } x_k \theta_{j_k}^* y_k \text{ for all } 1 \leq k \leq s \text{ and some } j_k \in I, \\ \text{at that } x_t = y_r. \end{cases}$$

It is not difficult to check that \sim is an equivalence relation. Denote by $[w, m]$ the equivalence class of \sim containing an element $(w, m) \in G((D_i, \circ_i))_{i \in I}$, and by $G^*((D_i, \circ_i))_{i \in I}$ the quotient set $G((D_i, \circ_i))_{i \in I} / \sim$.

Observe that (1) does not depend on the definition of operations on semigroups S_i , $i \in I$, and define operations \circ' and $*$ ' on $G^*((D_i, \circ_i))_{i \in I}$ by

$$[w_1, m_1] \circ' [w_2, m_2] = [w_1 \circ w_2, m_1],$$

$$[w_1, m_1] *' [w_2, m_2] = [w_1 * w_2, f_{(w_2, m_2)}^{w_1}]$$

for all $[w_1, m_1], [w_2, m_2] \in G^*((D_i, \circ_i))_{i \in I}$. The algebra $(G^*((D_i, \circ_i))_{i \in I}, \circ', *')$ will be denoted by $\check{G}(D_i)_{i \in I}$.

Theorem 2.3. $\check{G}(D_i)_{i \in I}$ is the free product of dimonoids $(D_i, \circ_i, *_i)$, $i \in I$.

Proof. First note that from the associativity of the operation of a free product semigroups it follows that

$$(w_1 \circ w_2) \circ w_3 = w_1 \circ (w_2 \circ w_3), \quad (8)$$

$$(w_1 * w_2) \circ w_3 = w_1 * (w_2 \circ w_3), \quad (9)$$

$$(w_1 * w_2) * w_3 = w_1 * (w_2 * w_3) \quad (10)$$

for all $w_1, w_2, w_3 \in Fr[(D_i, \circ_i)]_{i \in I}$.

The associativity of the operation \circ' follows from (8). Let further $[w_1, m_1], [w_2, m_2], [w_3, m_3] \in \check{G}(D_i)_{i \in I}$. Then

$$\begin{aligned} ([w_1, m_1] \circ' [w_2, m_2]) \circ' [w_3, m_3] &= [w_1 \circ w_2, m_1] \circ' [w_3, m_3] \\ &= [(w_1 \circ w_2) \circ w_3, m_1] = [w_1 \circ (w_2 \circ w_3), m_1] \\ &= [w_1 \circ (w_2 * w_3), m_1] = [w_1, m_1] \circ' [w_2 * w_3, f_{(w_3, m_3)}^{w_2}] \\ &= [w_1, m_1] \circ' ([w_2, m_2] *' [w_3, m_3]), \end{aligned}$$

by (8) and the condition $(w_1 \circ (w_2 \circ w_3), m_1) \sim (w_1 \circ (w_2 * w_3), m_1)$.

Moreover, using (9) and Lemma 2.1, we obtain

$$\begin{aligned}
([w_1, m_1] *' [w_2, m_2]) \circ' [w_3, m_3] &= [w_1 * w_2, f_{(w_2, m_2)}^{w_1}] \circ' [w_3, m_3] \\
&= [(w_1 * w_2) \circ w_3, f_{(w_2, m_2)}^{w_1}] \\
&= [w_1 * (w_2 \circ w_3), f_{(w_2 \circ w_3, m_2)}^{w_1}] \\
&= [w_1, m_1] *' [w_2 \circ w_3, m_2] \\
&= [w_1, m_1] *' ([w_2, m_2] \circ' [w_3, m_3]).
\end{aligned}$$

Further we get

$$\begin{aligned}
[w_1, m_1] *' ([w_2, m_2] *' [w_3, m_3]) &= [w_1, m_1] *' [w_2 * w_3, f_{(w_3, m_3)}^{w_2}] \\
&= [w_1 * (w_2 * w_3), f_{(w_2 * w_3, f_{(w_3, m_3)}^{w_2})}^{w_1}] \\
&= [(w_1 * w_2) * w_3, f_{(w_3, m_3)}^{w_1 \circ w_2}] = [(w_1 \circ w_2) * w_3, f_{(w_3, m_3)}^{w_1 \circ w_2}] \\
&= [w_1 \circ w_2, m_1] *' [w_3, m_3] \\
&= ([w_1, m_1] \circ' [w_2, m_2]) *' [w_3, m_3]
\end{aligned}$$

and

$$\begin{aligned}
([w_1, m_1] *' [w_2, m_2]) *' [w_3, m_3] &= [w_1 * w_2, f_{(w_2, m_2)}^{w_1}] *' [w_3, m_3] \\
&= [(w_1 * w_2) * w_3, f_{(w_3, m_3)}^{w_1 * w_2}],
\end{aligned}$$

according to (10), Lemma 2.2 and the fact that

$$((w_1 * w_2) * w_3, f_{(w_3, m_3)}^{w_1 \circ w_2}) \sim ((w_1 \circ w_2) * w_3, f_{(w_3, m_3)}^{w_1 \circ w_2}).$$

This shows that $\check{G}(D_i)_{i \in I}$ is a dimonoid. Moreover, for each $(D_i, \circ_i, *_i), i \in I$, we have

$$(D_i, \circ_i, *_i) \cong \tilde{D}_i = \{[w, 1] \in \check{G}(D_i)_{i \in I} \mid w \in D_i\}$$

and all subdimonoids $\tilde{D}_i, i \in I$, generate $\check{G}(D_i)_{i \in I}$.

In order to complete the proof we should check the condition of continuability of a homomorphism. For this let

$$\alpha_i : (D_i, \circ_i, *_i) \rightarrow (T, \circ'', *''),$$

where $i \in I$, be a homomorphism from $(D_i, \circ_i, *_i)$ to an arbitrary dimonoid $(T, \circ'', *'')$. Define a map

$$\alpha : \check{G}(D_i)_{i \in I} \rightarrow (T, \circ'', *'') : [x_1 \dots x_k \dots x_s, t] \mapsto [x_1 \dots x_k \dots x_s, t] \alpha,$$

assuming

$$[x_1 \dots x_k \dots x_s, t] \alpha = x_1 \gamma_1 *'' \dots *'' x_t \gamma_t \circ'' \dots \circ'' x_s \gamma_s,$$

where $\gamma_k = \alpha_k$ for $x_k \in D_k, 1 \leq k \leq s$.

A straightforward verification shows that $[w_1, m_1] \alpha = [w_2, m_2] \alpha$ for all $[w_1, m_1], [w_2, m_2] \in \check{G}(D_i)_{i \in I}$, if $(w_1, m_1) \in [w_2, m_2]$, and so, α is well-defined.

Using axioms of a dimonoid and homomorphisms $\alpha_i, i \in I$, one can show that α is a homomorphism continuing $\alpha_i, i \in I$. Thus, $\check{G}(D_i)_{i \in I}$ is the free product of dimonoids $(D_i, \circ_i, *_i), i \in I$. \square

From Theorem 2.3 we obtain

Corollary 2.4. *The free dimonoid is the free product $\check{G}(D_i)_{i \in I}$ of one-generated free dimonoids $(D_i, \circ_i, *_i), i \in I$.*

Proof. Observe that the free dimonoid $(D(X), \circ, *)$ of an arbitrary rank was constructed in ([1], p. 15) and the structure of one-generated free dimonoids was described in [3]. By Lemma 3 from [3] $\check{G}(D_i)_{i \in I} \cong (D(X), \circ, *)$. \square

3. The structure of $\check{G}(D_i)_{i \in I}$

Let $B(I)$ be the semilattice of all nonempty finite subsets of I with respect to the operation of the set theoretical union. For every $w = x_1 x_2 \dots x_l \dots x_k \in Fr[(D_i, \circ_i)_{i \in I}]$ assume $\tilde{c}(w) = \bigcup_{i=1}^k \{x_l j'\}$, where

$$j' : \bigcup_{i \in I} D_i \rightarrow I : a \mapsto i, \text{ if } a \in D_i, i \in I.$$

For every $Y \in B(I)$ and all $x, y \in Y$ let

$$H_Y = \{[w, m] \in \check{G}(D_i)_{i \in I} \mid \tilde{c}(w) = Y\},$$

$$H_Y^{(x,y)} = \{[w, m] \in H_Y \mid (w^{(0)} j', w^{(1)} j') = (x, y)\},$$

$Y \times Y$ be a rectangular band, that is, a semigroup with the operation $(x, y)(a, b) = (x, b)$. It is easy to see that H_Y is a subdimonoid of $\check{G}(D_i)_{i \in I}$ and $H_Y^{(x,y)}$ is a subdimonoid of H_Y .

In terms of dibands of subdimonoids (see, e.g., [4]) we obtain the following structure theorem.

Theorem 3.1. *The free product $\check{G}(D_i)_{i \in I}$ of dimonoids $(D_i, \circ_i, *_i), i \in I$, is a semilattice $B(I)$ of subdimonoids $H_Y, Y \in B(I)$. Every dimonoid $H_Y, Y \in B(I)$, is a rectangular band $Y \times Y$ of subdimonoids $H_Y^{(x,y)}, (x, y) \in Y \times Y$.*

Proof. Assuming

$$c' : [w, m] \mapsto \tilde{c}(w),$$

we obtain a homomorphism from $\check{G}(D_i)_{i \in I}$ to $B(I)$ as

$$\tilde{c}(w \star u) = \tilde{c}(w) \cup \tilde{c}(u)$$

for all $w, u \in Fr[(D_i, \circ_i)_{i \in I}]$ and $\star \in \{\circ, *\}$. Hence, $\check{G}(D_i)_{i \in I}$ is a semilattice $B(I)$ of subdimonoids $H_Y, Y \in B(I)$.

Now we shall prove the second part of the theorem.

Let

$$\pi : H_Y \rightarrow Y \times Y : [w, m] \mapsto (w^{(0)} j', w^{(1)} j').$$

As

$$(w \star u)^{(0)}j' = w^{(0)}j', \quad (w \star u)^{(1)}j' = u^{(1)}j'$$

for all $w, u \in Fr[(D_i, \circ_i)]_{i \in I}$ and $\star \in \{\circ, *\}$, then π is a homomorphism. From here, H_Y is a rectangular band $Y \times Y$ of subdimonoids $H_Y^{(x,y)}$, $(x, y) \in Y \times Y$. \square

We finish this section with the description of some congruence on $\check{G}(D_i)_{i \in I}$ when $\circ_i = *_i$ for all $i \in I$.

First observe that if $\circ_i = *_i$ for all $i \in I$, then \sim is the diagonal of $G(S_i)_{i \in I}$ and $G(S_i)_{i \in I} / \sim$ is identified with $G(S_i)_{i \in I}$. It is clear that in this case $\circ = *$.

Let α be an arbitrary fixed congruence on $Fr[S_i]_{i \in I}$. Define a relation $\tilde{\alpha}$ on $\check{G}(S_i)_{i \in I}$ by

$$[w_1, m_1] \tilde{\alpha} [w_2, m_2] \Leftrightarrow w_1 \alpha w_2$$

for all $[w_1, m_1], [w_2, m_2] \in \check{G}(S_i)_{i \in I}$.

It is not difficult to prove the following lemma.

Lemma 3.2. *The relation $\tilde{\alpha}$ is a congruence on the dimonoid $\check{G}(S_i)_{i \in I}$ and operations of the quotient dimonoid $\check{G}(S_i)_{i \in I} / \tilde{\alpha}$ coincide.*

From Lemma 3.2 we obtain

Corollary 3.3. *If α is the diagonal of $Fr[S_i]_{i \in I}$, then $\check{G}(S_i)_{i \in I} / \tilde{\alpha}$ is the free product of semigroups.*

Note that Theorem 3.1, Lemma 3.2 and Corollary 3.3 extend, respectively, Theorem 3, Lemma 5 and Corollary 1 from [3].

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