Quasigroups and Related Systems **21** (2013), 255 − 272

Automorphisms of abelian n-ary groups

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Abstract. We describe relations between automorphisms of an abelian n-ary group and automorphisms of their binary retracts.

1. Introduction and preliminary results

An algebra $\langle G, f\rangle$ with an n-ary operation f $(n \geqslant 2)$ is an *n-ary (polyadic) group*, if the operation f is associative, i.e.,

$$f(f(a_1,\dots,a_n),a_{n+1},\dots,a_{2n-1}) \\ = f(a_1,\dots,a_i,f(a_{i+1},\dots,a_{i+n}),a_{i+n+1},\dots,a_{2n-1})$$

for all $i = 1,\dots,n-1$, and the equation

$$f(a_1,\dots,a_{j-1},x_j,a_{j+1},\dots,a_n) = b$$

has a unique solution $x_j \in G$ for each $j = 1,\dots,n$ and $a_1,\dots,a_n, b \in G$.

Since for $n = 2$ we obtain a (binary) group, we will assume that $n > 2$.

n-Ary groups belong to a wide class of algebraic objects that are studied from various point of views. The importance of such groups was pointed out, for example, by A.G. Kurosh [14].

In an n-ary group $\langle G, f\rangle$ for each $a \in G$ the solution of the equation

$$f(a,\dots,a,x) = a$$

is denoted by $\bar{a}$ and is called the *skew element* for a. Since this element is uniquely determined, an n-ary group $\langle G, f\rangle$ can be considered (cf. [11]) as an algebra $\langle G, f, ^{-}\rangle$ with one associative n-ary operation f and one unary operation $^{-}\colon x \to \bar{x}$ such that the following identities:

$$f(y,\underbrace{x,\dots,x}_{n-2},\bar{x}) = f(y,\underbrace{x,\dots,x}_{n-3},\bar{x},x) = f(\bar{x},\underbrace{x,\dots,x}_{n-2},y) = f(x,\bar{x},\underbrace{x,\dots,x}_{n-3},y) = y$$

are satisfied.

2010 Mathematics Subject Classification: 20N15, 08N05
Keywords: n-ary group, abelian semicyclic n-ary group, retrect, automorphism, holomorph, automorphism group, congruence

Another weaker system of identities defining an n-ary group can be found in [3] and [5].

Note by the way that in some n-ary groups the map $^{-}\colon x \to \bar{x}$ is an endomorphism, i.e.,

$$\overline{f(x_1,\ldots,x_n)} = f(\bar{x}_1,\ldots,\bar{x}_n)$$

(cf. [7] and [9]). This situation take place, for example, in n-ary groups in which

$$f(x_1,\ldots,x_n) = f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

for all permutations $\sigma \in S_n$. Such n-ary groups are called *commutative* or *abelian*. Note that the term *abelian* is also used in another sense (cf. for example [9]).

With each n-ary group $\langle G, f\rangle$ there are associated binary groups $\langle G,+\rangle_c = ret_c\langle G, f\rangle$ defined by

$$a + b = f(a, \underbrace{c,\ldots,c}_{n-3}, \bar{c}, b)$$

where c is an arbitrary fixed element of G. The element c is a zero (neutral element) of the group $ret_c\langle G, f\rangle$. Moreover, all these groups (called *retracts* of $\langle G, f\rangle$) are isomorphic (cf. [6]). So, all retracts of an abelian n-ary group $\langle G, f\rangle$ will be identified with the group $\langle G,+\rangle$.

In the case of commutative n-ary groups we have

$$f(a_1,\ldots,a_n) = a_1 + \ldots + a_n + d, \tag{1}$$

where $\langle G,+\rangle = ret_c\langle G, f\rangle$ and $d = f(c,\ldots,c)$ (cf. [19] or [6]). In this case we say that an n-ary group $\langle G, f\rangle$ is *d-derived* from the group $\langle G,+\rangle$ and denote this fact by $\langle G, f\rangle = der_d\langle G,+\rangle$. If $d = 0$ (the neutral element of $\langle G,+\rangle$), then we say that an n-ary group $der_0\langle G,+\rangle$ is *derived* from the group $\langle G,+\rangle$.

The reverse is also true: if $\langle G,+\rangle$ is an arbitrary abelian group, then for every $d \in G$ the n-ary groupoid d-derived from the group $\langle G,+\rangle$ is an n-ary group. In this case, $\langle G,+\rangle = ret_0 der_d\langle G,+\rangle$, where 0 is a zero element of the group $\langle G,+\rangle$ (cf. [19]). Obviously $\langle G, f\rangle = der_d ret_c\langle G, f\rangle$, where $d = f(c,\ldots,c)$, for all commutative n-ary groups.

An n-ary group having a cyclic retract is called an *semicyclic* [8]. Each commutative semicyclic n-ary group is isomorphic to an n-ary group d-derived from some cyclic group (cf. [17]). So, commutative semicyclic n-ary groups will be called *abelian semicyclic n-ary groups*.

For other basic facts on n-ary groups see [4] and [8].

The composition of maps φ, ψ we define by the rule $(\varphi \circ \psi)(x) = \psi(\varphi(x))$. The cyclic group generated by a is denoted by (a); $\gcd(k, m)$ denotes the great common divisor of k and m.

2. Automorphisms of abelian n-ary groups

We start with the description of relations between automorphisms of abelian n-ary groups and automorphisms of their binary retracts.

Using ideas presented in the paper [6] we can prove the following two usaful propositions.

Proposition 2.1. *Let ψ be an automorphism of an abelian n-ary group $\langle G, f\rangle$ and $c \in G$. Then the map $\sigma : G \to G$ defined by $\sigma(x) = -\psi(c) + \psi(x)$ is an automorphism of the retract $ret_c\langle G, f\rangle$.*

Proof. Let $a, b \in G$. Then

$$\begin{aligned}\sigma(a+b) &= \sigma(f(a, c, \dots, c, \bar{c}, b)) = -\psi(c) + \psi(f(a, c, \dots, c, \bar{c}, b))\\ &= -\psi(c) + f(\psi(a), \psi(c), \dots, \psi(c), \overline{\psi(c)}, \psi(b))\\ &= -\psi(c) + \psi(a) + (n-3)\psi(c) + \overline{\psi(c)} + \psi(b) + d\\ &= (-\psi(c) + \psi(a)) + (-\psi(c) + \psi(b)) + (n-2)\psi(c) + \overline{\psi(c)} + d\\ &= \sigma(a) + \sigma(b) + (n-2)\psi(c) + \overline{\psi(c)} + d\\ &= f(\sigma(a) + \sigma(b), \psi(c), \dots, \psi(c), \overline{\psi(c)}) = \sigma(a) + \sigma(b),\end{aligned}$$

where $d = f(c, \dots, c)$. Hence the proposition. □

Proposition 2.2. *Let $\langle G, f\rangle = der_d\langle G, +\rangle$ and σ be an automorphism of the abelian group $\langle G, +\rangle$. If there is an element $u \in G$ such that $\sigma(d) = (n-1)u + d$, then the map $\psi : G \to G$, defined by $\psi(x) = u + \sigma(x)$, is an automorphism of the n-ary group $\langle G, f\rangle$. There are no more automorphisms of $\langle G, f\rangle$.*

Proof. Let $a_1, \dots, a_n \in G$. Then

$$\begin{aligned}\psi(f(a_1, \dots, a_n)) &= u + \sigma(a_1 + \dots + a_n + d) = u + \sigma(a_1) + \dots + \sigma(a_n) + \sigma(d)\\ &= u + \sigma(a_1) + \dots + \sigma(a_n) + (n-1)u + d\\ &= f(u + \sigma(a_1), \dots, u + \sigma(a_n)) = f(\psi(a_1), \dots, \psi(a_n)).\end{aligned}$$

Hence ψ is an automorphism of $\langle G, f\rangle$.

Now let τ be an arbitrary automorphism of $\langle G, f\rangle$. Then, according to Proposition 2.1, the map $\sigma : G \to G$ defined by $\sigma(x) = -u + \tau(x)$, where $u = \tau(0)$, is an automorphism of the group $\langle G, +\rangle = ret_0\langle G, f\rangle$. Moreover,

$$\begin{aligned}\sigma(d) &= -u + \tau(d) = -u + \tau(f(0, \dots, 0) = -u + f(u, \dots, u)\\ &= -u + nu + d = (n-1)u + d.\end{aligned}$$

Then τ is one of automorphisms of $\langle G, f\rangle$ obtained earlier from automorphisms of the group $\langle G, +\rangle$. Hence the proposition. □

Later we will need the following

Lemma 2.3. *Let d be a fixed element of an abelian group G and U_d be the set of all automorphisms σ of G such that $\sigma(d) = (n-1)u + d$ $(n > 2)$ for some $u \in G$. Then U_d is a subgroup of* $\operatorname{Aut} G$.

Proof. Let $\sigma_1, \sigma_2 \in \operatorname{Aut} G$ be such that $\sigma_1(d) = (n-1)u_1 + d$ and $\sigma_2(d) = (n-1)u_2 + d$ $(n > 2)$ for some $u_1, u_2 \in G$. Then

$$\begin{aligned}(\sigma_1 \circ \sigma_2)(d) &= \sigma_2(\sigma_1(d)) = \sigma_2((n-1)u_1 + d) = (n-1)\sigma_2(u_1) + \sigma_2(d) \\ &= (n-1)\sigma_2(u_1) + (n-1)u_2 + d = (n-1)(\sigma_2(u_1) + u_2) + d.\end{aligned}$$

Thus $(\sigma_1 \circ \sigma_2)(d) = (n-1)u_3 + d$, where $u_3 = \sigma_2(u_1) + u_2$. For an identity automorphism 1_G and for the zero element 0 of the group G we have $1_G(d) = (n-1)0 + d$. Finally

$$\begin{aligned}\sigma_1^{-1}(d) &= \sigma_1^{-1}(\sigma_1(d) - (n-1)u_1) = \sigma_1^{-1}(\sigma_1(d)) - (n-1)\sigma_1^{-1}(u_1) \\ &= (n-1)\sigma_1^{-1}(-u_1) + d.\end{aligned}$$

Hence $\sigma_1^{-1}(d) = (n-1)u_4 + d$, where $u_4 = \sigma_1^{-1}(-u_1)$. This completes the proof. □

Now we can study the automorphism group of an abelian n-ary group.

Theorem 2.4. *The automorphism group of abelian n-ary group $\langle G, f\rangle$ is embedded into the holomorph of the group $ret_c\langle G, f\rangle$.*

Proof. Consider the holomorph $Hol\, ret_c\langle G, f\rangle$ of the group $ret_c\langle G, f\rangle$. Define the map τ from $\operatorname{Aut}\langle G, f\rangle$ to $Hol\, ret_c\langle G, f\rangle$ by putting $\tau(\psi) = (\sigma, -\psi(c))$, where σ is an automorphism of $ret_c\langle G, f\rangle$ such that $\sigma(x) = -\psi(c) + \psi(x)$. By Proposition 2.1 the definition of τ is correct. Now we are going to show that τ is injective. Let $\tau(\psi_1) = \tau(\psi_2)$ for some $\psi_1, \psi_2 \in \operatorname{Aut}\langle G, f\rangle$, where $\tau(\psi_1) = (\sigma_1, -\psi_1(c))$ and $\tau(\psi_2) = (\sigma_2, -\psi_2(c))$. Then $\psi_1(c) = \psi_2(c)$, and for each $x \in G$ we have $\sigma_1(x) = \sigma_2(x)$ which implies $-\psi_1(c) + \psi_1(x) = -\psi_2(c) + \psi_2(x)$. Hence $\psi_1(x) = \psi_2(x)$, for each $x \in G$. So τ is injective. It also preserves the group operation. Indeed, if $\psi_1, \psi_2 \in \operatorname{Aut}\langle G, f\rangle$ and

$$\tau(\psi_1) = (\sigma_1, -\psi_1(c)), \quad \tau(\psi_2) = (\sigma_2, -\psi_2(c)), \quad \tau(\psi_1 \circ \psi_2) = (\sigma_3, -(\psi_1 \circ \psi_2)(c)),$$

where

$$\sigma_1(x) = -\psi_1(c) + \psi_1(x),\ \sigma_2(x) = -\psi_2(c) + \psi_2(x),\ \sigma_3(x) = -(\psi_1 \circ \psi_2)(c) + (\psi_1 \circ \psi_2)(x)$$

for each $x \in G$, then

$$\begin{aligned}(\sigma_1 \circ \sigma_2)(x) &= \sigma_2(\sigma_1(x)) = \sigma_2(-\psi_1(c) + \psi_1(x)) = -\sigma_2(\psi_1(c)) + \sigma_2(\psi_1(x)) \\ &= -(-\psi_2(c) + \psi_2(\psi_1(c))) + (-\psi_2(c) + \psi_2(\psi_1(x))) \\ &= -(\psi_1 \circ \psi_2)(c) + (\psi_1 \circ \psi_2)(x),\end{aligned}$$

hence $\sigma_3(x) = (\sigma_1 \circ \sigma_2)(x)$ for each $x \in G$. Then

$$\begin{aligned}\tau(\psi_1)\cdot\tau(\psi_2) &= (\sigma_1, -\psi_1(c))\cdot(\sigma_2, -\psi_2(c)) = (\sigma_1\circ\sigma_2, -\psi_1(c) + \sigma_1(-\psi_2(c)))\\ &= (\sigma_1\circ\sigma_2, -\psi_1(c) - \sigma_1(\psi_2(c)))\\ &= (\sigma_1\circ\sigma_2, -\psi_1(c) + \psi_1(c) - \psi_1(\psi_2(c)))\\ &= (\sigma_1\circ\sigma_2, -(\psi_1\circ\psi_2)(c)) = \tau(\psi_1\circ\psi_2),\end{aligned}$$

which completes the proof. □

3. Automorphisms of abelian semicyclic n-ary groups

Automorphisms of semicyclic n-ary groups (both abelian and non-abelian) are studied in [7]. Here we recall some facts from this paper.

Consider an additive group $\mathbb{Z}_k$ modulo k and a corresponding abelian semicyclic n-ary group $der_l\mathbb{Z}_k$, where $0 \leqslant l < k$. Finite cyclic groups of the same order are isomorphic but abelian semicyclic n-ary groups of the same order may not be isomorphic. It is known (cf. [10]) that two n-ary groups $der_{l_1}\mathbb{Z}_k$ and $der_{l_2}\mathbb{Z}_k$ are isomorphic if and only if $\gcd(l_1, n-1, k) = \gcd(l_2, n-1, k)$. It implies that the number of distinct (non-isomorphic) abelian semicyclic n-ary groups l-derived from the group $\mathbb{Z}_k$ is equal to the number of positive divisors $\tau(d)$ of $d = \gcd(n-1, k)$ and each such n-ary group is defined by a divisor l of d.

For example, three abelian semicyclic 5-ary groups can be defined on a cyclic group $\mathbb{Z}_4$ since $\gcd(4, 4)$ has three divisors: 1,2,4. So, they have the form $der_0\mathbb{Z}_4$, $der_1\mathbb{Z}_4$ and $der_2\mathbb{Z}_4$, where $der_1\mathbb{Z}_4$ is a cyclic 5-ary group.

Knowing automorphisms of a finite cyclic group one can find all automorphisms of the corresponding finite abelian semicyclic n-ary group.

Proposition 3.1. (Theorem 6.3, [7]) *Let $der_l\mathbb{Z}_k$ be a semicyclic n-ary group and $\sigma(x) = wx$, where w and k are coprime, be an automorphism of the group $\mathbb{Z}_k$. Then the map $\psi(x) = wx + t$, where t is a solution of the congruence $x(n-1) \equiv l(w-1) \pmod k$ and $\gcd(n-1, k)$ is a divisor of $l(w-1)$, is an automorphism of the n-ary group $der_l\mathbb{Z}_k$. There are no more automorphisms of $der_l\mathbb{Z}_k$.* □

Corollary 3.2. *If $\sigma(x) = wx$ is an automorphism of the group $\mathbb{Z}_k$, then $\psi(x) = wx + t$, where t is a solution of the congruence $x(n-1) \equiv 0 \pmod k$ is an automorphism of the n-ary group $der_0\mathbb{Z}_k$. There are no more automorphisms of $der_0\mathbb{Z}_k$.* □

It follows from Proposition 3.1 that each automorphism of a finite cyclic group $\mathbb{Z}_k$ defined by an integer w gives exactly $d = \gcd(n-1, k)$ distinct automorphisms of the semicyclic n-ary group $der_l\mathbb{Z}_k$, since the congruence $x(n-1) \equiv l(w-1) \pmod k$ has d solutions, that can be calculated using the formulas $t = t_0 + v\frac{k}{d}$ where $0 \leqslant v \leqslant d-1$ and t_0 is a solution of the congruence $x\frac{n-1}{d} \equiv \frac{l(w-1)}{d} \pmod{\frac{k}{d}}$. Thus each automorphism of $der_l\mathbb{Z}_k$ is defined uniquely by the integers w and t.

Example 3.3. Find all automorphisms of abelian semicyclic 5-ary groups defined on the cyclic group $\mathbb{Z}_4$. As it was mentioned earlier there are three such 5-ary groups: $der_0\mathbb{Z}_4$, $der_1\mathbb{Z}_4$ and $der_2\mathbb{Z}_4$.

The 5-ary group $der_0\mathbb{Z}_4$ has 8 automorphisms since there are two integers that are coprime to 4, and the congruence $4x \equiv 0 \pmod 4$ has four solutions. So, by Corollary 3.2, each automorphism is defined by one of the following rules: $\psi_1(x) = x$, $\psi_2(x) = x+1$, $\psi_3(x) = x+2$, $\psi_4(x) = x+3$, $\psi_5(x) = 3x$, $\psi_6(x) = 3x+1$, $\psi_7(x) = 3x+2$, $\psi_8(x) = 3x+3$.

The cyclic 5-ary group $der_1\mathbb{Z}_4$ has 4 automorphisms since there is exactly one integer that is coprime to 4 which satisfies the Proposition 3.1. Thus we have the congruence $4x \equiv 0 \pmod 4$. It has four solutions. According to Proposition 3.1, automorphisms of $der_1\mathbb{Z}_4$ have the form: $\psi_1(x) = x$, $\psi_2(x) = x+1$, $\psi_3(x) = x+2$, $\psi_4(x) = x+3$.

Finally, the 5-ary group $der_2\mathbb{Z}_4$ has 8 automorphisms since there are two integers w that are coprime to 4 and satisfy the Proposition 3.1. Both congruences: $4x \equiv 0 \pmod 4$ for $w = 1$, and $4x \equiv 4 \pmod 4$ for $w = 3$, have four solutions. So, by Proposition 3.1, these automorphisms coincide with automorphisms of the 5-ary group $der_0\mathbb{Z}_4$. □

Let $\mathbb{Z}_k^*$ be the multiplicative group of the ring $\mathbb{Z}_k$. Then the set

$$A^*_{\frac{d}{l}} = \left\{ w \in \mathbb{Z}_k^* \mid w \equiv 1 \ (\mathrm{mod}\ \frac{d}{l}) \right\},$$

where l divides d, is a subgroup of $\mathbb{Z}_k^*$ (see our discussion before Proposition 3.1).

Theorem 3.4. (Theorem 6.5, [7]) *The automorphism group of the abelian semicyclic n-ary group $der_l\mathbb{Z}_k$, provided $l|\gcd(n-1,k)$, is isomorphic to the extension of a cyclic group of order $d = \gcd(n-1,k)$ by the multiplicative group $A^*_{\frac{d}{l}}$.* □

Corollary 3.5. (Corollary 6.6, [7]) *The automorphism group of a cyclic n-ary group of a finite order k is isomorphic to the direct sum of A^*_d and a cyclic group $(\frac{k}{d})$, where $d = \gcd(n-1,k)$.* □

Corollary 3.6. *The automorphism group of an n-ary group derived from a cyclic group of a finite order k is isomorphic to the extension of a cyclic group of order $d = \gcd(n-1,k)$ by the multiplicative group $\mathbb{Z}_k^*$.*

Proof. Each n-ary group derived from a cyclic group of a finite order k is isomorphic to the n-ary group derived from the cyclic group $\mathbb{Z}_k$. Consequently, by Corollary 3.2, the multiplicative group $A^*_{\frac{d}{l}}$ from Theorem 3.4 is exactly the multiplicative group $\mathbb{Z}_k^*$. □

Corollary 3.7. (Corollary 6.8, [7]) *If $gcd(n-1,k) = 1$, then the n-ary group $der_l\mathbb{Z}_k$ is cyclic for each $l = 0, 1, 2, \ldots, k-1$ (see [18], Corollary 1) and its automorphism group is isomorphic to the multiplicative group $\mathbb{Z}_k^*$.* □

As is well known (see Theorem 3, [18]) each infinite abelian semicyclic n-ary group is isomorphic to the n-ary group $der_l\mathbb{Z}$, where $0 \leqslant l \leqslant \frac{n-1}{2}$ and $\mathbb{Z}$ is the additive group of integers.

Theorem 3.8. *Let $der_l\mathbb{Z}$ be an infinite semicyclic n-ary group. Then*

1) *for $l = 0$ it has only two automorphisms: $\varphi_1(x) = x$ and $\varphi_2(x) = -x$,*

2) *for $l = \frac{n-1}{2}$ it has only two automorphisms: $\varphi_1(x) = x$ and $\varphi_2(x) = -x-1$,*

3) *in other cases it has only the identity automorphism.*

Proof. If $l = 0$, then by Proposition 2.2 each automorphism of the group $\mathbb{Z}$ is an automorphisms of an n-ary group $der_0\mathbb{Z}$. So, $\varphi(x) = x$ or $\varphi(x) = -x$.

Now let $0 < l \leqslant \frac{n-1}{2}$. If τ is an automorphism of an n-ary group $der_l\mathbb{Z}$, then, by Proposition 2.1, the map $\sigma(x) = \tau(x) - t$, where $\tau(0) = t$, is an automorphism of the group $\mathbb{Z}$. So, either $\tau(x) = x + t$ or $\tau(x) = -x + t$. Furthermore, on one hand, either $\tau(f(0,\ldots,0)) = \tau(l) = l + t$ or $\tau(f(0,\ldots,0)) = \tau(l) = -l + t$; on the other hand, $f(\tau(0),\ldots,\tau(0)) = f(t,\ldots,t) = nt + l$. Hence, either $l + t = nt + l$ or $-l+t = nt+l$. The first equality implies $t = 0$, i.e., τ is the identity automorphism. The second equality gives two cases: (a) $l = 0$ and $t = 0$, (b) $l = \frac{n-1}{2}$ for odd n and $t = -1$. In the case (a) we have $\tau(x) = -x$; in the case (b) we get $\tau(x) = -x - 1$. Therefore, there are no other automorphisms. □

Since an n-ary group $der_l\mathbb{Z}$ is cyclic if and only if either $l \equiv 1 (\mathrm{mod}\ n-1)$ or $l \equiv -1 (\mathrm{mod}\ n-1)$ (see Proposition 8, [17]), as a consequence of the above theorem we obtain

Corollary 3.9. (Corollary 6.11, [7]) *For $n > 3$ the automorphism group of an infinite abelian cyclic n-ary group is trivial.* □

Corollary 3.10. (Corollary 4, [15]) *The automorphism group of an infinite abelian cyclic ternary group has only two elements: $\varphi(x) = x$ and $\varphi(x) = -x-1$.* □

4. Automorphisms of primary abelian n-ary groups

Following the group theory, we say that a finite n-ary group is an *n-ary p-group* if its order is a power of a prime number p. Such n-ary groups are also called *primary.*

Recall the following

Theorem 4.1. (Theorem 8, [2]) *Each finite abelian n-ary group is isomorphic to a direct product of semicyclic abelian n-ary p-groups.* □

Let $\langle G, f\rangle$ be an abelian n-ary group of an order $p^{\alpha_1}p^{\alpha_2}\ldots p^{\alpha_k}$, where p is prime and $\alpha_1 > \alpha_2 > \ldots > \alpha_k$. Consider the abelian group $\langle G,+\rangle = ret_c\langle G, f\rangle$. Since c is a zero of $\langle G,+\rangle$ it will be identified with 0.

Let $\langle G,+\rangle = \sum_{s=1}^{k} G_s$ be a direct sum of abelian p-groups G_s, where each group $G_s = \sum_{i=1}^{n_s}(a_{is})$ is a direct sum of cyclic groups (a_{is}) of the fixed order p^{α_s}. Then $d = f(0,\ldots,0) = \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}a_{is}$.

Consider the family of n-ary groups $der_{l_{is}a_{is}}(a_{is})$. The map ψ from an n-ary group $\langle G,f\rangle$ into the direct product $\prod_{s=1}^{k}\prod_{i=1}^{n_s} der_{l_{is}a_{is}}(a_{is})$ defined by

$$\psi(\textstyle\sum_{s=1}^{k}\sum_{i=1}^{n_s} x_{is}a_{is}) = \prod_{s=1}^{k}\prod_{i=1}^{n_s} x_{is}a_{is}$$

is an isomorphism (see the proof of Theorem 8 in [2]).

It is known (see, for example, §21, [13]), that the ring $End\langle G,+\rangle$ is isomorphic to the ring M of integer matrices (y_{is}^{jt}) of the order $n_1+n_2+\ldots+n_k$, where $1 \leqslant s,t \leqslant k$ and for given s,t the indexes i,j satisfy $\sum_{r=1}^{s-1} n_r + 1 \leqslant i \leqslant \sum_{r=1}^{s} n_r$ and $\sum_{r=1}^{t-1} n_r + 1 \leqslant j \leqslant \sum_{r=1}^{t} n_r$ (where in the case $s=1$ and $t=1$ we have $n_0 = 0$). The lower pair of indexes is denotes the number of the rows $\sum_{r=1}^{s-1} n_r + i$; the upper pair jt denotes the number of columns $\sum_{r=1}^{t-1} n_r + j$, where

$$y_{is}^{jt} = \begin{cases} x_{is}^{jt}, & \text{if either } s<t \text{ or } s=t \text{ and } i<j, \text{ where } 0 \leqslant x_{is}^{jt} < p^{\alpha_t}, \\ p^{\alpha_t-\alpha_s}x_{is}^{jt}, & \text{if either } s>t \text{ or } s=t \text{ and } i \geqslant j, \text{ where } 0 \leqslant x_{is}^{jt} < p^{\alpha_s}. \end{cases}$$

The addition and multiplication are defined as follows:

$$(y_{is}^{jt}) + (y'^{jt}_{is}) = ((y_{is}^{jt} + y'^{jt}_{is}) \pmod{p})^{\alpha_t}),$$
$$(y_{is}^{jt}) \times (y'^{jt}_{is}) = ((\textstyle\sum_{r=1}^{k}\sum_{v=1}^{n_r} y_{is}^{vr}\cdot y'^{jt}_{vr}) \pmod{p})^{\alpha_t}).$$

The isomorphism ψ maps every automorphism σ of the group $\langle G,+\rangle$ to the invertible matrix (y_{is}^{jt}) from the ring M, so σ acts on G by the following rule: if $g \in G$ and $g = \sum_{s=1}^{k}\sum_{i=1}^{n_s} q_{is}a_{is}$, then

$$\sigma(g) = \sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{s=1}^{k}\sum_{i=1}^{n_s} q_{is}y_{is}^{jt}\Big)a_{jt}. \tag{2}$$

Proposition 4.2. *Let $\langle G,f\rangle = \prod_{s=1}^{k}\prod_{i=1}^{n_s} der_{l_{is}a_{is}}(a_{is})$ be a direct product of n-ary groups $der_{l_{is}a_{is}}(a_{is})$, where $|(a_{is})| = p^{\alpha_s}$, $\alpha_1 > \alpha_2 > \ldots > \alpha_k$ and p is prime. If σ is an automorphism of the group $\langle G,+\rangle = \sum_{s=1}^{k}\sum_{i=1}^{n_s}(a_{is})$ that corresponds to the integer matrix (y_{is}^{jt}) of the order $\sum_{s=1}^{k} n_s$ and $\gcd(n-1,p^{\alpha_t})$ divides $l_{jt} - \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y_{is}^{jt}$ for each $t=1,\ldots,k$ and $j=1,\ldots,n_t$, then the map $\psi(g) = \sigma(g) + \sum_{t=1}^{k}\sum_{j=1}^{n_t} u_{jt}a_{jt}$, where u_{jt} are solutions of the congruences $\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y_{is}^{jt} \equiv (n-1)x + l_{jt} (\bmod\,)p^{\alpha_t}$, is an automorphism of the n-ary group $\langle G,f\rangle$.*

Proof. Since $\langle G,f\rangle = der_d\langle G,+\rangle$, where $d = \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}a_{is}$, and $\gcd(n-1,p^{\alpha_t})$ divides $l_{jt} - \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y_{is}^{jt}$ for all $t=1,\ldots,k$ and $j=1,\ldots,n_t$, then

$$\begin{cases} \gcd(n-1,p^{\alpha_1}) \mid (l_{11} - \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}p^{\alpha_1-\alpha_s}x_{is}^{11}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \gcd(n-1,p^{\alpha_1}) \mid (l_{n_1 1} - \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}p^{\alpha_1-\alpha_s}x_{is}^{n_1 1}) \\ \gcd(n-1,p^{\alpha_2}) \mid (l_{12} - \sum_{i=1}^{n_1} l_{i1}x_{i1}^{12} - \sum_{s=2}^{k}\sum_{i=1}^{n_s} l_{is}p^{\alpha_2-\alpha_s}x_{is}^{12}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \gcd(n-1,p^{\alpha_2}) \mid (l_{n_2 2} - \sum_{i=1}^{n_1} l_{i1}x_{i1}^{n_2 2} - \sum_{s=2}^{k}\sum_{i=1}^{n_s} l_{is}p^{\alpha_2-\alpha_s}x_{is}^{n_2 2}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \gcd(n-1,p^{\alpha_{k-1}}) \mid (l_{1k-1} - \sum_{s=1}^{k-1}\sum_{i=1}^{n_s} l_{is}x_{is}^{1k-1} - \sum_{i=1}^{n_k} l_{ik}p^{\alpha_{k-1}-\alpha_k}x_{ik}^{1k-1}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \gcd(n-1,p^{\alpha_{k-1}}) \mid (l_{n_{k-1}k-1} - \sum_{s=1}^{k-1}\sum_{i=1}^{n_s} l_{is}x_{is}^{n_{k-1}k-1} - \sum_{i=1}^{n_k} l_{ik}p^{\alpha_{k-1}-\alpha_k}x_{ik}^{n_{k-1}k-1}) \\ \gcd(n-1,p^{\alpha_k}) \mid (l_{1k} - \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}x_{is}^{1k}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \gcd(n-1,p^{\alpha_k}) \mid (l_{n_k k} - \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}x_{is}^{n_k k}). \end{cases}$$

This means that the following congruences

$$\begin{cases} \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}p^{\alpha_1-\alpha_s}x_{is}^{11} \equiv (n-1)x + l_{11} (\bmod\ p^{\alpha_1}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}p^{\alpha_1-\alpha_s}x_{is}^{n_1 1} \equiv (n-1)x + l_{n_1 1} (\bmod\ p^{\alpha_1}) \\ \sum_{i=1}^{n_1} l_{i1}x_{i1}^{12} + \sum_{s=2}^{k}\sum_{i=1}^{n_s} l_{is}p^{\alpha_2-\alpha_s}x_{is}^{12} \equiv (n-1)x + l_{12} (\bmod\ p^{\alpha_2}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \sum_{i=1}^{n_1} l_{i1}x_{i1}^{n_2 2} + \sum_{s=2}^{k}\sum_{i=1}^{n_s} l_{is}p^{\alpha_2-\alpha_s}x_{is}^{n_2 2} \equiv (n-1)x + l_{n_2 2} (\bmod\ p^{\alpha_2}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \sum_{s=1}^{k-1}\sum_{i=1}^{n_s} l_{is}x_{is}^{1k-1} + \sum_{i=1}^{n_k} l_{ik}p^{\alpha_{k-1}-\alpha_k}x_{ik}^{1k-1} \equiv (n-1)x + l_{1k-1} (\bmod\ p^{\alpha_{k-1}}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \sum_{s=1}^{k-1}\sum_{i=1}^{n_s} l_{is}x_{is}^{n_{k-1}k-1} + \sum_{i=1}^{n_k} l_{ik}p^{\alpha_{k-1}-\alpha_k}x_{ik}^{n_{k-1}k-1} \equiv (n-1)x + l_{n_{k-1}k-1} (\bmod\ p^{\alpha_{k-1}}) \\ \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}x_{is}^{1k} \equiv (n-1)x + l_{1k}\ (\bmod\ p^{\alpha_k}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}x_{is}^{n_k k} \equiv (n-1)x + l_{n_k k}\ (\bmod\ p^{\alpha_k}). \end{cases}$$

have solutions.

Let u_{jt} (where $t = 1,\dots,k$ and $j = 1,\dots,n_t$) be the solutions of the corresponding congruences from the above system. Then $\sigma(d) = (n-1)u + d$ for $u = \sum_{s=1}^{k}\sum_{i=1}^{n_s} u_{is}a_{is}$. Proposition 2.2 completes the proof. □

By Proposition 4.2 each automorphism σ of the group $\langle G,+\rangle = \sum_{s=1}^{k}\sum_{i=1}^{n_s}(a_{is})$ for which $d_t = \gcd(n-1,p^{\alpha_t}) \mid (l_{jt} - \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y_{is}^{jt})$ for $t = 1,\dots,k$ and $j = 1,\dots,n_t$, defines exactly $\prod_{t=1}^{k} d_t$ automorphisms of the abelian n-ary group $\langle G,f\rangle = \prod_{s=1}^{k}\prod_{i=1}^{n_s} der_{l_{is}a_{is}}(a_{is})$. Moreover, each of them is defined by the integers v_{jt} $(0 \leqslant v_{jt} \leqslant d_t - 1)$ such that $u_{jt}^{v_{jt}} = u_{jt}^{0} + v_{jt}\frac{p^{\alpha_t}}{d_t}$ is a solution of the congruence $\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y_{is}^{jt} \equiv (n-1)x + l_{jt}\,(\bmod\ p^{\alpha_t})$, where u_{jt}^{0} is the solution of the congruence $\frac{\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y_{is}^{jt}}{d_t} \equiv \frac{n-1}{d_t}x + \frac{l_{jt}}{d_t} (\bmod\ \frac{p^{\alpha_t}}{d_t})$. Thus each automorphism of $\prod_{s=1}^{k}\prod_{i=1}^{n_s} der_{l_{is}a_{is}}(a_{is})$ is uniquely determined by the ordered set

$$V = \{v_{jt} \mid t = 1, \ldots, k,\ j = 1, \ldots, n_t\}$$

and an automorphism σ of the direct sum of cyclic groups $\sum_{s=1}^{k}\sum_{i=1}^{n_s}(a_{is})$ such that $d_t = \gcd(n-1, p^{\alpha_t}) \mid (l_{jt} - \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y_{is}^{jt})$ for $t = 1, \ldots, k$ and $j = 1, \ldots, n_t$. Thus we denote such automorphism by $\psi_{\sigma,V}$.

Theorem 4.3. *Let $\langle G, f\rangle = \prod_{s=1}^{k}\prod_{i=1}^{n_s} der_{l_{is}a_{is}}(a_{is})$ be the direct product of n-ary groups $der_{l_{is}a_{is}}(a_{is})$, where $|(a_{is})| = p^{\alpha_s}$, $\alpha_1 > \alpha_2 > \ldots > \alpha_k$ and p is prime. If U_d is an automorphism group of the direct sum of cyclic groups $\sum_{s=1}^{k}\sum_{i=1}^{n_s}(a_{is})$ having the corresponding integer matrices (y_{is}^{jt}) of the degree $\sum_{s=1}^{k} n_s$ such that $d_t = \gcd(n-1, p^{\alpha_t}) \mid (l_{jt} - \sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y_{is}^{jt})$ for each $t = 1, \ldots, k$ and $j = 1, \ldots, n_t$, then the automorphism group of the n-ary group $\langle G, f\rangle$ is isomorphic to the extension of the direct sum $\sum_{t=1}^{k}\sum_{j=1}^{n_t}(\frac{p^{\alpha_t}}{d_t}a_{jt})$ of cyclic subgroups $(\frac{p^{\alpha_t}}{d_t}a_{jt})$ of cyclic groups (a_{jt}) by the group U_d.*

Proof. For each $\sigma \in U_d$ corresponds an invertible matrix (y_{is}^{jt}) from the ring M (defined earlier) such that σ acts on G by the rule (2). For each index $s \in \{1, \ldots, k\}$ and each index $i \in \{1, \ldots, , n_s\}$ (for each fixed s) we can calculate the image of the generating element $\frac{p^{\alpha_s}}{d_s}a_{is}$ of the cyclic subgroup $(\frac{p^{\alpha_s}}{d_s}a_{is})$. Namely,

$$\sigma(\frac{p^{\alpha_s}}{d_s}a_{is}) = \sum_{t=1}^{k}\sum_{j=1}^{n_t}\frac{p^{\alpha_s}}{d_s}y_{is}^{jt}a_{jt}. \tag{3}$$

Now we fix indexes s and i and show that for any indexes t and j from (3) the integer $\frac{p^{\alpha_s}}{d_s}y_{is}^{jt}$ is divided by $\frac{p^{\alpha_t}}{d_t}$. Indeed, if $s < t$ or $s = t$ and $i < j$, then $\alpha_s \geqslant \alpha_t$ and, consequently, $\frac{p^{\alpha_s}}{d_s}$ is divided by $\frac{p^{\alpha_t}}{d_t}$. If $s > t$ or $s = t$ and $i \geqslant j$, then $\alpha_s \leq \alpha_t$ and hence $\frac{p^{\alpha_s}}{d_s}p^{\alpha_t-\alpha_s}$ is divided by $\frac{p^{\alpha_t}}{d_t}$. So in both cases $\frac{p^{\alpha_s}}{d_s}y_{is}^{jt}$ is divided by $\frac{p^{\alpha_t}}{d_t}$. Let $\frac{p^{\alpha_s}}{d_s}y_{is}^{jt} = z_{is}^{jt}\frac{p^{\alpha_t}}{d_t}$. Since $z_{is}^{jt} = q_{is}^{jt}d_t + r_{is}^{jt}$, where $0 \leqslant r_{is}^{jt} < d_t$, from (3) we obtain

$$b_{is} = \sigma(\tfrac{p^{\alpha_s}}{d_s}a_{is}) = \textstyle\sum_{t=1}^{k}\sum_{j=1}^{n_t} r_{is}^{jt}\frac{p^{\alpha_t}}{d_t}a_{jt}.$$

Let us show that all elements b_{is} form the basis of the direct sum $\sum_{s=1}^{k}\sum_{i=1}^{n_s}(\frac{p^{\alpha_s}}{d_s}a_{is})$. Let $\sum_{s=1}^{k}\sum_{i=1}^{n_s} m_{is}b_{is} = 0$, then $\sum_{s=1}^{k}\sum_{i=1}^{n_s} m_{is}(\sum_{t=1}^{k}\sum_{j=1}^{n_t} r_{is}^{jt}\frac{p^{\alpha_t}}{d_t}a_{jt}) = 0$ or $\sum_{t=1}^{k}\sum_{j=1}^{n_t}(\sum_{s=1}^{k}\sum_{i=1}^{n_s} m_{is}r_{is}^{jt})\frac{p^{\alpha_t}}{d_t}a_{jt} = 0$. Since all the elements $\frac{p^{\alpha_t}}{d_t}a_{jt}$ form the basis of the direct sum $\sum_{s=1}^{k}\sum_{i=1}^{n_s}(\frac{p^{\alpha_s}}{d_s}a_{is})$, then $\sum_{s=1}^{k}\sum_{i=1}^{n_s} m_{is}r_{is}^{jt} \equiv 0 (\bmod\ d_t)$ for all t and j. Since $z_{is}^{jt} \equiv r_{is}^{jt} (\bmod\ d_t)$, then $\sum_{s=1}^{k}\sum_{i=1}^{n_s} m_{is}z_{is}^{jt} \equiv 0 (\bmod\ d_t)$ for all t, j. Multiplying the last congruence by $\frac{p^{\alpha_t}}{d_t}$ we get $\sum_{s=1}^{k}\sum_{i=1}^{n_s} m_{is}z_{is}^{jt}\frac{p^{\alpha_t}}{d_t} \equiv 0$

(mod p^{α_t}) for all t and j. Since $\frac{p^{\alpha_s}}{d_s} y_{is}^{jt} = z_{is}^{jt} \frac{p^{\alpha_t}}{d_t}$, then $\sum_{s=1}^{k} \sum_{i=1}^{n_s} m_{is} \frac{p^{\alpha_s}}{d_s} y_{is}^{jt} \equiv 0$ (mod p^{α_t}). Thus $\sum_{t=1}^{k} \sum_{j=1}^{n_t} (\sum_{s=1}^{k} \sum_{i=1}^{n_s} m_{is} \frac{p^{\alpha_s}}{d_s} y_{is}^{jt}) a_{jt} = 0$. According to (2) we have $\sigma(\sum_{s=1}^{k} \sum_{i=1}^{n_s} m_{is} \frac{p^{\alpha_s}}{d_s} a_{is}) = 0$. Hence $\sum_{s=1}^{k} \sum_{i=1}^{n_s} m_{is} \frac{p^{\alpha_s}}{d_s} a_{is} = 0$ since σ is bijective. But the elements a_{is} form the basis of the group $\sum_{s=1}^{k} \sum_{i=1}^{n_s} (a_{is})$, thus, $m_{is} \frac{p^{\alpha_s}}{d_s} \equiv 0 (\text{mod } p^{\alpha_s})$ for any s and i. Then $m_{is} \equiv 0 \pmod{d_s}$ for any indexes s and i. Thus we have proved that the elements b_{is} form the basis of the direct sum $B = \sum_{s=1}^{k} \sum_{i=1}^{n_s} (\frac{p^{\alpha_s}}{d_s} a_{is})$, and therefore the map σ^B defined by the following rule: if $g \in B$ and $g = \sum_{s=1}^{k} \sum_{i=1}^{n_s} q_{is} \frac{p^{\alpha_s}}{d_s} a_{is}$, then

$$\sigma^B(g) = \textstyle\sum_{t=1}^{k} \sum_{j=1}^{n_t} (\sum_{s=1}^{k} \sum_{i=1}^{n_s} q_{is} r_{is}^{jt}) \frac{p^{\alpha_t}}{d_t} a_{jt},$$

is an automorphism of the group B.

Now we fix the homomorphism $\zeta : U_d \to \operatorname{Aut} B$ such that $\zeta(\sigma) = \sigma^B$. We construct the extension $U_d \cdot B$ of the group B by the group U_d with the operation acting in the following way: let $\sigma_1, \sigma_2 \in U_d$, $g_1, g_2 \in B$, $g_1 = \sum_{s=1}^{k} \sum_{i=1}^{n_s} v'_{is} \frac{p^{\alpha_s}}{d_s} a_{is}$, $g_2 = \sum_{s=1}^{k} \sum_{i=1}^{n_s} v''_{is} \frac{p^{\alpha_s}}{d_s} a_{is}$ and the automorphism σ_2 from U_d be defined by the matrix $({y''}_{is}^{jt})$. Moreover,

$$\sigma_2\Big(\frac{p^{\alpha_s}}{d_s} a_{is}\Big) = \sum_{t=1}^{k} \sum_{j=1}^{n_t} \frac{p^{\alpha_s}}{d_s} {y''}_{is}^{jt} a_{jt} = \sum_{t=1}^{k} \sum_{j=1}^{n_t} {r''}_{is}^{jt} \frac{p^{\alpha_t}}{d_t} a_{jt} \tag{4}$$

for all elements $\frac{p^{\alpha_s}}{d_s} a_{is}$ of B. Therefore,

$$\zeta(\sigma_2)(g_1) = \sigma_2^B(g_1) = \sum_{t=1}^{k} \sum_{j=1}^{n_t} \Big(\sum_{s=1}^{k} \sum_{i=1}^{n_s} v'_{is} {r''}_{is}^{jt} \Big) \frac{p^{\alpha_t}}{d_t} a_{jt}.$$

Thus,

$$\begin{aligned} \sigma_1 g_1 \cdot \sigma_2 g_2 &= (\sigma_1 \circ \sigma_2)(\zeta(\sigma_2)(g_1) + g_2) = (\sigma_1 \circ \sigma_2)(\sigma_2^B(g_1) + g_2) \\ &= (\sigma_1 \circ \sigma_2)\Big(\sum_{t=1}^{k} \sum_{j=1}^{n_t} \Big(\sum_{s=1}^{k} \sum_{i=1}^{n_s} v'_{is} {r''}_{is}^{jt} + v''_{jt} \Big) \frac{p^{\alpha_t}}{d_t} a_{jt} \Big) \end{aligned}$$

(see, for example, [12]). Hence, $\sigma_1 g_1 \cdot \sigma_2 g_2 = (\sigma_1 \circ \sigma_2) g_3$, where

$$g_3 = \sum_{t=1}^{k} \sum_{j=1}^{n_t} \Big(\sum_{s=1}^{k} \sum_{i=1}^{n_s} v'_{is} {r''}_{is}^{jt} + v''_{jt} \Big) \frac{p^{\alpha_t}}{d_t} a_{jt}. \tag{5}$$

We define the map $\tau : \operatorname{Aut}\langle G, f\rangle \to U_d \cdot B$ by putting $\tau : \psi_{\sigma,V} \to \sigma g$, where $g = \sum_{s=1}^{k} \sum_{i=1}^{n_s} v_{is} \frac{p^{\alpha_s}}{d_s} a_{is}$. It is clear that τ is a bijection.

Let $\psi_{\sigma_1,V_1}, \psi_{\sigma_2,V_2} \in \operatorname{Aut}\langle G, f\rangle$, where the automorphisms σ_1 and σ_2 are defined by matrices $({y'}_{is}^{jt})$ and $({y''}_{is}^{jt})$, respectively. Consider the ordered set V_1

of integers v'_{jt} taken from the solutions $u_{jt}^{v'_{jt}} = u'^{0}_{jt} + v'_{jt}\frac{p^{\alpha_t}}{d_t}$ of the congruence $\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y'^{jt}_{is} \equiv (n-1)x + l_{jt} \,(\mathrm{mod}\; p^{\alpha_t})$, where u'^{0}_{jt} is a solution of the congruence $\frac{\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y'^{jt}_{is}}{d_t} \equiv \frac{n-1}{d_t}x + \frac{l_{jt}}{d_t} \;(\mathrm{mod}\; \frac{p^{\alpha_t}}{d_t})$. Similarly, V_2 is an ordered set of integers v''_{jt} taken from the solutions $u_{jt}^{v''_{jt}} = u''^{0}_{jt} + v''_{jt}\frac{p^{\alpha_t}}{d_t}$ of the congruence $\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y''^{jt}_{is} \equiv (n-1)x + l_{jt} \;(\mathrm{mod}\; p^{\alpha_t})$, where u''^{0}_{jt} is a solution of the congruence $\frac{\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y''^{jt}_{is}}{d_t} \equiv \frac{n-1}{d_t}x + \frac{l_{jt}}{d_t} \;(\mathrm{mod}\; \frac{p^{\alpha_t}}{d_t})$. Here $t = 1,\ldots,k$ and $j = 1,\ldots,n_t$ for any fixed t.

For each $g \in G$, $g = \sum_{s=1}^{k}\sum_{i=1}^{n_s} q_{is}a_{is}$, we have

$$(\psi_{\sigma_1,V_1} \circ \psi_{\sigma_2,V_2})(g) = \psi_{\sigma_2,V_2}(\psi_{\sigma_1,V_1}(g)) = \psi_{\sigma_2,V_2}\Big(\sigma_1(g) + \sum_{r=1}^{k}\sum_{v=1}^{n_r} u_{vr}^{v'_{vr}}a_{vr}\Big)$$
$$= \psi_{\sigma_2,V_2}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}\Big(\sum_{s=1}^{k}\sum_{i=1}^{n_s} q_{is}y'^{vr}_{is}\Big)a_{vr} + \sum_{r=1}^{k}\sum_{v=1}^{n_r} u_{vr}^{v'_{vr}}a_{vr}\Big)$$
$$= \psi_{\sigma_2,V_2}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}\Big(\sum_{s=1}^{k}\sum_{i=1}^{n_s} q_{is}y'^{vr}_{is} + u_{vr}^{v'_{vr}}\Big)a_{vr}\Big)$$
$$= \sigma_2\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}\Big(\sum_{s=1}^{k}\sum_{i=1}^{n_s} q_{is}y'^{vr}_{is} + u_{vr}^{v'_{vr}}\Big)a_{vr}\Big) + \sum_{t=1}^{k}\sum_{j=1}^{n_t} u_{jt}^{v''_{jt}}a_{jt}$$
$$= \sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}\Big(\sum_{s=1}^{k}\sum_{i=1}^{n_s} q_{is}y'^{vr}_{is} + u_{vr}^{v'_{vr}}\Big)y''^{jt}_{vr}\Big)a_{jt} + \sum_{t=1}^{k}\sum_{j=1}^{n_t} u_{jt}^{v''_{jt}}a_{jt}$$
$$=\sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}\Big(\sum_{s=1}^{k}\sum_{i=1}^{n_s} q_{is}y'^{vr}_{is}\Big)y''^{jt}_{vr}\Big)a_{jt} + \sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r} u_{vr}^{v'_{vr}}y''^{jt}_{vr}\Big)a_{jt} + \sum_{t=1}^{k}\sum_{j=1}^{n_t} u_{jt}^{v''_{jt}}a_{jt}$$
$$= (\sigma_1 \circ \sigma_2)(g) + \sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r} u_{vr}^{v'_{vr}}y''^{jt}_{vr} + u_{jt}^{v''_{jt}}\Big)a_{jt}.$$

Let us show that $c = \sum_{r=1}^{k}\sum_{v=1}^{n_r} u_{vr}^{v'_{vr}}y''^{jt}_{vr} + u_{jt}^{v''_{jt}}$ is a solution of the congruence

$$\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r} y'^{vr}_{is}y''^{jt}_{vr}\Big) \equiv (n-1)x + l_{jt} \;(\mathrm{mod}\; p^{\alpha_t}). \tag{6}$$

By the hypothesis, the following $n_1 + \ldots + n_k$ congruences

$$\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y'^{vr}_{is} \equiv (n-1)u_{vr}^{v'_{vr}} + l_{vr} \;\;(\mathrm{mod}\; p^{\alpha_r})$$

is valid for $r = 1,\ldots,k$ and $v = 1,\ldots,n_r$.

Multiplying each of these congruences by the corresponding y''^{jt}_{vr} (for fixed t and j) we obtain $(n_1 + \ldots + n_k)^2$ congruences

$$\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}y'^{vr}_{is}y''^{jt}_{vr} \equiv (n-1)u_{vr}^{v'_{vr}}y''^{jt}_{vr} + l_{vr}y''^{jt}_{vr} \;\;(\mathrm{mod}\; p^{\alpha_r}).$$

Adding (with respect to r and v) obtained congruences for fixed t and j we obtain $n_1+\ldots+n_k$ true congruences

$$\sum_{r=1}^{k}\sum_{v=1}^{n_r}\Big(\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y'}_{is}^{vr}{y''}_{vr}^{jt}\Big)\equiv\sum_{r=1}^{k}\sum_{v=1}^{n_r}(n-1)u_{vr}^{v'_{vr}}{y''}_{vr}^{jt}+\sum_{r=1}^{k}\sum_{v=1}^{n_r} l_{vr}{y''}_{vr}^{jt}\,(\mathrm{mod}\,p^{\alpha_t}). \quad (7)$$

But by the hypothesis for each t and j we also get $n_1+\ldots+n_k$ true congruences

$$\sum_{r=1}^{k}\sum_{v=1}^{n_r} l_{vr}{y''}_{vr}^{jt}\equiv(n-1)u_{jt}^{v''_{jt}}+l_{jt}\ (\mathrm{mod}\ p^{\alpha_t}).$$

So, (7) gives

$$\sum_{r=1}^{k}\sum_{v=1}^{n_r}\Big(\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y'}_{is}^{vr}{y''}_{vr}^{jt}\Big)\equiv\sum_{r=1}^{k}\sum_{v=1}^{n_r}(n-1)u_{vr}^{v'_{vr}}{y''}_{vr}^{jt}+(n-1)u_{jt}^{v''_{jt}}+l_{jt}\ (\mathrm{mod}\ p^{\alpha_t})\quad \text{or}$$

$$\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}{y'}_{is}^{vr}{y''}_{vr}^{jt}\Big)\equiv(n-1)\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}u_{vr}^{v'_{vr}}{y''}_{vr}^{jt}+u_{jt}^{v''_{jt}}\Big)+l_{jt}\ (\mathrm{mod}\ p^{\alpha_t}).$$

Hence c satisfies the congruence (6). Therefore $c=u_{jt}^{v'''_{jt}}={u'''}_{jt}^{0}+v'''_{jt}\frac{p^{\alpha_t}}{d_t}$ is a solution of the congruence $\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y'''}_{is}^{jt}\equiv(n-1)x+l_{jt}(\mathrm{mod}\ p^{\alpha_t})$, where ${u'''}_{jt}^{0}$ is a solution of the congruence $\frac{\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y'''}_{is}^{jt}}{d_t}\equiv\frac{n-1}{d_t}x+\frac{l_{jt}}{d_t}(\mathrm{mod}\ \frac{p^{\alpha_t}}{d_t})$, ${y'''}_{is}^{jt}=\sum_{r=1}^{k}\sum_{v=1}^{n_r}{y'}_{is}^{vr}{y''}_{vr}^{jt}$ and $0\leqslant v'''_{jt}\leqslant d_t-1$.

Consequently, the composition $\psi_{\sigma_1,V_1}\circ\psi_{\sigma_2,V_2}$ of the automorphisms ψ_{σ_1,V_1} and ψ_{σ_2,V_2} of the n-ary group $\langle G,f\rangle$ is the automorphism $\psi_{\sigma_1\circ\sigma_2,V_3}$, where V_3 is a collection of integers v'''_{jt} from the solutions $u_{jt}^{v'''_{jt}}$ of (6).

Now let us prove that

$$\tau(\psi_{\sigma_1,V_1}\circ\psi_{\sigma_2,V_2})=\tau(\psi_{\sigma_1,V_1})\cdot\tau(\psi_{\sigma_2,V_2}).$$

We have $\tau(\psi_{\sigma_1,V_1}\circ\psi_{\sigma_2,V_2})=\tau(\psi_{\sigma_1\circ\sigma_2,V_3})=(\sigma_1\circ\sigma_2)g_4$, where g_4 has the form $g_4=\sum_{s=1}^{k}\sum_{i=1}^{n_s}v'''_{is}\frac{p^{\alpha_s}}{d_s}a_{is}$. On the other hand $\tau(\psi_{\sigma_1,V_1})\cdot\tau(\psi_{\sigma_2,V_2})=(\sigma_1\circ\sigma_2)g_3$, where g_3 is from (5). Let us show $g_3=g_4$. Indeed, considering (4) we have

$$g_4=\sum_{t=1}^{k}\sum_{j=1}^{n_t}v'''_{jt}\frac{p^{\alpha_t}}{d_t}a_{jt}=\sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}u_{vr}^{v'_{vr}}{y''}_{vr}^{jt}+u_{jt}^{v''_{jt}}-{u'''}_{jt}^{0}\Big)a_{jt}$$

$$=\sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}({u'}_{vr}^{0}+v'_{vr}\frac{p^{\alpha_r}}{d_r}){y''}_{vr}^{jt}+{u''}_{jt}^{0}+v''_{jt}\frac{p^{\alpha_t}}{d_t}-{u'''}_{jt}^{0}\Big)a_{jt}$$

$$=\sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}{u'}_{vr}^{0}{y''}_{vr}^{jt}+\sum_{r=1}^{k}\sum_{v=1}^{n_r}v'_{vr}\frac{p^{\alpha_r}}{d_r}{y''}_{vr}^{jt}+{u''}_{jt}^{0}+v''_{jt}\frac{p^{\alpha_t}}{d_t}-{u'''}_{jt}^{0}\Big)a_{jt}$$

$$=\sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}{u'}_{vr}^{0}{y''}_{vr}^{jt}+{u''}_{jt}^{0}-{u'''}_{jt}^{0}\Big)a_{jt}+\sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}v'_{vr}\frac{p^{\alpha_r}}{d_r}{y''}_{vr}^{jt}+v''_{jt}\frac{p^{\alpha_t}}{d_t}\Big)a_{jt}$$

$$= \sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r} {u'}^{0}_{vr}{y''}^{jt}_{vr} + {u''}^{0}_{jt} - {u'''}^{0}_{jt}\Big)a_{jt} + \sum_{t=1}^{k}\sum_{j=1}^{n_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r} v'_{vr}{r''}^{jt}_{vr} + v''_{jt}\Big)\frac{p^{\alpha_t}}{d_t}a_{jt}.$$

It is now sufficient to show that the first component of the last sum is equal to zero. In fact, we have $n_1 + \ldots + n_k$ true congruences

$$\frac{\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y'}^{vr}_{is}}{d_r} \equiv \frac{n-1}{d_r}{u'}^{0}_{vr} + \frac{l_{vr}}{d_r} \pmod{\frac{p^{\alpha_r}}{d_r}},$$

where $r = 1, \ldots, k$ and $v = 1, \ldots, n_r$. Multiplying each congruence by the corresponding ${y''}^{jt}_{vr}$ (for fixed t and j), we obtain $(n_1 + \ldots + n_k)^2$ congruences

$$\frac{\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y'}^{vr}_{is}{y''}^{jt}_{vr}}{d_r} \equiv \frac{n-1}{d_r}{u'}^{0}_{vr}{y''}^{jt}_{vr} + \frac{l_{vr}}{d_r}{y''}^{jt}_{vr} \pmod{\frac{p^{\alpha_r}}{d_r}}.$$

Adding (with respect to r and v) obtained congruences for fixed t and j and get $n_1 + \ldots + n_k$ true congruences

$$\frac{\sum_{r=1}^{k}\sum_{v=1}^{n_r}(\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y'}^{vr}_{is}{y''}^{jt}_{vr})}{d_t} \equiv \frac{\sum_{r=1}^{k}\sum_{v=1}^{n_r}(n-1){u'}^{0}_{vr}{y''}^{jt}_{vr}}{d_t} + \frac{\sum_{r=1}^{k}\sum_{v=1}^{n_r} l_{vr}{y''}^{jt}_{vr}}{d_t} \pmod{\frac{p^{\alpha_t}}{d_t}}. \tag{8}$$

Since, by the hypothesis, for each t and j we have $n_1 + \ldots + n_k$ true congruences

$$\frac{\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y''}^{jt}_{is}}{d_t} \equiv \frac{n-1}{d_t}{u''}^{0}_{jt} + \frac{l_{jt}}{d_t} \pmod{\frac{p^{\alpha_t}}{d_t}},$$

from (8) we obtain

$$\frac{\sum_{r=1}^{k}\sum_{v=1}^{n_r}(\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y'}^{vr}_{is}{y''}^{jt}_{vr})}{d_t} \equiv \frac{n-1}{d_t}\Big(\sum_{r=1}^{k}\sum_{v=1}^{n_r}{u'}^{0}_{vr}{y''}^{jt}_{vr} + {u''}^{0}_{jt}\Big) + \frac{l_{jt}}{d_t} \pmod{\frac{p^{\alpha_t}}{d_t}}.$$

But

$$\frac{\sum_{r=1}^{k}\sum_{v=1}^{n_r}(\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y'}^{vr}_{is}{y''}^{jt}_{vr})}{d_t} \equiv \frac{n-1}{d_t}{u'''}^{0}_{jt} + \frac{l_{jt}}{d_t} \pmod{\frac{p^{\alpha_t}}{d_t}}.$$

Thus the congruence

$$\frac{\sum_{r=1}^{k}\sum_{v=1}^{n_r}(\sum_{s=1}^{k}\sum_{i=1}^{n_s} l_{is}{y'}^{vr}_{is}{y''}^{jt}_{vr})}{d_t} \equiv \frac{n-1}{d_t}x + \frac{l_{jt}}{d_t} \pmod{\frac{p^{\alpha_t}}{d_t}}$$

has a unique solution. Therefore,

$$\sum_{r=1}^{k}\sum_{v=1}^{n_r}{u'}^{0}_{vr}{y''}^{jt}_{vr} + {u''}^{0}_{jt} \equiv {u'''}^{0}_{jt} \pmod{\frac{p^{\alpha_t}}{d_t}}.$$

Consequently, $\sum_{t=1}^{k}\sum_{j=1}^{n_t}(\sum_{r=1}^{k}\sum_{v=1}^{n_r}{u'}^{0}_{vr}{y''}^{jt}_{vr} + {u''}^{0}_{jt} - {u'''}^{0}_{jt})a_{jt} = 0$, which completes the proof. □

If a prime p does not divide $n-1$, then a finite abelian n-ary p-group is isomorphic to a direct product of some cyclic n-ary p-groups (see Corollary 3, [2]). Thus we have the following

Corollary 4.4. *If a prime p does not divide $n-1$, then the automorphism group of a finite abelian n-ary p-group $\langle G, f\rangle$ is isomorphic to the automorphism group of $ret_c\langle G, f\rangle$.* □

Following the group theory we say that an n-ary p-group $\langle G, f\rangle$ is an *elementary abelian* n-ary p-group if it is isomorphic to the n-ary group $der_{l_1}\mathbb{Z}_p \times \ldots \times der_{l_k}\mathbb{Z}_p$. Such n-ary groups will be denoted by $\langle G_k(p), f\rangle$.

In binary case each elementary abelian p-group $\langle G_k(p), +\rangle$ of the rank k can be viewed as the vector space of dimension k over the field $\mathbb{Z}/p\mathbb{Z}$ with p elements. Its automorphism group is isomorphic to the group $GL(k, \mathbb{Z}/p\mathbb{Z})$.

Corollary 4.5. (Corollary 1, [16]) *The automorphism group of the elementary abelian n-ary p-group*

$$\langle G_k(p), f\rangle = der_0\mathbb{Z}_p \times \ldots \times der_0\mathbb{Z}_p,$$

where $p \mid (n-1)$, and the automorphism group of any elementary n-ary p-group of order p^k, where $p \nmid (n-1)$, are isomorphic to the group $GL(k, \mathbb{Z}/p\mathbb{Z})$. □

Corollary 4.6. (Theorem 4, [16]) *The automorphism group of the elementary abelian n-ary p-group*

$$\langle G_k(p), f\rangle = der_{l_1}\mathbb{Z}_p \times \ldots \times der_{l_k}\mathbb{Z}_p,$$

where at least one of $l_1, \ldots, l_k$ is non-zero and $p \mid (n-1)$, is isomorphic to the extension of the group

$$G_k(p) = \underbrace{\mathbb{Z}_p + \ldots + \mathbb{Z}_p}_{k}$$

by the stationary subgroup $St(d) \subseteq \operatorname{Aut} G_k(p)$ of the element $d = \sum_{i=1}^{k} l_i$. □

5. Automorphisms of free abelian n-ary groups

Free n-ary groups are described in [1]. In this section we describe the automorphism group of finitely generated free abelian n-ary groups.

We start with the following result which will be used later.

Theorem 5.1. (Corollary 1, [18]) *Each free abelian n-ary group $\langle F, f\rangle$ generated by a finite set X is isomorphic to a direct product of one infinite cyclic n-ary group $der_1\mathbb{Z}$ and $|X|-1$ copies of an n-ary group $der_0\mathbb{Z}$.* □

Theorem 5.2. *The automorphism group of the n-ary group $der_1\mathbb{Z}\times\prod_{i=1}^{k-1} der_0\mathbb{Z}$ is isomorphic to the group of all automorphisms σ of the free abelian group $\sum_{i=1}^{k}\mathbb{Z}$ such that*

$$\sigma((1,0,\dots,0)) = (t_1,t_2,\dots,t_k),$$

where $t_1\equiv 1(\mathrm{mod}\ n-1)$ and $t_i\equiv 0(\mathrm{mod}\ n-1)$ for $i=2,\dots,k$.

Proof. Let $\langle P,f\rangle = der_1\mathbb{Z}\times\prod_{i=1}^{k-1} der_0\mathbb{Z}$. Consider the abelian group $ret_c\langle P,f\rangle$ determined by the element $c=(0,\dots,0)$ and put $d=f(c,\dots,c)=(1,0,\dots,0)$. Then $\langle P,f\rangle = der_d\sum_{i=1}^{k}\mathbb{Z}$ and $ret_c\langle P,f\rangle=\sum_{i=1}^{k}\mathbb{Z}$.

It is clear that the set U of all automorphisms σ of the group $\sum_{i=1}^{k}\mathbb{Z}$ satisfying conditions mentioned in the theorem forms a subgroup of the group $\mathrm{Aut}\sum_{i=1}^{k}\mathbb{Z}$.

Moreover, $\sigma(d)=(n-1)u+d$ for every $\sigma\in U$ and $u=(\frac{t_1-1}{n-1},\frac{t_2}{n-1},\dots,\frac{t_k}{n-1})$. Indeed,

$$\begin{aligned}\sigma(d)&=\sigma((1,0,\dots,0))=(t_1,t_2,\dots,t_k)=(t_1-1,t_2,\dots,t_k)+(1,0,\dots,0)\\&=(n-1)\big(\tfrac{t_1-1}{n-1},\tfrac{t_2}{n-1},\dots,\tfrac{t_k}{n-1}\big)+(1,0,\dots,0)=(n-1)u+d.\end{aligned}$$

It follows from Proposition 2.2 that $\psi(x)=u+\sigma(x)$ is an automorphism of $\langle P,f\rangle$.

Consider the map $\phi: U\to\mathrm{Aut}\langle P,f\rangle$ defined by $\phi(\sigma)=\psi$. This map is surjective. In fact, by Proposition 2.1, for every $\psi\in\mathrm{Aut}\langle P,f\rangle$, the map $\sigma(x)=-\psi(c)+\psi(x)$ is an automorphism of the group $ret_c\langle P,f\rangle=\sum_{i=1}^{k}\mathbb{Z}$. Moreover,

$$\begin{aligned}\sigma(d)&=-\psi(c)+\psi(d)=-\psi(c)+\psi(f(c,\dots,c))=-\psi(c)+f(\psi(c),\dots,\psi(c))\\&=-\psi(c)+n\psi(c)+d=(n-1)\psi(c)+d.\end{aligned}$$

If $\sigma(d)=\sigma((1,0,\dots,0)=(t_1,t_2,\dots,t_k)$ and $\psi(c)=(r_1,r_2,\dots,r_k)$, then

$$(t_1,t_2,\dots,t_k)=(n-1)(r_1,r_2,\dots,r_k)+(1,0,\dots,0).$$

Therefore $t_1=(n-1)r_1+1$ and $t_i=(n-1)r_i$ $(i=2,\dots,k)$, i.e., $\sigma\in U$. Thus $\phi(\sigma)=\psi$. So, ϕ is surjective.

It is also is injective. Indeed, since for any automorphism $\sigma_j\in U$ we have $\sigma_j(d)=(t_{j1},t_{j2},\dots,t_{jk})$, where $t_{j1}\equiv 1(\mathrm{mod}\ n-1)$ and $t_{ji}\equiv 0(\mathrm{mod}\ n-1)$, $i=2,\dots,k$, from $\phi(\sigma_1)=\phi(\sigma_2)$ it follows $u_1+\sigma_1(x)=u_2+\sigma_2(x)$ for any $x\in P$, where $u_j=(\frac{t_{j1}-1}{n-1},\frac{t_{j2}}{n-1},\dots,\frac{t_{jk}}{n-1})$, $j=1,2$. Thus, $u_1+\sigma_1(d)=u_2+\sigma_2(d)$, i.e.,

$$\Big(\frac{t_{11}-1}{n-1}+t_{11},\frac{t_{12}}{n-1}+t_{12},\dots,\frac{t_{1k}}{n-1}+t_{1k}\Big)=\Big(\frac{t_{21}-1}{n-1}+t_{21},\frac{t_{22}}{n-1}+t_{22},\dots,\frac{t_{2k}}{n-1}+t_{2k}\Big).$$

Then $\frac{t_{11}-1}{n-1}+t_{11}=\frac{t_{21}-1}{n-1}+t_{21}$ and $\frac{t_{1i}}{n-1}+t_{1i}=\frac{t_{2i}}{n-1}+t_{2i}$ for $i=2,\dots,k$. This means that $nt_{1i}=nt_{2i}$ for $i=1,2,\dots,k$, i.e., $t_{1i}=t_{2i}$. Hence, $u_1=u_2$. Therefore $\sigma_1=\sigma_2$, so ϕ is injective.

Now we have to check that ϕ preserves the group operation. Let $\sigma_1, \sigma_2 \in U$ and $\phi(\sigma_1) = u_1 + \sigma_1(x)$, $\phi(\sigma_2)(x) = u_2 + \sigma_2(x)$. Then

$$(\phi(\sigma_1)\circ\phi(\sigma_2))(x) = \phi(\sigma_1)(\phi(\sigma_2)(x)) = \phi(\sigma_1)(u_2+\sigma_2(x)) = u_1+\sigma_1(u_2)+(\sigma_1\circ\sigma_2)(x).$$

On the other side, if $\phi(\sigma_1 \circ \sigma_2)(x) = u_3 + (\sigma_1 \circ \sigma_2)(x)$, then $u_3 = u_1 + \sigma_1(u_2)$ since the automorphism $\phi(\sigma_1 \circ \sigma_2)$ uniquely determines the automorphism from $\mathrm{Aut}\langle P, f\rangle$. Thus $\phi(\sigma_1) \circ \phi(\sigma_2) = \phi(\sigma_1 \circ \sigma_2)$, which completes the proof. □

The automorphism group of the free abelian group of a finite rank k is isomorphic to the group $GL_k(\mathbb{Z})$ of invertible matrices of order k over the ring of integers $\mathbb{Z}$. Denote by U_k the set of all matrices $[a_{ij}]_k$ from $GL_k(\mathbb{Z})$ such that the element a_{11} is a solution of the congruence $x \equiv 1 \,(\mathrm{mod}\ n-1)$ and other elements of the first row are the solutions of the congruence $x \equiv 0 \,(\mathrm{mod}\ n-1)$, provided $n > 2$.

The set U_k is a subgroup of $GL_k(\mathbb{Z})$ and it is isomorphic to the group U of all automorphisms σ of the free abelian group of a finite rank k satisfying the conditions given in Theorem 5.2. Then from Theorem 5.1 and Theorem 5.2 we get

Corollary 5.3. *The automorphism group of the free k-generated abelian n-ary group is isomorphic to a multiplicative group of invertible matrices U_k of the order k over the ring of integers $\mathbb{Z}$ such that the first element of the first row is congruent to* 1 *modulo* $n-1$ *and the rest of elements in the first row are congruent to* 0 *modulo* $n-1$. □

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Received August 2, 2013

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