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Groups with the same orders and large character degrees as $\mathbf{PGL}(2,9)$

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Abstract. An interesting class of problems in character theory arises from considering how the structure of a group G and the set $\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ are related. It is proved that some finite groups are uniquely determined by their character tables. Recently Xu, Chen and Yan proved that if G is a simple K_3-group, then G is uniquely determined by its order and one or both of its largest irreducible character degrees.

In this paper, we determine groups with the same order and the same largest and second largest irreducible character degrees as $\mathrm{PGL}(2,9)$ and as a consequence of our result it follows that $\mathrm{PGL}(2,9)$ is characterizable by the structure of its complex group algebra.

1. Introduction and preliminary results

Let G be a finite group, $\mathrm{Irr}(G)$ be the set of irreducible characters of G, and denote by $\mathrm{cd}(G)$, the set of irreducible character degrees of G. An interesting class of problems in character theory arises from considering the relation between the structure of a group G and the set $\mathrm{cd}(G)$. Many authors were recently concerned with the following question: What can be said about the structure of a finite group G, if some information is known about the arithmetical structure of the degrees of the irreducible characters of G?

We know that there are 2328 groups of order 2^7 and these groups have only 30 different degree patterns, and there are 538 of them with the same character degrees (see [4]). However, in the late 1990s, Bertram Huppert conjectured that each finite nonabelian simple group G is essentially determined by $\mathrm{cd}(G)$, and he posed the following conjecture:

Conjecture. *Let H be a nonabelian simple group and G be a finite group such that $cd(G) = cd(H)$. Then $G \cong H \times A$, where A is an abelian group.*

In [1, 4, 5, 8, 9, 10, 11, 12, 13] it is proved that this conjecture holds for some simple groups.

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Later in [2] it is proved that all finite simple groups can be uniquely determined by their character tables. A finite group G is called a K_3-group if $|G|$ has exactly three distinct prime divisors. Recently Chen et. al. in [14] characterized all simple K_3-groups by less information and they proved that these groups are uniquely determined by their orders and one or both of its largest and second largest irreducible character degrees. Also in [15] groups with the same order, largest and second largest irreducible character degrees as an automorphism group of a K_3 simple group are determined. It is proved that A_5, $L_2(7)$, $L_2(17)$, $L_3(3)$ and $U_4(2)$ can be uniquely determined by their orders and the largest degrees of their irreducible characters. Also it is proved that A_6 is characterized by its order and the second largest degree of its irreducible characters. Finally it is proved that $L_2(8)$ and $U_3(3)$ are characterizable by their orders and the largest and the second largest degrees of their irreducible characters.

In this paper we determine all finite simple groups with the same order, the largest and the second largest degrees of their irreducible characters as $\mathrm{PGL}(2,9)$. As a consequence of our results, it follows that $\mathrm{PGL}(2,9)$ is uniquely determined by the structure of its complex group algebra.

Throughout the paper, for a finite group G we denote by χ, the irreducible character with the largest irreducible character degree, and by β, the irreducible character with the second largest irreducible character degree. Specially we used the notations of [3]. For a prime number p, we denote by $n_p(G)$, the number of Sylow p-subgroups of G. All other notations are standard and again we refer to [3].

Lemma 1. [14, Lemma] *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.* □

2. Main Results

Theorem 1. *Let G be a finite group such that $|G| = |\mathrm{PGL}(2,9)| = 2^4 3^2 5$, $\chi(1) = 10$ and $\beta(1) = 9$. Then $G \cong \mathrm{PGL}(2,9)$, $G \cong \mathbb{Z}_2 \times A_6$ or $2 \cdot A_6$.*

Proof. First we prove that G is a nonsolvable group.

Let G be a solvable group. If $Q = O_5(G) \neq 1$, then $|Q| = 5$ and by Ito's theorem we have $10 = \chi(1) \mid |G : Q|$, which is a contradiction. Therefore $O_5(G) = 1$. Also if $P = O_3(G) \neq 1$, then $|P| = 3$ or $|P| = 9$. Since in each case P is abelian, again by Ito's theorem we get that $9 = \beta(1) \mid |G : P|$, which is impossible. Therefore $O_3(G) = 1$ and since G is a solvable group we conclude that $O_2(G) \neq 1$.

Now let M be the normal minimal subgroup of G. Then M is a 2-elementary abelian subgroup of G. Therefore $|O_2(G)| = 2^\alpha$, where $1 \leqslant \alpha \leqslant 4$.

If $|M| = 2^4$, then since M is abelian, by Ito's theorem we get $10 = \chi(1) \mid |G : M|$, which is a contradiction.

If $|M| = 2^\alpha$, where $\alpha = 2$ or 3, then $|G/M| = 2^{4-\alpha}3^2 5$. Let S/M be a Hall subgroup of G/M of order $3^2 5$. Then $|G/M : S/M| = 2^{4-\alpha} \leqslant 4$. Therefore if $T/M = \mathrm{Core}_{G/M}(S/M)$, then $|G/M : T/M| \mid |S_4|$, which implies that $15 \mid |T/M|$, since $3^2 \nmid |S_4|$. Also $|T/M| \mid |S/M| = 3^2 5$. Therefore $|T/M| = 3^2 5$ or $|T/M| = 15$. Let $Q/M \in \mathrm{Syl}_5(T/M)$. Then Q/M ch $T/M \lhd G/M$, which implies that $Q \lhd G$, where $|Q| = 2^\alpha 5$ and $2 \leqslant \alpha \leqslant 3$. Let $P \in \mathrm{Syl}_5(Q)$. Then $P \lhd Q$ and so $P \lhd G$, which implies that $O_5(G) \neq 1$, a contradiction.

Hence $|M| = 2$. Let H be a Hall subgroup of G of order $2^4 3^2$. Then $|G : H| = 5$ and so $G/H_G \hookrightarrow S_5$. Also G/H_G is a solvable group and we know that the orders of solvable subgroups of S_5 which is divisible by 5 are 5, 10 or 20. So we conclude that $|H_G| = 2^4 3^2$, $2^3 3^2$ or $2^2 3^2$. Consider $\theta \in \mathrm{Irr}(H_G)$ such that $[\beta_{H_G}, \theta] \neq 0$. Then

$$\frac{9}{\theta(1)} = \frac{\beta(1)}{\theta(1)} \mid |G : H_G|,$$

implies that $9 \mid \theta(1)$. Therefore $81 = \theta(1)^2 \leqslant |H_G| \leqslant |H|$, and so the only possibility is $|H_G| = |H| = 2^4 3^2$. Hence $H \lhd G$ and $M \leqslant H$. Also $|H/M| = 2^3 3^2$. Then $n_3(H/M) = 1$ or 4. Let $Q/M \in \mathrm{Syl}_3(H/M)$. If $n_3(H/M) = 1$, then Q/M ch $H/M \lhd G/M$ and so $Q \lhd G$ and $|Q| = 18$. Now if P is the Sylow 3-subgroup of Q, then $P \lhd G$ and so $O_3(G) \neq 1$, a contradiction. Therefore $n_3(H/M) = 4$. Let $R/M = N_{H/M}(Q/M)$. Then $|H/M : R/M| = 4$ and so $|R/M| = 18$. Also $Q/M \lhd R/M \leqslant H/M$. Let $T/M = \mathrm{Core}_{H/M}(R/M)$. Then $(H/M)/(T/M) \hookrightarrow S_4$ and since $3^2 \nmid |S_4|$, it follows that $3 \mid |T/M|$. Let $L/M \in \mathrm{Syl}_3(T/M)$. Since $|T/M| \mid |R/M| = 18$, we conclude that L/M ch $T/M \lhd H/M$ and so $L/M \lhd H/M$. Therefore $O_3(H/M) \neq 1$ and since $n_3(H/M) = 4$, we get that $|O_3(H/M)| = 3$ and so $O_3(H/M) = L/M$ ch H/M, which implies that $L \lhd G$. Now if $P \in \mathrm{Syl}_3(L)$, then $|L| = 2|P|$ and so $P \lhd G$, which implies that $O_3(G) \neq 1$, a contradiction.

Therefore G is not a solvable group. Now using Lemma 1 we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups we get that $K/H \cong A_5$ or A_6.

Case 1. Let $K/H \cong A_5$. Then $|G/K| \mid 2$ and so $|H| = 2^\alpha 3$, where $\alpha = 1$ or $\alpha = 2$.

For $|H| = 6$, there exists $\theta \in \mathrm{Irr}(H)$ such that $[\beta_H, \theta] \neq 0$. Then we have $9/\theta(1) \mid |G : H|$, which implies that $\theta(1) = 3$. Hence $9 = \theta(1)^2 < |H| = 6$, a contradiction. Therefore $|H| = 12$ and $|G/K| = 1$. Similarly to the above we conclude that there exists $\theta \in \mathrm{Irr}(H)$ such that $\theta(1) = 3$. Now $|H| = 12$, implies that H has three linear character and $|\mathrm{Irr}(H)| = 4$. Let Q be a Sylow 3-subgroup of H. Then $n_3(H) = 1$ or 4.

If $n_3(H) = 1$, then $Q \lhd H$ and so there exists $\varphi \in \mathrm{Irr}(Q)$ such that $[\theta_Q, \varphi] \neq 0$. Hence $3 = \theta(1) \mid |H : Q| = 4$, which is impossible. Therefore $n_3(H) = 4$ and since H acts on its four Sylow 3-subgroups by conjugation, it follows that $H \hookrightarrow S_4$ and so $H \cong A_4$.

On the other hand, there exists $\psi \in \mathrm{Irr}(H)$ such that $e = [\chi_H, \psi] \neq 0$. Since $10/\psi(1) = \chi(1)/\psi(1) \mid |G : H|$, it follows that ψ is a linear character of H. Therefore $10 = \chi(1) = et\psi(1) = et$, where $t = |G : I_G(\psi)|$. We know that $C_G(H) \subseteq I_G(\psi)$ and so $t \mid |Aut(H)| = 24$. Then $t = 1$ or $t = 2$. If $t = 1$, then $e = 10$ and so

$$[\chi_H, \chi_H] = e^2 t = 100 > |G : H| = 60,$$

which is a contradiction. Therefore $t = 2$ and $e = 5$. Now $t = |G : I_G(\psi)| = 2$ and since $H \leqslant I_G(\psi)$, it follows that $I_G(\psi)/H$ is a normal subgroup of $G/H \cong A_5$ such that $|G/H : I_G(\psi)/H| = 2$, which is a contradiction since A_5 is a simple group. Therefore $G/H \ncong A_5$.

Case 2. Let $K/H \cong A_6$. Then $|G/K| = 2$ or $|H| = 2$.

If $|G/K| = 2$, then $G \cong A_6 \cdot 2_1 \cong S_6$, $A_6 \cdot 2_2 \cong \mathrm{PGL}(2,9)$ or $A_6 \cdot 2_3$. Now using [3] we can see that S_6 and $A_6 \cdot 2_3$ have irreducible characters of degree 16.

If $|H| = 2$, then since the Schur multiplier of A_6 is 6, we get that $G \cong A_6 \times \mathbb{Z}_2$ or $2 \cdot A_6$. Now using [3] we can see that these groups have order 120 and the same $\chi(1)$ and $\beta(1)$ as $\mathrm{PGL}(2,9)$. □

The degree pattern of G, which is denoted by $X_1(G)$ is the set of all irreducible complex character degrees of G counting multiplicities. We note that $X_1(G)$ is the first column of the ordinary character table of G. As a consequence of these results we get the following result:

Corollary 1. *Let G be a finite group such that $X_1(G) = X_1(\mathrm{PGL}(2,9))$. Then $G \cong \mathrm{PGL}(2,9)$.* □

The following result is an answer to the question arose in [7].

Corollary 2. *Let $H = \mathrm{PGL}(2,9)$. If G is a group such that $\mathbb{C}G \cong \mathbb{C}H$, then $G \cong \mathrm{PGL}(2,9)$. Thus $\mathrm{PGL}(2,9)$ is uniquely determined by the structure of its complex group algebra.* □

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