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Coset diagrams of the action of a certain Bianchi group on $PL(F_p)$

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Abstract. In this paper, we investigate actions of a certain Bianchi group $PSL_2(O_2)$ on the projective line over the finite field, $PL(F_p)$, by drawing coset diagrams. We prove that $PSL_2(O_2)$ acts on $PL(F_p)$ only if $p-2$ is a perfect square in F_p. We prove that the permutation group (emerging from this) of the action is a subgroup of A_{p+1}, and describe how the connectors connect different fragments occuring in the coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$. We also show that the group each orbit after removing the connectors from these coset diagrams is isomorphic to A_4 and establish formulae to count the number of orbits for each p and prove that the action is transitive.

1. Introduction

A large portion of work on finite group theory, in particular combinatorial group theory, depends upon subgroups of Mobius group. There are several methods which generate some important and interesting subgroups of Mobius group. Let d be a positive square free integer. We suppose O_d is a ring of algebraic integers over the imaginary quadratic numbers $Q(\sqrt{(-d)})$. A *Bianchi group* denoted by $PSL_2(O_d)$ (or Γ_d) is defined as

$$PSL_2(O_d) = \left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} : w, x, y, z \in O_d, wz - xy = 1 \right\}.$$

It is well known that the Bianchi groups are classified into two classes on the basis of the rings O_d having the Euclidean algorithm. The Bianchi group $PSL_2(O_2)$ is one of the five Euclidean Bianchi groups. A good refrence for the Bianchi groups is [5]. The finite presentation of the group $PSL_2(O_2)$ due to [3] is given by

$$PSL_2(O_2) = < a, t, u : a^2 = (at)^3 = (u^{-1}aua)^2 = [t, u] = 1 >$$

where $a : z \to (-1)/z,\ t : z \to z+1,\ u : z \to z + \sqrt{(-2)}$ are the linear fractional transformations. The matrix representation corresponding to each respective lin-

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ear fractional transformation is given as

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & \sqrt{-2} \\ 0 & 1 \end{bmatrix}.$$

By the application of *Tietz transformations* namely,

$$s = at, \quad m = u^{-1}au, \quad v = u^{-1}su,$$

we obtain a new presentation of $PSL_2(O_2)$ as $< a, s, m, v, u : a^2 = s^3 = m^2 = v^3 = (am)^2 = (sv^{-1})^2 = 1,\ m = u^{-1}au, v = u^{-1}su, am = sv^{-1} >$, where $a : z \to (-1)/z$, $s : z \to \frac{-1}{z+1}$, $m : z \to \frac{-\sqrt{-2}z+1}{z+\sqrt{-2}}$, $v : z \to \frac{-\sqrt{-2}z+(1-\sqrt{-2})}{z+(1+\sqrt{-2})}$, $u : z \to z + \sqrt{-2}$ are the linear fractional transformations.

The main idea behind the theory of group amalgams is to decompose, if possible, an infinite group into an amalgam of some of its subgroups. Therefore, amalgam decomposition is equally important in infinite group theory as a prime factorization theorem in number theory, although the amalgam decomposition of a group need not be unique. There are two main approaches in the theory of a group amalgams. One of them is combinatorial approach which deals with presentation for the groups and its factors. A complete behavior of the combinatorial approach is given in [7]. Whereas, the second one is a powerful geometric method due to [2]. It is well known that $PSL_2(O_2)$ can be decomposed as a free product of G_1 and G_2 with amalgamated subgroup H written as $\Gamma_2 = G_1 *_H G_2$, where G_1 and G_2 are HNN groups of Klien-4 group D_2 and the alternating group A_4 and $H = Z * Z_2$.

One of the graphical methods to study groups is the coset diagrams. It is used to study various properties of a group by taking its actions on fields, quadratic fields and sets. The Euclidean Bianchi groups are the natural algebraic generalization of the extensively studied modular group. G. Higman initiated the study of coset diagrams for the modular group and the extended modular group. Further Q. Mushtaq [8] proved many important results for the modular group using coset diagrams. For more on coset diagrams one can refer to [1], [4] and [8].

2. Action of $\mathbf{PSL_2(O_2)}$ on $\mathbf{PL(F_p)}$

Every odd prime of the sequence in which -2 is a perfect square can be expressed as $p = 4n + 1$ if n is even, or $p = 4n - 1$ if n is odd, $n \in N$. The only even prime 2 is also in the sequence. Such primes are called the *$M - S$ primes.*

We observe from the following theorem that $PSL_2(O_2)$ acts on $PL(F_p)$ only if $p - 2$ is a perfect square in F_p.

Theorem 2.1. *The group $PSL_2(O_2)$ acts on $PL(F_p)$ only if $p - 2$ is a perfect square in F_p, where p is an M-S prime.*

Proof. $PSL_2(O_2) = < a, s, m, v, u : a^2 = s^3 = m^2 = v^3 = (am)^2 = (sv^{-1})^2 = 1,\ m = u^{-1}au,\ v = u^{-1}su,\ am = sv^{-1} >$, where $a : z \to \frac{-1}{z}$, $s : z \to \frac{-1}{z+1}$, $m = z \to \frac{-\sqrt{-2}z+1}{z+\sqrt{-2}}$, $v : z \to \frac{-\sqrt{-2}z+(1-\sqrt{-2})}{z+(1+\sqrt{-2})}$.

The linear fractional transformations a, s, m, v convert into because of the modular calculations.

Since $-2 \equiv p-2 (\text{mod } p)$ therefore, implies that $\sqrt{-2} \equiv \sqrt{p-2} (\text{mod } p)$, yielding m and v as $m = \frac{-\sqrt{p-2}z+1}{z+\sqrt{p-2}}$ and $v : z \to \frac{-\sqrt{p-2}z+(1-\sqrt{p-2})}{z+(1+\sqrt{p-2})}$.

The transformations a and s map elements of $PL(F_p)$ onto the elements of $PL(F_p)$ without any condition on p, whereas transformations m and v map elements of $PL(F_p)$ onto elements of $PL(F_p)$ only if $p-2$ is a perfect square in F_p because of the occurence of $\sqrt{p-2}$ in the transformations $m(z)$ and $v(z)$. Hence $PSL_2(O_2)'$ acts on $PL(F_p)$ only if $p-2$ is a perfect square in F_p. □

Remark 2.2. In the action of $PSL_2(O_2)$ on $PL(F_p)$,

(i) fixed points of the transformations a and m exist only if -1 is a perfect square (mod p),

(ii) fixed points of the transformations s and v exist only if -3 is a perfect square (mod p).

Theorem 2.3. *There does not exist X such that $X^2 = (AX)^2 = (SX)^2 = (MX)^2 = (VX)^2 = 1$.*

Proof. Let on the contrary, X be such that $X^2 = (AX)^2 = (SX)^2 = (MX)^2 = (VX)^2 = 1$. Suppose that $X(z) = \frac{az+b}{cz+d}$, that is, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The matrix representations of the linear fractional transformations a, s, m and v are as follow

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},\ M = \begin{bmatrix} -\sqrt{-2} & 1 \\ 1 & \sqrt{-2} \end{bmatrix},\ V = \begin{bmatrix} -\sqrt{-2} & 1-\sqrt{-2} \\ 1 & 1+\sqrt{-2} \end{bmatrix},$$

$X^2 = 1$, and a matrix of $GL(2, \mathbb{C})$ is of order 2 if and only if its trace is zero, or $tr(X) = a + d = 0$.

Also $AX = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix}$ and since it is assumed that $(AX)^2 = 1$, therefore $tr(AX) = 0$, that is, $b = c$.

Similarly $SX = \begin{bmatrix} -c & -d \\ a+c & b+d \end{bmatrix}$, and supposition, $(SX)^2 = 1$ implies that $tr(SX) = 0$, that is, $b + d = c$.

Again $MX = \begin{bmatrix} -\sqrt{-2}a+c & -\sqrt{-2}+b \\ a+\sqrt{-2}c & b+\sqrt{-2}d \end{bmatrix}$ and due to the supposition, $(MX)^2 = 1$, we get $tr(MX) = 0$, that is, $\sqrt{-2}a + c + b + \sqrt{-2}d = 0$.

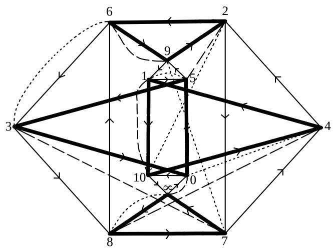
Similarly, $VX = \begin{bmatrix} -\sqrt{-2}a + (1-\sqrt{-2})c & -\sqrt{-2}b + (1-\sqrt{-2})d \\ a + (1+\sqrt{-2})c & b + (1+\sqrt{-2})d \end{bmatrix}$ and by supposition, $(VX)^2 = 1$, we get $tr(VX) = 0$, that is, $-\sqrt{-2}a + (1-\sqrt{-2})c + b + (1+\sqrt{-2})d = 0$.

From $b = c$ and $b + d + c = 0$, we get $d = 0$. But $a + d = 0$ implies that $a = 0$. This together with the case of MX shows that X is a zero matrix.

Thus, there does not exist any non-zero transformation like $X(z)$. □

The coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$, where p is an $M-S$ prime, are made of four generators $\overline{a}$, $\overline{s}$, $\overline{m}$ and $\overline{v}$, where $\overline{a}$, $\overline{s}$, $\overline{m}$ and $\overline{v}$ are images of $a, s.m$ and v under the action. We denote these generators graphically as follows. The three cycles of the permutations $\overline{s}$ are represented by triangles having solid lines whereas $\overline{v}$ is represented by triangles having edges consisiting of bold solid lines. The involution $\overline{a}$ is denoted by broken edges and $\overline{m}$ is denoted by dotted edges. Fixed points are represented by heavy dots if they exist. Each diagram represents finite, non-abelian and simple subgroups of A_{p+1}, for $p \geqslant 11$. We denote the permutation subgroup emerging from the action of $PSL_2(O_2)$ on $PL(F_p)$ by $\overline{\Gamma_2}$.

For example the following coset diagram depicts action of $PSL_2(O_2)$ on $PL(F_{11})$



Theorem 2.4. *The action of $PSL_2(O_2)$ on $PL(F_p)$ gives a permutation group generated by $\overline{a}, \overline{s}, \overline{m}$ and $\overline{v}$ with relations $(\overline{a})^2 = (\overline{s})^3 = (\overline{m})^2 = (\overline{v})^3 = (\overline{am})^2 = (\overline{sv^{-1}})^2 = 1$, as a subgroup of A_{p+1}, where p is an M-S prime.*

Proof. Note that $\overline{\Gamma_2}$ is generated by the permutations $\overline{a}$, $\overline{s}$, $\overline{m}$ and $\overline{v}$ corresponding to the linear fractional transformations a, s, m and v in $PSL_2(O_2)$, where $\overline{s}$ and $\overline{v}$ are products of cycles each of length three. So every cycle can be decomposed into even number of transpositions. Thus $\overline{s}$ and $\overline{v}$ are even permutations. Since $\overline{a}$ and $\overline{m}$ are involutions, therefore, the number of transpositions of $\overline{a}$ and $\overline{m}$ is $\frac{p+1}{2}$ which is an even number if $\overline{a}$ and $\overline{m}$ have no fixed points in F_p. If $\overline{a}$ and $\overline{m}$ have fixed points which exist only if -1 is a perfect square (mod p), then the number of transpositions inf $\overline{a}$ and $\overline{m}$ is $\frac{p-1}{2}$ which is also an even number. This implies that $\overline{\Gamma_2}$ is generated by even permutations and hence $\overline{\Gamma_2}$ is a subgroup of A_{p+1}. □

3. Connectors for the coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$

There are two connectors, namely, C_1 and C_2 of the coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$. The connectors C_1 and C_2 graphically represent the behavior of the linear fractional transformations a and m respectively in these diagrams. Since the mappings $a : z \to (-1)/z$, and $m : z \to \frac{-\sqrt{-2}z+1}{z+\sqrt{-2}}$ are bijective, therefore, the corresponding connectors C_1 and C_2 join each vertex of a fragment to the other vertex in a unique way. Initially, the coset diagrams contain different orbits. Each orbit represent the alternating group A_4. When we start joining these orbits through the connectors C_1 and C_2, the diagrams start becoming connected. Once all these orbits are completely joined by C_1 and C_2, we obtain connected coset diagrams depicting subgroups of A_{p+1}. The connected diagrams also show that the action of $PSL_2(O_2)$ on $PL(F_p)$ is transitive. The connector C_1 is represented by broken edges and the connector C_2 is represented by doted edges in the coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$. The algebraic effects of these connectors on the diagrams are same. This means that if we drop one of these connectors from the diagrams, the algebraic properties of the groups depicting from these diagrams do not change. However, if we remove both the connectors from the diagrams, we obtain different orbits of the complete coset diagram. Let G denote the group represented by each such orbit then

$$G =< s, v : s^3 = v^3 = (sv^{-1})^2 = 1 > .$$

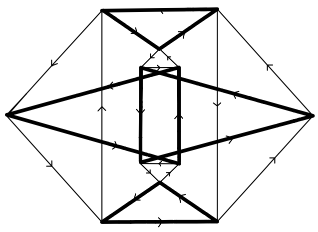
In the following result, we note the above stated fact.

Theorem 3.1. *The group* $G=<s, v : s^3 = v^3 = (sv^{-1})^2 = 1>$ *is isomorphic to* A_4.

Proof. Since $G =< s, v : s^3 = v^3 = (sv^{-1})^2 = 1 >$, we put $s = ab$, $v = b$ and obtain $G =< a, b, s, v : s^3 = v^3 = (sv^{-1})^2 = 1, s = ab, v = b >=< a, b : (ab)^3 = b^3 = (abb^{-1})^2 = 1 >=< a, b : a^2 = b^3 = (ab)^3 = 1 >$, which is isomorphic to A_4. □

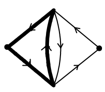
There are three types of fragments of G which occurr in the coset diagrams for the action of $PSL_2(O_2)$ on $PL(F_p)$. Some details of these fragments are given below.

The fragment Γ_1 :



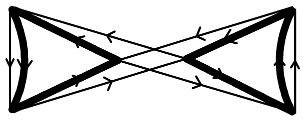
This fragment exists in each coset diagram for the action of $PSL_2(O_2)$ on $PL(F_p)$ for all p except $p = 3$.

The fragment Γ_2 :



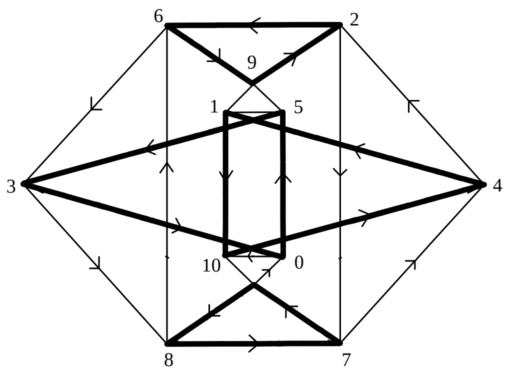
This fragment occurs along with fragment Γ_1 in those coset diagrams in which -3 is a perfect square modulo p.

The fragment Γ_3 :



This fragment occurs along with fragment Γ_1 in those coset diagrams when -1 is a perfect square modulo p.

Consider the following fragment of the action of $PSL_2(O_2)$ on $PL(F_{11})$ after removing all connectors.



The following result is about the number of orbits as copies of A_4 occurring in the coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ after removing the connectors.

Theorem 3.2. *In the action of $PSL_2(O_2)$ on $PL(F_p)$,*

(*i*) *if -1 is a perfect square* (mod p), *then number of copies of A_4 are* $(\frac{p-5}{12})+1$,

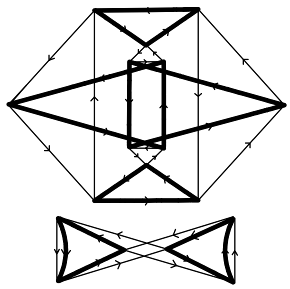
(*ii*) *if -3 is a perfect square* (mod p), *then number of copies of A_4 are* $(\frac{p+5}{12})+1$, *except for $p=3$,*

(*iii*) *if neither -1 nor -3 are perfect squares* (mod p), *then number of copies of A_4 are* $(\frac{p+1}{12})$,

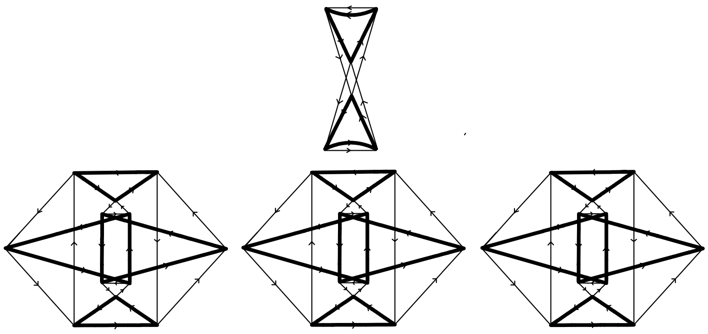
(*iv*) *if both -1 and -3 are perfect squares* (mod p), *then number of copies of A_4 are* $(\frac{p-1}{12})+2$.

Proof. Case (i): The subsequence of the sequence of M-S prime in which -1 is a perfect square (mod p) is given by $x_1=\{17,41,89,\ldots\}$.

The number of copies of A_4 for $p=17$ is 2 that is $(\frac{17-5}{12})+1$. The following diagram indicates the above fact



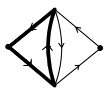
Number of copies of A_4 for $p = 41$ is 4 that is $(\frac{41-5}{12})+1$. The following diagram indicates the above fact.



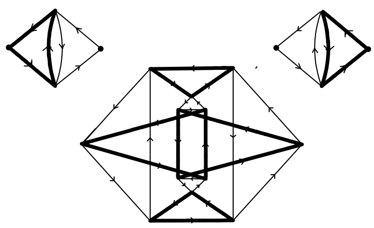
Continuation of the above process leads us to note that there exist $(\frac{p-5}{12}) + 1$ copies of A_4 in this case.

Case (ii): The subsequence of the sequence of M-S primes in which -3 is a perfect square (mod p) is given by $\pi_2 = \{3, 19, 43, 67, \ldots\}$.

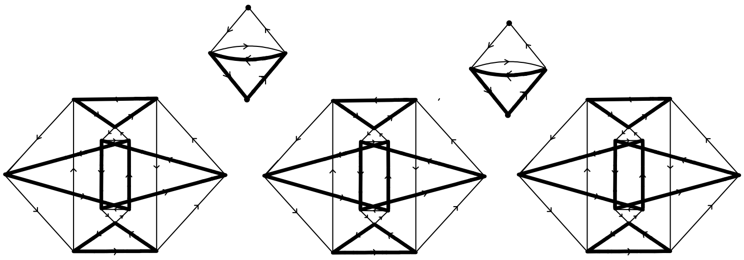
One can easily observe that only one copy of A_4 exists for $p = 3$. The following diagram indicates the above argument.



Number of copies of A_4 for $p = 19$ is 3, that is, $(\frac{19+5}{12}) + 1$. The following diagram depicts the above fact.



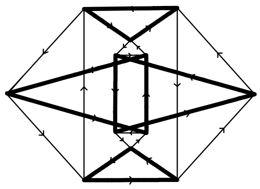
Number of copies of A_4 for $p = 43$ is 5, that is, $(\frac{43+5}{12}) + 1$. The following diagram depicts the above fact.



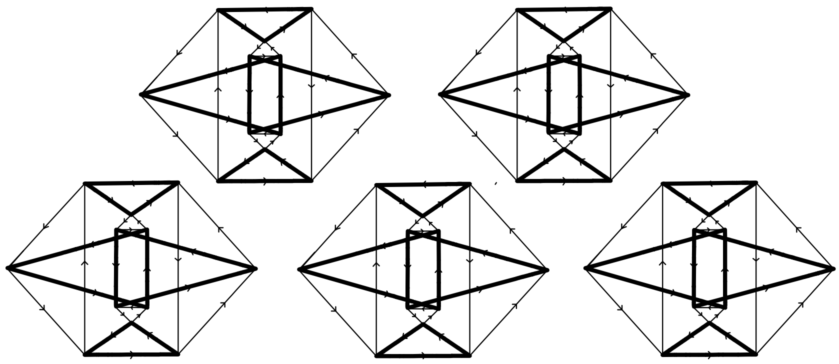
Continuation of the above process leads us to the conclusion that in this case there exist $(\frac{p+5}{12}) + 1$ copies of A_4.

Case (iii): The subsequence of the sequence of M-S primes in which neither -1 nor -3 are perfect squares (mod p) is given by $\pi_3 = \{11, 59, 83, \ldots\}$.

Number of copies of A_4 exist for $p = 11$ is one that is, $(\frac{11+1}{12})$. The following diagram indicates the above fact.

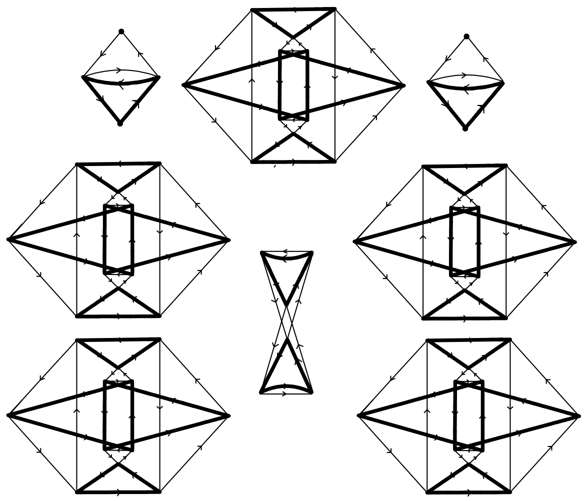


The number of copies of A_4 for $p = 59$ is 5, that is $(\frac{59+1}{12})$. The following diagram depicts the above fact.



Continuation of the above process leads us to the conclusion that in this case there exist $(\frac{p+1}{12})$ copies of A_4.

Case (iv): The subsequence of the sequence of M-S primes in which both -1 and -3 are perfect squares (mod p) is given by $\{73, 97, \ldots\}$ copies of A_4 exist for $p = 73$ is 8 that is $(\frac{73-1}{12}) + 2$. The following diagram illustrates the above fact.



The continuation of this process shows that in this case there exist $(\frac{p-1}{12})+2$ copies of A_4. □

4. Transitivity of the action of $PSL_2(O_2)$ on $PL(F_p)$

In this section we study transitivity of the action of $PSL_2(O_2)$ on $PL(F_p)$.

Theorem 4.1. *The function $f: F_p \times \{p\} \to F_p \backslash \{0\}$, defined by $f(n,p) = \frac{p-1}{(p-1)+n}$, is injective, for a prime p.*

Proof. Let $m, n \in F_p$ and $f(m) = f(n)$. Then $\frac{p-1}{(p-1)+m} = \frac{p-1}{(p-1)+n}$ implies that $m = n$, that is f is injective. □

Theorem 4.2. *The action of $PSL_2(O_2)$ on $PL(F_p)$ is transitive.*

Proof. We begin from vertex 1. Since the transformation sa maps 1 to $\frac{p-1}{p}$, so there is a path from 1 to $\frac{p-1}{p} = \frac{p-1}{(p-1)+1}$. Again application of the transformation sa on $\frac{p-1}{p}$ yields $\frac{p-1}{p+1} = \frac{p-1}{(p-1)+2}$. Therefore, we find a path from 1 to $\frac{p-1}{p+1} = \frac{p-1}{(p-1)+2}$. The continuation of the above process provides us a path from the vertex 1 to $\frac{p-1}{(p-1)+n}$, where $n = 0, 1, 2, \ldots, p-1$. By Theorem 4.1, all the vertices $\frac{p-1}{(p-1)+n}$, where $n = 0, 1, 2, \ldots, p-1$ are distinct and none of these is equal to 0. By applying the transformation a on $\frac{p-1}{(p-1)+1}$, we reach at the vertex 0. Hence the action is transitive. □

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