A still shorter axiom for trimedial quasigroups

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Abstract. In this brief note, we exhibit an identity in product only that characterizes the variety of trimedial quasigroups that is shorter than the shortest one currently known. Our identity was found with the aid of the automated theorem-prover Prover9.

1. Introduction

A *quasigroup* consists of a non-empty set Q equipped with a binary operation, which we simply denote by juxtaposition, such that for all $a, b \in Q$, there exist unique $x, y \in Q$ such that $ax = b$ and $ya = b$. Alternatively, a quasigroup is an algebra $(Q; \cdot, \backslash, /)$ of type $(2, 2, 2)$ such that (We sometimes write xy instead of $x \cdot y$. The order of operations is juxtaposition, $\cdot$, $/$, and $\backslash$.)

$$x\backslash xy = y \tag{1}$$

$$xy/y = x \tag{2}$$

$$x(x\backslash y) = y \tag{3}$$

$$(x/y)y = x. \tag{4}$$

A quasigroup Q is *medial* if $wx \cdot yz = wy \cdot xz$ (medial quasigroups are sometimes called *abelian*, *entropic*, etc.). A quasigroup is *trimedial* if every subquasigroup generated by three elements is medial (trimedial quasigroups are sometimes called *triabelian*, *terentropic*, etc.).

In [1], [2], Kepka showed that the trimedial quasigroups can be characterized as the quasigroups that satisfy the three identities in product only below.

$$xx \cdot yz = xy \cdot xz \tag{5}$$

$$yz \cdot xx = yx \cdot zx \tag{6}$$

$$(x \cdot xx) \cdot uv = xu \cdot (xx \cdot v) \tag{7}$$

In [1], [2], Kepka also showed that the trimedial quasigroups can be characterized as the quasigroups that satisfy the single identity in product only below (note that this identity has 13 variable occurrences on each side and six distinct variables).

$$(xx \cdot yz)((xy \cdot uu)((w \cdot ww) \cdot zv)) = (xy \cdot xz)((xu \cdot yu)(wz \cdot (ww \cdot v))) \tag{8}$$

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In [3], Kinyon and Phillips showed that any quasigroup (in fact, any left cancellative groupoid) satisfying (6) and (7) must also satisfy (5). Therefore, (6) and (7) are enough to characterize the trimedial quasigroups. They used this result to improve upon (8) by showing that the trimedial quasigroups can be characterized as the quasigroups that satisfy the single identity in product only below (note that this identity has nine variable occurrences on each side and six distinct variables).

$$(xy \cdot uu)((w \cdot ww) \cdot zv) = (xu \cdot yu)(wz \cdot (ww \cdot v)) \tag{9}$$

In this brief note, we improve upon (9) by exhibiting a still shorter identity in product only that characterizes the trimedial quasigroups. Our identity was found with the aid of the automated theorem-prover Prover9 [4], a resolution theorem-prover for first-order logic with equality. We also used the scripting language Perl to further automate our search.

2. A shorter axiom for trimedial quasigroups

In this section, we describe our search for a shorter axiom in product only for trimedial quasigroups.

We began by generating identities that are valid in the variety of trimedial quasigroups by running Prover9 on the input

```
set(input_sos_first).  % sos clauses become given before derived clauses
set(print_kept).  % print derived and retained clauses
assign(max_given, 1000).  % terminate after 1000 iterations of main loop
formulas(sos).  % set of support formulas
x \ (x * y) = y.(x * y) / y = x.x * (x \ y) = y.(x/y)*y=x.  % quasigroup
(x * x) * (y * z) = (x * y) * (x * z).
(y * z) * (x * x) = (y * x) * (z * x).
(x * (x * x)) * (u * v) = (x * u) * ((x * x) * v).  % trimedial
end_of_list.
```

and simply letting it derive consequences of (5), (6), and (7) in the variety of quasigroups for 1000 iterations of its main loop. We then extracted from the output (using Perl) the derived identities with the following properties: they were derived in the seventh or later iteration of the main loop (this way all of the defining identities for quasigroups as well as (5), (6), and (7) have begun participating in the search), there are at most eight variable occurrences on each side, and there are no occurrences of / or \. This resulted in 357 identities. Below is a proof that (6) and (7) imply the identity $((x \cdot xx) \cdot yz) \cdot uu = (xy \cdot u)((xx \cdot z)u)$.

Proof. By (7),

$$((x \cdot xx) \cdot yz) \cdot uu = (xy \cdot (xx \cdot z)) \cdot uu. \tag{10}$$

By (6),

$$(xy \cdot (xx \cdot z)) \cdot uu = (xy \cdot u)((xx \cdot z)u). \tag{11}$$

By (10) and (11),

$$((x \cdot xx) \cdot yz) \cdot uu = (xy \cdot u)((xx \cdot z)u), \tag{12}$$

which completes the proof. □

Next, we sent each of these identities to Prover9 to attempt to prove that it characterizes the trimedial quasigroups. For example, running Prover9 on the input

```
formulas(sos).  % set of support formulas
x \ (x * y) = y.(x * y) / y = x.x * (x \ y) = y.(x/y)*y=x.  % quasigroup
((x * (x * x)) * (y * z)) * (u * u) = ((x*y)*u)*(((x*x)*z)*u).
                                                % candidate identity
(b * c) * (a * a) != (b * a) * (c * a) |
 (a * (a * a)) * (b * c) != (a * b) * ((a * a) * c).  % not trimedial
end_of_list.
```

produces a proof that (12) characterizes the trimedial quasigroups (note that our identity has only seven variable occurrences on each side and only four distinct variables). This was the only proof that was found. Below is a "humanized" version of Prover9's proof that (2), (3), and (12) imply (6) and (7).

Proof. By (12),

$$((x \cdot xx) \cdot yz) \cdot uu/uu = (xy \cdot u)((xx \cdot z)u)/uu. \tag{13}$$

By (2) and (13),

$$(x \cdot xx) \cdot yz = (xy \cdot u)((xx \cdot z)u)/uu$$

or

$$(xy \cdot z)((xx \cdot u)z)/zz = (x \cdot xx) \cdot yu. \tag{14}$$

Substituting $xx\backslash z$ for z in (12),

$$(x \cdot xx)(y(xx\backslash z)) \cdot uu = (xy \cdot u) \cdot (xx \cdot (xx\backslash z))u. \tag{15}$$

By (3) and (15),

$$((x \cdot xx)(y(xx\backslash z))) \cdot uu = (xy \cdot u) \cdot zu. \tag{16}$$

Substituting z for u in (14),

$$(x \cdot xx) \cdot yz = (xy \cdot z)((xx \cdot z)z)/zz. \tag{17}$$

By (12),

$$(xy \cdot z)((xx \cdot z)z) = (xy \cdot (xx \cdot z)) \cdot zz. \tag{18}$$

By (17) and (18),

$$(x \cdot xx) \cdot yz = (xy \cdot (xx \cdot z)) \cdot zz/zz. \tag{19}$$

By (2) and (19)

$$(x \cdot xx) \cdot yz = xy \cdot (xx \cdot z) \tag{20}$$

which implies (7)

By (20),

$$(x \cdot xx)(y(xx\backslash z)) \cdot uu = (xy \cdot (xx \cdot (xx\backslash z))) \cdot uu. \tag{21}$$

By (3), (16), and (21),

$$(xy \cdot z) \cdot uu = (xy \cdot u) \cdot zu. \tag{22}$$

Substituting x/y for x in (22),

$$((x/y)y \cdot z) \cdot uu = ((x/y)y \cdot u) \cdot zu. \tag{23}$$

By (2) and (23),

$$xz \cdot uu = xu \cdot zu$$

which implies (6). □

Theorem 2.1. *The trimedial quasigroups can be characterized as the quasigroups that satisfy the single identity in product only below*

$$((x \cdot xx) \cdot yz) \cdot uu = (xy \cdot u)((xx \cdot z)u).$$

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