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E-disjunctive semigroups and idempotent pure congruences

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Abstract. An equivalence relation ρ on a semigroup S is called *idempotent pure* if $e\rho \subseteq E_S$ for all $e \in E_S$, where

$$E_S = \{a \in S : a^2 = a\}$$

is the set of *idempotents* of S, and S is said to be *E-disjunctive* if the identity relation 1_S is the largest idempotent pure congruence on S. Further, we say that an element a of S is *E-inversive* if $ax \in E_S$ for some $x \in S$, and S is *E-inversive* if each of its elements is E-inversive.

We first prove that an arbitrary equivalence class of the largest idempotent pure congruence on a semigroup S either consists entirely of E-inversive elements or has no E-inversive elements of S, and then that each E-disjunctive semigroup is necessarily E-inversive. Moreover, in some special classes $\mathcal{C}$ of semigroups, which are contained in the class of E-inversive semigroups, we investigate the connections between the largest idempotent pure congruence, the least group congruence and the relation $(S \in \mathcal{C})$

$$\chi = \{(a, b) \in S \times S : I(a) = I(b)\},$$

where $I(s) = \{x \in S : sx, xs \in E_S\}$ $(s \in S)$. Using some of those connections, we give certain new characterizations of the least group (Clifford) congruence.

1. Preliminaries

Groups are obvious examples of E-disjunctive semigroups. Recall from [11] that in an arbitrary semigroup S the relation

$$\tau = \{(a, b) \in S \times S : (\forall x, y \in S^1)\ xay \in E_S \Leftrightarrow xby \in E_S\},$$

where S^1 denotes the monoid obtained from S by adjoining the identity 1, is the largest idempotent pure congruence. In the case $\tau = 1_S$, S is *E-disjunctive* [16]. In [7] it has been shown that if S is *idempotent-surjective* (i.e., each idempotent congruence class of S contains an idempotent of S [1]), then S/τ is E-disjunctive. In fact, for every congruence ρ on S, $S/\tau(\rho)$ is E-disjunctive, where the relation

$$\tau(\rho) = \{(a, b) \in S \times S : (a\rho, b\rho) \in \tau\}$$

is the greatest congruence on S with respect to $\ker(\rho) = \bigcup_{e \in E_S} e\rho$.

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Clearly, a congruence ρ on an idempotent-surjective semigroup S is idempotent pure if and only if $\ker(\rho) = E_S$.

The concept of an E-inversive semigroup was introduced by Thierrin in the paper [17]. It is well-known that S is E-inversive if and only if the set

$$W(a) = \{x \in S : x = xax\}$$

is non-empty for every $a \in S$. Notice that if $x \in W(a)$, then both elements ax, xa are idempotents, therefore, S is E-inversive if and only if $I(a) \neq \emptyset$ for every $a \in S$. Known examples of E-inversive semigroups are:

(a) *idempotent-surjective semigroups* [10];

(b) *regular-surjective semigroups*, see [5] for the definition of such semigroups;

(c) *regular semigroups* (a semigroup S is said to be *regular* if the set

$$\mathrm{Reg}(S) = \{a \in S : a \in aSa\}$$

of all regular elements of S coincides with S);

(d) *eventually regular semigroups* (a semigroup S is called *eventually regular* if every element of S has a regular power [1]; in particular, all finite semigroups are eventually regular);

(e) *structurally regular semigroups* (for the definition and numerous examples of such semigroups, see [12]).

The semigroups from (c) through (e) are idempotent-surjective [1, 12].

For some interesting results concerning E-inversive semigroups we refer the reader to the papers [2], [14] and [15].

In [9] Hall observed that the set $\mathrm{Reg}(S)$ of a semigroup S with $E_S \neq \emptyset$ forms a regular subsemigroup of S if and only if $E_S E_S \subseteq \mathrm{Reg}(S)$. In that case, we say that S is an *R-semigroup*. Recall from [5] that all structurally regular semigroups are (idempotent-surjective) R-semigroups. Moreover, if E_S forms a subsemigroup of S, then S is called an *E-semigroup*. Obviously, E-semigroups are R-semigroups. Finally, (eventually) regular E-semigroups are said to be *(eventually) orthodox*.

Recall from [11] that a semigroup S with $E_S \neq \emptyset$ is *left E-unitary* if for all $a \in S$ and $e \in E_S$, the condition $ea \in E_S$ implies $a \in E_S$. The notion of a *right E-unitary* semigroup is defined dually. Finally, S is *E-unitary* if it is both left and right unitary. In [4] it has been proved that an E-inversive semigroup is E-unitary if and only if it is left (right) unitary.

It is well-known that any E-inversive semigroup S possesses the least *group* congruence σ (that is, S/σ is a group), see e.g. [4].

Result 1.1. [4] *The following conditions concerning an E-inversive semigroup S are equivalent*:

(a) *S is E-unitary*;

(b) $\tau = \sigma$;

(c) $\ker(\sigma) = E_S$.

In particular, every E-unitary E-inversive semigroup is an E-semigroup. □

In the light of the above result, an E-inversive semigroup is a group if and only if it is both E-unitary and E-disjunctive. We shall generalize this statement to an arbitrary semigroup (Corollary 2.3, below).

By a *semilattice* we shall mean a commutative semigroup in which every element is an idempotent. Let $\mathcal{C}$ be a fixed class of semigroups (call its elements *$\mathcal{C}$-semigroups*). Recall that a semigroup is a *semilattice of $\mathcal{C}$-semigroups* if there exists a semilattice congruence ρ on S (i.e., S/ρ is a semilattice) such that each ρ-class of S is a $\mathcal{C}$-semigroup. In particular, if every ρ-class of S is a group, then we say that S is a *semilattice of groups*. It is very well-known that S is a semilattice of groups if and only if S is a *Clifford* semigroup [11] (that is, S is regular and its idempotents are central). LaTorre in [13] studied the least semilattice of groups congruence ξ on regular semigroups, and then in [3, 8] the author described all Clifford congruences on idempotent-surjective R-semigroups and on eventually regular semigroups, as well as on perfect semigroups [6].

Finally, we shall say that a semigroup S is:

(a) *E-reflexive* if for all $a, b \in S$,

$$ab \in E_S \implies ba \in E_S;$$

(b) *strongly E-reflexive* if for all $a, b \in S, e \in E_{S^1}$,

$$eab \in E_S \implies eba \in E_S.$$

It is obvious that every strongly E-reflexive semigroup is E-reflexive. Moreover, it is easily seen (cf. (c) in Result 1.1) that each E-unitary E-inversive semigroup is strongly E-reflexive. An another example of a strongly E-reflexive semigroup is a semilattice of E-unitary (eventually) regular semigroups, see [8, 13] (notice that these semigroups are both E-semigroups). In particular, all Clifford semigroups are strongly E-reflexive.

2. E-disjunctive semigroups

Recall from [11] that if $\mathcal{R}$ is an equivalence relation on a semigroup S, then the relation

$$\mathcal{R}^\flat = \{(a, b) \in S \times S : (\forall x, y \in S^1)\ xay\, \mathcal{R}\, xby\}$$

is the greatest congruence contained in $\mathcal{R}$.

Let S be a semigroup. Denote by A the set of all elements of S which are *not* E-inversive. Evidently, A is either empty or it is an ideal of S. Furthermore, put $B = S \setminus (A \cup E_S)$ and consider the equivalence relation λ on S induced by the partition $\{A, B, E_S\}$. Obviously, λ is idempotent pure.

In the following result we give a new characterization of the largest idempotent pure congruence in an arbitrary semigroup.

Theorem 2.1. *Let ρ be a congruence on a semigroup S. Then ρ is idempotent pure if and only if $\rho \subseteq \lambda$. In particular, $\tau = \lambda^{\flat}$. Moreover, any τ-class of S either consists entirely of E-inversive elements of S or has no E-inversive elements of S. Finally, if $A \neq \emptyset$, then A is a τ-class of S.*

Proof. We show that $\tau \subseteq \lambda$. Let $(a, b) \in \tau$. If neither a nor b is E-inversive, then $a, b \in A$, therefore, $(a, b) \in \lambda$. If either a or b is E-inversive (say a), then $ax\ E_S$ for some x in S and $(ax, bx) \in \tau$. This implies that $bx \in E_S$, therefore, $(a, b) \in \lambda$. Consequently, if ρ is idempotent pure, then $\rho \subseteq \tau \subseteq \lambda$. Conversely, if $\rho \subseteq \lambda$, then obviously ρ is idempotent pure. Clearly, $\tau = \lambda^{\flat}$.

Suppose that $A \neq \emptyset$ and let $\{B_i : i \in I\}$ and $\{E_j : j \in J\}$ be the collections of all $\lambda^{\flat}$-classes contained in B and E_S, respectively (notice that these collections may be both empty). Next, observe that the partition

$$\{A, B_i\,(i \in I), E_j\,(j \in J)\}$$

induces an idempotent pure congruence on S containing τ. Thus A must be a τ-class. The rest of the theorem is obvious. □

Corollary 2.2. *Every E-disjunctive semigroup S is E-inversive.*

Proof. Suppose by way of contradiction that A is non-empty. Since $\tau = \lambda^{\flat} = 1_S$, then $|A| = 1$ (by the last part of Theorem 2.1), say $A = \{a\}$. On the other hand, A is an ideal of S. In particular, $a^2 = a$, a contradiction with a is not E-inversive. Consequently, $A = \emptyset$ and so S is E-inversive. □

Corollary 2.3. *A semigroup is a group if and only if it is both E-disjunctive and E-unitary.*

Proof. This follows from Corollary 2.2 and Result 1.1. □

3. The connections between τ, σ and χ

Let S be a semigroup. Recall that

$$\chi = \{(a, b) \in S \times S : I(a) = I(b)\},$$

where $I(s) = \{x \in S : sx, xs \in E_S\}$ $(s \in S)$.

Proposition 3.1. *If S is an E-reflexive semigroup, then χ is a congruence on S. Moreover, χ is an idempotent pure congruence on S if S is E-unitary.*

Proof. Let S be E-reflexive, $a\,\chi\,b$, $c \in S$ and let $x \in I(ac)$. Then $(ac)x \in E_S$. Hence $a(cx) \in E_S$. Because S is E-reflexive, then we have $cx \in I(a) = I(b)$ and so $(bc)x = b(cx) \in E_S$, i.e., $x \in I(bc)$ (since S is E-reflexive). Thus $I(ac) \subseteq I(bc)$. We may equally well show the opposite inclusion, therefore $I(ac) = I(bc)$. Similarly, $I(ca) = I(cb)$. Consequently, χ is a congruence on S.

Suppose further that S is E-unitary. In particular, S is E-reflexive, so χ is a congruence on S. Let $I(a) = I(e)$, where $a \in S$ and $e \in E_S$. Then $e \in I(e) = I(a)$. Hence $ea \in E_S$, so $a \in E_S$. Thus χ is idempotent pure. □

Remark 3.2. The assumption that S is E-unitary is important, that is, in the class of E-reflexive semigroups (or even in the class of commutative semigroups) the congruence χ is not (in general) idempotent pure. Indeed, in the semigroup of non-negative integers with respect to multiplication, χ is the universal relation.

Also, if S is not E-reflexive, then (in general) χ is not a congruence. Indeed, consider the Brandt semigroup B_2 with the multiplication table given below:

$\cdot$	0	e	f	a	b
0	0	0	0	0	0
e	0	e	0	a	0
f	0	0	f	0	b
a	0	0	a	0	e
b	0	b	0	f	0

Then B_2 is an E-semigroup with $E_{B_2} = \{0, e, f\}$. Since $ae \in E_{B_2}$ and $ea = a$, then B_2 is not E-reflexive. Finally,

$$I(0) = B_2,\ I(e) = \{0, e, f\} = I(f),\ I(a) = \{0, a, b\} = I(b),$$

so $(a, b) \in \chi$. On the other hand, $(ae, be) = (0, b) \notin \chi$.

Proposition 3.3. *In an arbitrary semigroup S,*

$$\tau \subseteq \chi.$$

If S is E-unitary, then

$$\tau = \chi.$$

Finally, if S is E-inversive, then

$$\tau \subseteq \chi \subseteq \sigma.$$

Proof. Let $(a, b) \in \tau$ (in view of Theorem 2.1, $I(a) = \emptyset \Leftrightarrow I(b) = \emptyset$, so we may assume that $I(a), I(b) \neq \emptyset$) and $x \in I(a)$. Because $(ax, bx), (xa, xb) \in \tau$, then $bx, xb \in E_S$, since τ is idempotent pure and so $x \in I(b)$, therefore, $I(a) \subseteq I(b)$. By symmetry we may conclude that $I(a) = I(b)$. Consequently, $\tau \subseteq \chi$.

If S is E-unitary, then $\tau = \chi$ by the above and Proposition 3.1.

Let S be E-inversive, $(a, b) \in \chi$. Then $xa, xb \in E_S$ for some $x \in S$. It follows that $(xa)\sigma = (xb)\sigma$. Thus $(x\sigma)(a\sigma) = (x\sigma)(b\sigma)$ and so $a\sigma = b\sigma$ (by cancellation), that is, $\chi \subseteq \sigma$, as required. □

The first part of Remark 3.2 says that χ is not idempotent pure in the class of all E-reflexive semigroups. However, in some special classes of such semigroups, χ is idempotent pure.

Proposition 3.4. *Let S be an E-reflexive orthodox semigroup. Then $\tau = \chi$.*

Proof. Let $a \in S$ and $e \in E_S$. Note that $a = axa$ for some $x \in S$. It is very well-known that $xax \in V(a) = \{a^* \in S : a = aa^*a, a^* = a^*aa^*\}$. Take $a^* \in V(a)$ and suppose that $I(a) = I(e)$. Then $a^*a \in I(e) = I(a)$ (since S is an E-semigroup). Hence $a = a(a^*a) \in E_S$. Consequently, the congruence χ (see Proposition 3.1) is idempotent pure. Because $\tau \subseteq \chi$ (Proposition 3.3), then $\tau = \chi$. □

It has been shown [8] that if S is a semilattice of E-unitary (eventually) regular semigroups, then S is (eventually) orthodox, and that the least Clifford congruence ξ on S is the intersection of the least semilattice congruence η and the largest idempotent pure congruence τ.

Theorem 3.5. *The least Clifford congruence on a semilattice of E-unitary regular semigroups is given by*

$$\xi = \{(a, b) \in \eta : I(a) = I(b)\}.$$

Proof. Indeed, we have mentioned above that such a semigroup is an E-reflexive orthodox semigroup. Thus $\tau = \chi$ (Proposition 3.4). Consequently, $\xi = \eta \cap \chi$. □

The following theorem gives some new characterizations of the least group congruence on an E-unitary E-inversive semigroup.

Theorem 3.6. *The following conditions on an E-unitary E-inversive semigroup S are equivalent:*

(*a*) $(a, b) \in \sigma$;
(*b*) $(a, b) \in \tau$;
(*c*) $(a, b) \in \chi$;
(*d*) $I(a) \subseteq I(b)$;
(*e*) $I(b) \subseteq I(a)$;
(*f*) $I(a) \cap I(b) \neq \emptyset$.

Proof. Indeed, (a) and (b) are equivalent by Result 1.1. In view of the last part of Proposition 3.3, the conditions (a), (b) and (c) are equivalent.

Obviously, $(c) \Rightarrow (d) \Rightarrow (f)$.

By symmetry of the conditions (d) and (e), it is sufficient to show that (f) implies (a). The proof of this implication is very similar to the proof of the last part of Proposition 3.3. □

Consider now the case when $\chi = \sigma$.

Lemma 3.7. *Let S be an E-inversive semigroup S. If $\chi = \sigma$, then S is an E-semigroup.*

Proof. Let $e, f \in E_S$. Then $(e, f) \in \sigma = \chi$. Thus $e \in I(f)$, so $ef \in E_S$. □

Remark 3.8. The converse of the above result is not true. Indeed, consider the semigroup S with the multiplication table given below:

$\cdot$	e	f	a	b	c
e	e	e	e	b	b
f	e	f	a	b	b
a	e	a	f	b	b
b	b	b	b	e	e
c	b	c	c	e	e

Then S is an E-reflexive E-semigroup with $E_S = \{e, f\}$; $I(e) = \{e, f, a\}$ and $I(f) = \{e, f\}$, so $\sigma \neq \chi$. Also, $I(a) = \{e, a\}$ and $I(b) = \{b, c\} = I(c)$. Thus χ is idempotent pure, so $\chi = \tau$ (Proposition 3.1). Finally, notice that $S/\tau \cong \{e, f, a, b\}$ is E-disjunctive.

The proof of the following proposition is similar to the proof of Theorem 3.6 and is omitted.

Proposition 3.9. *Let S be an E-inversive semigroup such that $\chi = \sigma$. Then the following conditions are equivalent*:

(a) $(a, b) \in \sigma$;
(b) $I(a) \subseteq I(b)$;
(c) $I(b) \subseteq I(a)$;
(d) $I(a) \cap I(b) \neq \emptyset$. $\square$

In the class of regular semigroups a much stronger statement holds.

Theorem 3.10. *The following conditions on a regular semigroup S are equivalent*:

(a) $\chi = \sigma$;
(b) χ *is a group congruence on* S;
(c) $\tau = \sigma$;
(d) S *is E-unitary.*

Proof. Since $\chi \subseteq \sigma$, then the conditions (a) and (b) are equivalent.

(a) $\Rightarrow$ (c). Let $\chi = \sigma$. We claim that S *is strongly E-reflexive.* Indeed, suppose that $eab \in E_S$, where $a, b \in S$ and $e \in E_S$. Then

$$e\sigma = (eab)\sigma = (ab)\sigma = (ba)\sigma.$$

Hence $I(e) = I(ba)$. In particular, $eba \in E_S$. On the other hand, if $ab \in E_S$, then $I(bb^*) = I(ba)$ for some $b^* \in V(b)$, and so $ba \in E_S$, as required. Consequently, S is a strongly E-reflexive orthodox semigroup (Lemma 3.7). Hence $\tau = \chi$ (by Proposition 3.4). Thus $\tau = \sigma$.

(c) $\Rightarrow$ (d). This is clear.

(d) $\Rightarrow$ (a). It is a consequence of Theorem 3.6. $\square$

Remark 3.11. We have proved above that $\chi = \sigma$ in an arbitrary E-unitary E-inversive semigroup S. The converse is not valid without regularity of S. For example, one may consider the semigroup of all non-negative integers with respect to multiplication.

Recall from [3] that an idempotent-surjective R-semigroup S is a semilattice of E-unitary E-inversive semigroups if and only if the equality

$$\xi = \eta \cap \tau$$

holds in the lattice of all congruences on S.

From the above equality follows immediately the following theorem.

Theorem 3.12. *The following conditions on an arbitrary idempotent-surjective R-semigroup S are equivalent*:

(a) *S is an E-disjunctive semilattice of E-unitary E-inversive semigroups*;

(b) *S is an E-disjunctive semilattice of groups.* □

The following result generalizes Proposition 3.4.

Proposition 3.13. *Let an idempotent-surjective R-semigroup S be a semilattice of E-unitary E-inversive semigroups. Then $\tau = \chi$.*

Proof. We show first that S is strongly E-reflexive. Let $eab \in E_S$, where $a, b \in S$ and $e \in E_{S^1}$. Then $(eab)\xi$ is an idempotent of the Clifford semigroup S/ξ. Since S/ξ is strongly E-reflexive, then $(eba)\xi \in E_{S/\xi}$. On the other hand, ξ is idempotent pure, therefore, $eba \in E_S$, as required. Hence χ is a congruence on S (Prop. 3.1). Also, S is an E-semigroup. Indeed, because S is an R-semigroup, then $\mathrm{Reg}(S)$ is a regular semigroup which is strongly E-reflexive (since we have just shown that S is strongly E-reflexive). In the light of Corollary 3 [13], $\mathrm{Reg}(S)$ is an orthodox semigroup, so S is an E-semigroup, as exactly required.

Finally, let $I(a) = I(e)$ (where $a \in S$, $e \in E_S$). Then $f \in I(e) = I(a)$ for some $f \in E_{a\eta}$, where $a\eta$ is an E-unitary subsemigroup of S. Thus $fa \in E_{a\eta}$, so

$$a \in E_{a\eta} \subseteq E_S,$$

that is, χ is idempotent pure. Consequently, $\tau = \chi$. □

Theorem 3.14. *Suppose that an idempotent-surjective R-semigroup S is a semilattice of E-unitary E-inversive semigroups. Then the least Clifford congruence on S is given by*

$$\xi = \{(a, b) \in \eta : I(a) = I(b)\}.$$ □

Remark 3.15. We have mentioned above that the class of idempotent-surjective R-semigroups contains the class of structurally regular semigroups, therefore, the last three results remain true for all structurally regular semigroups.

From the first part of the proof of Proposition 3.13 we are able to extract some overall result. First, we shall need certain definitions. Let $a_1, a_2, \dots, a_n$ be elements of a semigroup S and let α be a non-identical permutation of the set $\{1, 2, \dots, n\}$. We shall say that S is (E, α)-*reflexive* if

$$(a_1 a_2 \cdots a_n \in E_S) \implies (a_{1\alpha} a_{2\alpha} \cdots a_{n\alpha} \in E_S).$$

Further, a congruence ρ on S is *idempotent-surjective* if every idempotent ρ-class of S contains some idempotent [1].

Then we get the following lemma which may be at times useful.

Lemma 3.16. *Let ρ be an idempotent pure and idempotent-surjective congruence on a semigroup S such that S/ρ is (E, α)-reflexive. Then S is (E, α)-reflexive.*

Proof. Indeed, let $a_1 a_2 \cdots a_n \in E_S$. Then $(a_1 a_2 \cdots a_n)\rho \in E_{S/\rho}$. Hence

$$(a_{1\alpha} a_{2\alpha} \cdots a_{n\alpha})\rho \in E_{S/\rho}.$$

Thus $(a_{1\alpha} a_{2\alpha} \cdots a_{n\alpha}, e) \in \rho$ for some $e \in E_S$ (since ρ is idempotent-surjective), so $a_{1\alpha} a_{2\alpha} \cdots a_{n\alpha} \in E_S$ (because ρ is idempotent pure). $\square$

In [8] it has been shown that if an eventually regular semigroup is strongly E-reflexive, then it is eventually orthodox, so the following theorem holds.

Theorem 3.17. *Let S be a semilattice of E-unitary eventually regular semigroups. Then $\tau = \chi$ and*

$$\xi = \{(a, b) \in \eta : I(a) = I(b)\}. \qquad \square$$

Remark 3.18. Notice that if an arbitrary (E-inversive) *E-semigroup* S is a semilattice S/η of E-unitary E-inversive semigroups $a\eta$ ($a \in S$), then the following equality holds (cf. Section 5 of [6]):

$$\xi = \eta \cap \tau,$$

so S is strongly E-reflexive. Moreover, from the proof of Proposition 3.13 we obtain the following result.

Theorem 3.19. *Suppose that an arbitrary E-semigroup S is a semilattice S/η of E-unitary E-inversive semigroups $a\eta$ ($a \in S$). Then $\tau = \chi$ and*

$$\xi = \{(a, b) \in \eta : I(a) = I(b)\}. \qquad \square$$

Finally, we have the following open problem.

Problem 3.20. Is an analogous result to Theorem 3.19 valid, when S is a semilattice S/η of E-unitary E-inversive semigroups $a\eta$ ($a \in S$)?

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