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On 2-absorbing semimodules

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Abstract. In this paper, we introduce the concept of 2-absorbing semimodules over a commutative semiring with non-zero identity which is a generalization of prime semimodules and give some characterizations related to the same. We also prove the 2-absorbing avoidance theorem for semimodules and give an application of them.

1. Introduction

Badawi [2], has introduced the concept of 2-absorbing ideals in a commutative ring with a non-zero identity element, which is a generalization of prime ideals and investigated some properties. Darani and Soheilnia [3], Payrovi and Babaei [6] have studied the notion of 2-absorbing submodules and gave some characterizations. In [1], R. Ameri have studied the concept of prime submodules of multiplication modules over rings.

By a *semiring* we mean a semigroup $(S, \cdot)$ and a commutative monoid $(S, +, 0_S)$ in which 0_S is the additive identity and $0_S \cdot x = x \cdot 0_S = 0_S$ for all $x \in S$, both are connected by the ring like distributivity. A subset I of a semiring S is called an *ideal* of S if $a, b \in I$ and $r \in S$, $a + b \in I$ and $ra, ar \in I$. An ideal I of a semiring S is called *subtractive* if $a, a + b \in I$, $b \in S$ implies $b \in I$. Let S be a semiring. A *left S-semimodule* M is a commutative monoid $(M, +)$ which has a zero element 0_M, together with an operation $S \times M \to M$, denoted by $(a, x) \to ax$ such that for all $a, b \in S$ and $x, y \in M$,

(i) $a(x + y) = ax + ay$,

(ii) $(a + b)x = ax + bx$,

(iii) $(ab)x = a(bx)$,

(iv) $0_S \cdot x = 0_M = a \cdot 0_M$.

A non-empty subset N of an S-semimodule M is a *subsemimodule* of M if N is closed under addition and scalar multiplication. A proper subsemimodule N of an S-semimodule M is called *subtractive* if $a, a + b \in N$, $b \in M$ implies $b \in N$. A left S-semimodule M is called *cyclic* if M can be generated by a single

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element, that is, $M = (m) = Sm = \{sm \mid s \in S\}$ for some $m \in M$. Let M be an S-semimodule. Then M is said to be a *multiplication semimodule* if for all subsemimodules N of M there exists an ideal I of S such that $N = IM$. For example, every cyclic semimodule M is a multiplication semimodule. From this definition, it is clear that $I \subseteq (N : M)$ and also $N = IM \subseteq (N : M)M \subseteq N$ and therefore $N = (N : M)M$. Let M be a multiplication S-semimodule and N, K are subsemimodules of M. Then there exist ideals I, J of S such that $N = IM$ and $K = JM$. Define the multiplication of two subsemimodules N and K, denoted by NK, as $NK = (IM)(JM) = (IJ)M$.

A non-zero proper ideal I of S is called a *2-absorbing ideal* if whenever $a, b, c \in S$ and $abc \in I$ then $ab \in I$ or $ac \in I$ or $bc \in I$. It is proved that a non-zero proper ideal I of S is a 2-absorbing ideal if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$. It is easy to proved that every prime ideal of a semiring S is a 2-absorbing ideal of S but converse need not be true. For example it is easy to see that every ideal generated by $\langle 4 \rangle$ of a semiring E of even integers is 2-absorbing but not a prime ideal of E.

Throughout this paper, S will always denote a commutative semiring with identity $1 \neq 0$ and left S-semimodules means semimodules.

2. 2-absorbing subsemimodules

Definition 2.1. Let M be an S-semimodule and N be a proper subsemimodule of M. An *associated ideal* of N is defined as

$$(N : M) = \{a \in S : aM \subseteq N\}.$$

Result 2.2. Let M be an S-semimodule and N be a proper subsemimodule of M. If N is a subtractive subsemimodule of M, and let $m \in M$. Then the following hold:

(i) $(N : M)$ is a subtractive ideal of S.

(ii) $(0 : M)$ and $(N : m)$ are subtractive ideals of S.

Proof. Proof is straightforward. □

Definition 2.3. A proper subsemimodule N of M is called *prime* if $ax \in N$, $a \in S$, $x \in M$ then either $x \in N$ or $a \in (N : M)$.

Definition 2.4. Let S be a semiring. Let M be an S-semimodule and N be a proper subsemimodule of M. Then N is called a *2-absorbing subsemimodule* of M, if for $a, b \in S$ and $x \in M$, $abx \in N$ implies that $ab \in (N : M)$ or $ax \in N$ or $bx \in N$.

It is easy to verify that every prime subsemimodule of M is a 2-absorbing subsemimodule of M but converse need not be true. This can be illustrated as follows

Example 2.5. Let S be $Z^* = Z^+ \cup \{0\}$. Then $M = Z^* \times Z^*$ is an S-semimodule. If we take the subsemimodule $N = \{0\} \times 4Z^*$ of M then the associated ideal of N is $\{0\}$. Here, N is a 2-absorbing subsemimodule of M but N is not prime subsemimodule of M because $2 \cdot (0, 2) \in N$ but $2 \notin (N : M)$ and $(0, 2) \notin N$.

Definition 2.6. If I is an ideal of S, then a *radical* of I is defined as

$$\mathrm{Rad}(I) = \sqrt{I} = \{a \in S : a^2 \in I\}.$$

It is easy to prove that if I is a 2-absorbing ideal of S, then $J = \sqrt{I}$ is a 2-absorbing ideal of S with $J^2 \subseteq I \subseteq J$.

Proposition 2.7. *Let M be an S-semimodule and let N be a 2-absorbing subtractive subsemimodule of M with $\sqrt{N : M} = J$. Then $(N : M)$ and J are 2-absorbing ideals of S with $J^2 \subseteq (N : M) \subseteq J$, where*

$$J = \sqrt{N : M} = \{r \in S : r^2 \in (N : M)\}.$$

Proof. Clearly, $(N : M)$ is a subtractive ideal of S. Now, we show that $(N : M)$ is a 2-absorbing ideal of S. Let $u, v, w \in S$ be such that $uvw \in (N : M)$. Suppose $uw, vw \notin (N : M)$. Then there exist $x, y \in M \setminus N$ such that $uwx, vwy \notin N$. Also, $uv(w(x + y)) \in N$ gives $uw(x + y) \in N$ or $vw(x + y) \in N$ or $uv \in (N : M)$. If $uw(x + y) \in N$ and since $uwx \notin N$ then we have $uwy \notin N$(as N is a subtractive subsemimodule of M). Since $uv(wy) \in N$ and N is a 2-absorbing subsemimodule of M, therefore, either $uv \in (N : M)$ or $vwy \in N$ or $uwy \in N$. Thus $uv \in (N : M)$. Similarly, if $vw(x + y) \in N$, then we have $uv \in (N : M)$. Hence $(N : M)$ is a 2-absorbing ideal of S. Next, since $(N : M)$ is a 2-absorbing ideal of S, therefore, we have $J = \sqrt{N : M}$ is also a 2-absorbing ideal with $J^2 \subseteq (N : M) \subseteq J$. □

Remark 2.8. In general, suppose M be an S-semimodule and let N be a subtractive subsemimodule of M. If $(N : M)$ is a 2-absorbing ideal of S, then N need not be a 2-absorbing subsemimodule of M.

Example 2.9. Let S be $Z^* = Z^+ \cup \{0\}$ then $M = Z^* \times Z^*$ is an S-semimodule. Consider the subsemimodule $N = \{0\} \times 8Z^*$ of M. Then the associated ideal of N is $\{0\}$, which is a 2-absorbing ideal of S but N is not 2-absorbing subsemimodule of M because $2 \cdot 2 \cdot (0, 2) \in N$ but $2 \cdot 2 \notin (N : M)$ and $2 \cdot (0, 2) \notin N$.

Note: The converse of the above remark is true in the case of cyclic semimodules.

Proposition 2.10. *Let M be a cyclic S-semimodule and let N be a 2-absorbing subsemimodule of M. Then N is 2-absorbing subsemimodule of M if and only if $(N : M)$ is a 2-absorbing ideal of S.*

Proof. The proof is similar to the proof of Proposition 2.9 in [3]. □

Proposition 2.11. *Let N be a 2-absorbing subtractive subsemimodule of M with $\sqrt{(N:M)} = J$. If $(N:M) \neq J$, for every $r \in J \setminus (N:M)$, then*

$$N_r = \{x \in M : rx \in N\}$$

is a prime subsemimodule of M containing N with $J \subseteq (N_r : M)$.

Proof. Let $ux \in N_r$, where $u \in S \setminus (N_r : M)$ and $x \in M$. Then $rux \in N$ and N is a 2-absorbing subsemimodule of M. Therefore, $ru \in (N:M)$ or $rx \in N$ or $ux \in N$. If $ru \in (N:M)$, then $u \in (N_r : M)$, which is a contradiction. If $rx \in N$, by the definition of N_r, $x \in N_r$, then nothing to prove. If $ux \in N$ and also, $r^2 \in J^2 \subseteq (N:M)$. This gives $rx \in N_r$ for particular $x \in M$. Now, we have $(r+u)x \in N_r$, that is, $r(r+u)x \in N$ and since N is a 2-absorbing subsemimodule of M, therefore $rx \in N$ or $(r+u)x \in N$ or $r(r+u) \in (N:M)$.

Again, if $rx \in N$, then $x \in N_r$, which is required. If $(r+u)x \in N$ and $ux \in N$ and as N is a subtractive, therefore, $rx \in N$. This gives $x \in N_r$, which is required.

If $r(r+u) \in (N:M)$ and since $r^2 \in J^2 \subseteq (N:M)$, this gives $ru \in (N:M)$ that is, $u \in (N_r : M)$, a contradiction. Hence, N_r is a prime subsemimodule of M. □

Corollary 2.12. *Let N be a 2-absorbing subtractive subsemimodule of M with $\sqrt{N:M} = J$. If $(N:M) \neq J$, for every $r \in J \setminus (N:M)$, then N_r is a 2-absorbing subsemimodule of M containing N with $J \subseteq (N_r : M)$.* □

Proposition 2.13. *If N is a subtractive subsemimodule of M, then N_r is a subtractive subsemimodule of M and hence $(N_r : M)$ is a subtractive ideal of S.*

Proof. Let $a, (a+b) \in N_r$ and $b \in M$. Then we have $ra, (ra+rb) \in N$ and N is a subtractive subsemimodule of M. Therefore, we have $rb \in N$, this gives $b \in N_r$. Hence, N_r is a subtractive subsemimodule of M. It can easily be prove that $(N_r : M)$ is a subtractive ideal of S. □

Proposition 2.14. *If N is a 2-absorbing subsemimodule of M and K is any subsemimodule of M, then $K \cap N$ is a 2-absorbing subsemimodule of K.*

Proof. Proof is straightforward. □

Theorem 2.15. *If N is an intersection of two prime subsemimodules of M, then N is 2-absorbing.*

Proof. Let N_1 and N_2 be two prime subsemimodules of M. Then to show that $N_1 \cap N_2$ is a 2-absorbing subsemimodule of M. Let $abm \in N_1 \cap N_2$ for $a, b \in S$, $m \in M$. Then $abm \in N_1$ and $abm \in N_2$. Now $abm \in N_1$ implies $a \in (N_1 : M)$ or $b \in (N_1 : M)$ or $m \in N_1$. Similarly, $abm \in N_2$ gives $a \in (N_2 : M)$ or $b \in (N_2 : M)$ or $m \in N_2$. If $a \in (N_1 : M)$ and $a \in (N_2 : M)$, then $a \in (N_1 \cap N_2 : M)$ and so

$ab \in (N_1 \cap N_2 : M)$. Again, if $a \in (N_1 : M)$ and $m \in N_2$, then $am \in N_1 \cap N_2$. Similarly, we can prove the other cases. □

It is easy to see that the intersection of two distinct nonzero 2-absorbing subsemimodules need not be a 2-absorbing subsemimodule of M. For example $\{0\} \times 4Z$ and $\{0\} \times 3Z$ are 2-absorbing subsemimodules of $Z \times Z$ but their intersection $(\{0\} \times 4Z) \cap (\{0\} \times 3Z) = (\{0\} \times 12Z)$ is not a 2-absorbing subsemimodule of $Z \times Z$. Similarly, we can find that an intersection of a prime semimodule and a 2-absorbing semimodule need not be a 2-absorbing semimodule of $Z \times Z$, where Z is the set of positive integers with zero.

Theorem 2.16. *Let M be a cyclic S-semimodule and N be a subsemimodule of M. Then N is a 2-absorbing subsemimodule of M if and only if for any subsemimodules U, V and W of M, $UVW \subseteq N$ implies $UV \subseteq N$ or $VW \subseteq N$ or $UW \subseteq N$.*

Proof. Suppose N is a 2-absorbing subsemimodule of M. Let $UVW \subseteq N$ for some subsemimodules U, V, W of M. Since M is cyclic therefore the multiplication semimodule over S, therefore, there exist ideals I, J and K of S such that $U = IM$, $V = JM$ and $W = KM$. Then, we have $UVW = (IJK)M \subseteq N$. This implies $IJK \subseteq (N : M)$. Since N is a 2-absorbing subsemimodule of M therefore $(N : M)$ is a 2-absorbing ideal of S, by Proposition 2.10. Therefore, either $IJ \subseteq (N : M)$ or $JK \subseteq (N : M)$ or $IK \subseteq (N : M)$. This gives $IJM \subseteq N$ or $JKM \subseteq N$ or $IKM \subseteq N$. That is, $(IM)(JM) \subseteq N$ or $(JM)(KM) \subseteq N$ or $(IM)(KM) \subseteq N$, Hence, we have either $UV \subseteq N$ or $VW \subseteq N$ or $UW \subseteq N$.

Conversely, suppose that $IJK \subseteq (N : M)$, where I, J, K are ideals of S. Then $IJKM \subseteq N$. Since M is a cyclic therefore it is a multiplication semimodule. Therefore $(IJK)M \subseteq N$ implies $(IM)(JM)(KM) \subseteq N$. Therefore $(IM)(JM) \subseteq N$ or $(JM)(KM) \subseteq N$ or $(IM)(KM) \subseteq N$, that is, $IJ \subseteq (N : M)$ or $JK \subseteq (N : M)$ or $IK \subseteq (N : M)$. Therefore $(N : M)$ is a 2-absorbing ideal of S. Hence, by Proposition 2.10, N is a 2-absorbing subsemimodule of M. □

Proposition 2.17. *Let M be a cyclic S-semimodule. Then the following statements are equivalent:*

- (i) *N is a 2-absorbing subsemimodule;*
- (ii) *$(N : M)$ is a 2-absorbing ideal of S;*
- (iii) *$N = PM$, where P is a 2-absorbing ideal of S which is maximal with respect to the property, that is, $IM \subseteq N$ implies that $I \subseteq P$.*

Proof. (i) and (ii) are equivalent by Proposition 2.10.

$(ii) \Rightarrow (iii)$. Since M is a cyclic therefore M is a multiplicative semimodule. Now since N is a subsemimodule of a multiplicative semimodule M, then there exists an ideal P of S such that $N = PM$. This implies $P = (N : M)$ which is a 2-absorbing by (ii). Suppose there exists an ideal I of S such that $IM \subseteq N$. This implies $I \subseteq (N : M) = P$. So P is maximal with respect to the property, that is, if $IM \subseteq N$, then $I \subseteq P$.

$(iii) \Rightarrow (i)$. To show that N is a 2-absorbing subsemimodule we show that $(N : M)$ is a 2-absorbing ideal in S. Suppose that $IJK \subseteq (N : M)$, where ideals $I, J, K \subseteq S$. Then $IJKM \subseteq N$. Since M is a cyclic therefore M is a multiplicative semimodule. Therefore $(IJK)M \subseteq N = PM$, where P is a 2-absorbing ideal of S. This implies $IJK \subseteq P$ (by maximality of ideal P with respect to the property $IM \subseteq N$. Hence, $I \subseteq P$. Since P is a 2-absorbing ideal of S, therefore, $IJ \subseteq P$ or $JK \subseteq P$ or $IK \subseteq P$. This gives $IJM \subseteq PM = N$ or $JKM \subseteq PM = N$ or $IKM \subseteq PM = N$. Consequently, $IJ \subseteq (N : M)$ or $JK \subseteq (N : M)$ or $IK \subseteq (N : M)$. Thus $(N : M)$ is a 2-absorbing ideal of S. Therefore N is a 2-absorbing semimodule of M. □

Definition 2.18. Let M be an S-semimodule and N be a subsemimodule of M. Then N is called *pure* if $aN = N \cap aM$ for every $a \in S$.

Definition 2.19. Let M be an S-semimodule. Then a semimodule M is *M-cancellative semimodule* if whenever $rm = rn$ for elements $m, n \in M$ and $r \in S$ then $m = n$.

Theorem 2.20. *Let M be an M-cancellative S-semimodule and N be a proper subsemimodule of M. Then N is a pure subsemimodule of M if and only if N is a 2-absorbing subsemimodule of M with $(N : M) = \{0\}$.*

Proof. Suppose that N is a pure subsemimodule of M and $abm \in N$ such that $ab \notin (N : M)$, where $a, b \in S$ and $m \in M$. Then $abm \in abM \cap N = abN$, so $abm = abn$ for some $n \in N$. This implies $bm = bn \in N$ (as M is a M-cancellative semimodule). Thus N is a 2-absorbing subsemimodule of M. Next, suppose that $a \in (N : M)$ with $a \neq 0$. Since $N \neq M$ there exists $x \in M \setminus N$ such that $ax \in aM \cap N = aN$, so there exists $y \in N$ such that $ax = ay$. Therefore $x = y$, a contradiction. Thus $(N : M) = \{0\}$.

Conversely, assume that N is a 2-absorbing subsemimodule of M. Let $abz \in abM \cap N$, where $z \in M$ and $a, b \in S$. We may assume that $ab \neq 0$. Since N is a 2-absorbing subsemimodule of M we have either $az \in N$ or $bz \in N$. If $bz \in N$, for $a \in S$ we have $abz \in abN$. Therefore $abM \cap N \subseteq abN$. Similarly, we can prove the case for $az \in N$. Converse is obvious. Hence $abM \cap N = abN$ and therefore N is a pure subsemimodule of M. □

3. The 2-absorbing avoidance theorem

In this section, we prove the 2-absorbing avoidance theorem for semimodules. Before proving this theorem, we first define an efficient covering of subsemimodules.

Let $N_1, N_2, \ldots, N_n$ be subsemimodules of M. Define a covering $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$ is efficient if no N_i is superfluous. In other words, we say that

$N = N_1 \cup N_2 \cup \ldots \cup N_n$ is an efficient union if none of the N_i may be excluded. Any cover or union consisting of subsemimodules of M be reduced to an efficient one, called an *efficient reduction*, by deleting any unnecessary terms.

Theorem 3.1. *Let N be a subsemimodule of an S-semimodule M such that $N \subseteq N_1 \cup N_2$ for some subtractive subsemimodules N_1, N_2 of M. Then either $N \subseteq N_1$ or $N \subseteq N_2$.*

Proof. Let $N \subseteq N_1 \cup N_2$ but $N \nsubseteq N_1$ and $N \nsubseteq N_2$. Then there exist $x \in N \setminus N_1$ and $y \in N \setminus N_2$ such that $x \in N_2$ and $y \in N_1$. Also $x + y \in N$ gives either $x + y \in N_1$ or $x + y \in N_2$. If $x + y \in N_1$ and $y \in N_1$ then $x \in N_1$ as N_1 is a subtractive subsemimodule of M, a contradiction. Similarly, if $x + y \in N_2$ and $x \in N_2$ we get $y \in N_2$, a contradiction. Hence, $N \subseteq N_1$ or $N \subseteq N_2$. □

Theorem 3.2. *Let $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$ be an efficient union of subtractive subsemimodules of an S-semimodule M. Then for any $j \in \{1, 2, \ldots, n\}$ we have*

$$\bigcap_{i=1}^{n} N_i = \bigcap_{\substack{i=1 \\ i \neq j}}^{n} N_i.$$

Proof. Clearly, for $j = 1$ we have $N_1 \cap N_2 \cap \ldots \cap N_n \subseteq N_2 \cap N_3 \cap \ldots \cap N_n$. Therefore $\bigcap_{i=1}^{n} N_i \subseteq \bigcap_{i=2}^{n} N_i$. Now, let $\ell_1 \in \bigcap_{i=2}^{n} N_i$ and $\ell_2 \in N \setminus \bigcup_{i=2}^{n} N_i$. Therefore, $\ell_1 \in N$ and $\ell_2 \in N_1$. Then $\ell_1 + \ell_2 \in N$, which gives $\ell_1 + \ell_2 \in N_j$ for some $j \in \{1, 2, \ldots, n\}$. If $j \in \{2, \ldots, n\}$ and since N_j is a subtractive subsemimodule of M, we have $\ell_2 \in N_j$, a contradiction. If $j = 1$, then $\ell_1 + \ell_2 \in N_1$ gives $\ell_1 \in N_1$. Hence $\ell_1 \in \bigcap_{i=1}^{n} N_i$. □

Lemma 3.3. *Let $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$ be an efficient union of subtractive subsemimodules of an S-semimodule M, where $n > 1$. If $(N_k : m) = (N_k : M)$ for all $m \in M \setminus N_k$, $\sqrt{(N_k : M)} \neq (N_k : M)$ and there exists $r \in \sqrt{(N_k : M)} \setminus (N_k : M)$ such that $((N_j)_r : M) \nsubseteq ((N_k)_r : M)$ for every $j \neq k$, then for $k \in \{1, 2, \ldots, n\}$ no N_k is a 2-absorbing subsemimodule of M.*

Proof. Suppose that N_k is a 2-absorbing subsemimodule of M for some $1 \leqslant k \leqslant n$. Since $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$ is an efficient covering,

$$N = (N \cap N_1) \cup (N \cap N_2) \cup \ldots \cup (N \cap N_n)$$

is an efficient union, otherwise for some $i \neq j$, $N \cap N_i \subseteq N \cap N_j$ and this would imply

$$N = (N \cap N_1) \cup \ldots \cup (N \cap N_{i-1}) \cup (N \cap N_{i+1}) \cup \ldots (N \cap N_n)$$

and thus we would get $N \subseteq N_1 \cup \ldots \cup N_{i-1} \cup N_{i+1} \cup \ldots \cup N_n$, a contradiction. Hence for every $k \leqslant n$ there exists an element $\ell_k \in N \setminus N_k$. Moreover, $\bigcap_{j \neq k} (N \cap N_j) \subseteq N \cap N_k$ by Theorem 3.2. If $j \neq k$, then $((N_j)_r : M) \nsubseteq ((N_k)_r : M)$ so there exists an $s_j \in ((N_j)_r : M) \setminus ((N_k)_r : M)$. Now, $s = \prod_{j\neq k} s_j \in ((N_j)_r : M)$ but $s = \prod_{j\neq k} s_j \notin ((N_k)_r : M)$ (as $((N_k)_r : M)$ is a prime ideal by Proposition 2.11). Consequently, $rs\ell_k \in N \cap N_j$ for every $j \neq k$ but $rs\ell_k \notin N \cap N_k$, which contradicts to $\bigcap_{j\neq k} (N \cap N_j) \subseteq N \cap N_k$. Therefore no N_k is a 2-absorbing subsemimodule of M. □

Theorem 3.4 (The 2-absorbing avoidance theorem). *Let $N_1, N_2, \ldots, N_n$ be finite number of subtractive subsemimodules of M such that at most two of N_i's are not 2-absorbing subsemimodule of M and let N be a subsemimodule of M such that $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$. If $(N_k : m) = (N_k : M)$ for all $m \in M \setminus N_k$, $\sqrt{(N_k : M)} \neq (N_k : M)$ and there exists $r \in \sqrt{(N_k : M)} \setminus (N_k : M)$ such that $((N_j)_r : M) \nsubseteq ((N_k)_r : M)$ for every $j \neq k$. Then $N \subseteq N_k$ for some k.*

Proof. Suppose that for given $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$, $N \subseteq N_{i_1} \cup N_{i_2} \cup \ldots \cup N_{i_m}$ is its efficient reduction. Then $1 \leqslant m \leqslant n$ and $m \neq 2$. If $m > 2$, then there exists at least one N_{i_j} to be a 2-absorbing subsemimodule of M. Thus, by Lemma 3.3, no N_k is a 2-absorbing as $((N_j)_r : M) \nsubseteq ((N_k)_r : M)$ for $j \neq k$. Hence $m = 1$, namely $N \subseteq N_k$ for some k. □

Now we prove the following result [7, Theorem 3.64] to the semimodule case by consequence of the 2-absorbing avoidance theorem of semimodules.

Theorem 3.5. *Let $N_1, N_2, \ldots, N_\ell$ be finite number of 2-absorbing subtractive subsemimodules of an S-semimodule M. If $(N_k : m) = (N_k : M)$ for all $m \in M \setminus N_k$, $\sqrt{(N_k : M)} \neq (N_k : M)$ and there exists $r \in \sqrt{(N_k : M)} \setminus (N_k : M)$ such that $((N_j)_r : M) \nsubseteq ((N_k)_r : M)$ for every $j \neq k$ and $k = 1, 2, \ldots, l$. If N is a subsemimodule of M and $m \in M$ be such that $mS + N \nsubseteq \bigcup_{i=1}^{\ell} N_i$, then $m + n \notin \bigcup_{i=1}^{\ell} N_i$ for some $n \in N$.*

Proof. Assume that m lies in each of $N_1, \ldots, N_k$ but in none of $N_{k+1}, N_{k+2}, \ldots, N_\ell$. If $k = 0$ then $m = m + 0 \notin \bigcup_{i=1}^{\ell} N_i$, and so nothing to prove. Suppose our claim is true for $k \geqslant 1$.

Now $N \nsubseteq \bigcup_{i=1}^{k} N_i$, for otherwise, by the 2-absorbing avoidance theorem of semimodules we would get a contradiction. Therefore, there exists $d \in N \setminus \bigcup_{i=1}^{k} N_i$. Hence we have $N_{k+1} \cap \ldots \cap N_\ell \nsubseteq N_1 \cup \ldots \cup N_k$. Otherwise, by the 2-absorbing avoidance theorem we get a contradiction. Therefore, there exists

$$s \in (N_{k+1} : M) \cap \ldots \cap (N_\ell : M) \setminus (N_1 : M) \cup \ldots \cup (N_k : M).$$

Hence, we have for $r \in S$,

$$s \in ((N_{k+1})_r : M) \cap \ldots \cap ((N_\ell)_r : M) \setminus ((N_1)_r : M) \cup \ldots \cup ((N_k)_r : M).$$

Let $n = (rs)d \in N$. Also, $n \in \bigcap_{j=k+1}^{\ell} N_j$. Then $n = (rs)d \notin N_1 \cup \ldots \cup N_k$. Otherwise, $n = rsd \in N_i$ for $1 \leqslant i \leqslant k$. This implies that either $rs \in (N_i : M)$ or $rd \in N_i$ or $sd \in N_i$ since N_i is 2-absorbing. Then

$$n \in (N_{k+1} \cap \ldots \cap N_\ell) \setminus (N_1 \cup \ldots \cup N_k).$$

Therefore, since $m \in (N_1 \cup \ldots \cup N_k)$, it follows that $m + n \notin \bigcup_{i=1}^{\ell} N_i$. □

Proposition 3.6. *Let N be a 2-absorbing subsemimodule of M and $N_1, N_2, \ldots, N_k$ are subtractive subsemimodules of the multiplication semimodule M over the semiring S. Then $\bigcap_{i=1}^{k} N_i \subseteq N$ if and only if $N_j \subseteq N$ for some $1 \leqslant j \leqslant k$.*

Proof. Let $N_j \subseteq N$ for some $1 \leqslant j \leqslant k$. Then $\bigcap_{i=1}^{k} N_i \subseteq N_j \subseteq N$. Conversely, let $\bigcap_{i=1}^{k} N_i \subseteq N$. Then $(\bigcap_{i=1}^{k} N_i : M) \subseteq (N : M)$. Since N is a 2-absorbing subsemimodule of M. Therefore, $(N : M)$ is a 2-absorbing ideal of semiring S. Also, $(\bigcap_{i=1}^{k} N_i : M) = \bigcap_{i=1}^{k} (N_i : M)$. Therefore, we have $(N_j : M) \subseteq (N : M)$. This gives $N_j \subseteq N$. □

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