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Contractions of quasigroups and Latin squares

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Abstract. By a contraction (compression) we mean a method of reduction of the multiplication table of a quasigroup having n elements to the multiplication table of a quasigroup having $n-1$ elements. We describe such method and characterize quasigroups allowing a contraction.

1. Introduction

By a *prolongation* of a quasigroup we mean a process which shows how from a given quasigroup $Q(\cdot)$ of order n we can obtain a new quasigroup $Q'(\circ)$ of order $n+1$. In other words, it is a process which shows how a given Latin square can be extended to a new Latin square containing one more column and row. The first method of a prolongation was given by R. H. Bruck [5] for Steiner quasigroups. More general method was proposed V. D. Belousov in [1] (see also [2]).

G. B. Belyavskaya studied this problem in [3] together with the inverse problem, called *contraction* or *compression of quasigroups*, i.e., how from a given Latin square of order n we can obtain a Latin square of order $n-1$. Her method of prolongation is based on the existence of a complete map. In [7] we extend this metod to a quasicomplete map. The method of contractions proposed by G. B. Belyavskaya is based on the identity (3), where $a \cdot b = c$. In our two methods, presented below, the condition $a \cdot b = c$ is omitted. In the second our method we also replace the identity (3) by a weaker identity (5).

2. Preliminaries

The composition of permutations is defined in the usual way as $\varphi\psi(x) = \varphi(\psi(x))$. Permutations are written in the form of cycles, cycles are separated by points:

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix} = (132.45.6.)\,.$$

Each quasigroup $Q(\cdot)$ admits several special permutations. The left multiplication by $a \in Q$, i.e., the map $L_a(x) = a \cdot x$ is called a *left translation* of Q. The right

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multiplication by $a \in Q$, i.e., the map $R_a(x) = x \cdot a$ is called a *right translation* of Q. A permutation φ_i defined by

$$x \cdot \varphi_i(x) = i, \quad i \in Q \tag{1}$$

is called the *middle translation* by i or a *track* of an element $i \in Q$.

3. D-contractions

Consider a quasigroup $Q(\cdot)$ in which one can find three elements $a, b, c \in Q$ such that for all $x, y \in Q - \{c\}$, $x \neq a$, $y \neq b$ we have

$$x \cdot y = c \Longleftrightarrow x \cdot b = a \cdot y. \tag{2}$$

This condition means that in the multiplication table of $Q(\cdot)$ the element c has the same projection onto the row a and the column b except the case when c is in the row a or in the column b.

Comparing $x \cdot y = c$ with (1) we see that $y = \varphi_c(x)$. Thus, the right side of (2) can be written in the form $x \cdot b = a \cdot \varphi_c(x)$. Therefore for $x \in Q - \{a, c, \varphi_c^{-1}(b)\}$ we have

$$R_b(x) = L_a \varphi_c(x). \tag{3}$$

In a similar way we can see that (3) implies (2). Hence for $x, y \in Q - \{c\}$, $x \neq a$, $y \neq b$, the above two conditions are equivalent. Since $x \neq a$ they do not imply $a \cdot b = c$.

It is not difficult to verify that the construction presented in the following theorem gives a quasigroup.

Theorem 3.1. *Any quasigroup $Q(\cdot)$ containing three elements $a, b, c \in Q$ satisfying* (3) *allows a contraction to a quasigroup $Q'(\circ)$, where $Q' = Q - \{c\}$ and*

$$x \circ y = \begin{cases} x \cdot y & if \quad x \cdot y \neq c, \ x \neq a, \ y \neq b, \\ x \cdot b & if \quad x \cdot y = c, \ x \neq a, \ y \neq b, \\ c \cdot y & if \quad c \cdot y \neq c, \ x = a, \ y \neq b, \\ x \cdot c & if \quad x \cdot c \neq c, \ x \neq a, \ y = b, \\ a \cdot y & if \quad c \cdot y = c, \ x = a, \ y \neq b, \\ x \cdot b & if \quad x \cdot c = c, \ x \neq a, \ y = b, \\ c \cdot c & if \quad x = a, \ y = b. \end{cases} \tag{4}$$

In the sequel, for simplicity, this contraction will be called the *D-contraction*. It can be described by the following four steps:

1. all elements c are replaced by elements of the b-column located in the same row as c,
2. elements of the b-column are replaced by elements of the c-column,
3. elements of the a-row are replaced by elements of the c-row,
4. the c-row and the c-column are deleted.

Example 3.2. Consider the quasigroup $Q(\cdot)$ defined by the following table:

$\cdot$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	1	7	4	5	6
3	3	4	2	6	7	1	5
4	4	5	6	3	2	7	1
5	5	6	7	1	3	4	2
6	6	7	5	2	1	3	4
7	7	1	4	5	6	2	3

In this table the element 3 has the same projections onto the 5th row and the 7th column except the case where 3 occurs in the 5th row or in the 7th column. So, (3) is satisfied for $a = 5$, $b = 7$ and $c = 3$. This means that the quasigroup $Q(\cdot)$ can be contracted to the quasigroup $Q' = Q - \{3\}$ with the following multiplication table:

$\circ$	1	2	4	5	6	7
1	1	2	4	5	6	7
2	2	$\boxed{6}$	7	4	5	$\boxed{1}$
4	4	5	$\boxed{1}$	2	7	$\boxed{6}$
5	5	$\boxed{4}$	$\boxed{6}$	$\boxed{7}$	$\boxed{1}$	$\boxed{2}$
6	6	7	2	1	$\boxed{4}$	$\boxed{5}$
7	7	1	5	6	2	$\boxed{4}$

$\sim$

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	5	6	3	4	1
3	3	4	1	2	6	5
4	4	3	5	6	1	2
5	5	6	2	1	3	4
6	6	1	4	5	2	3

where entries in the box are calculated according to the above procedure. □

Example 3.3. The quasigroup $Q(\cdot)$ from the previous example allows also three other contractions. Namely, it allows contractions determined by elements

(i) $a = 3$, $b = 7$, $c = 7$,
(ii) $a = 6$, $b = 7$, $c = 2$,
(iii) $a = 1$, $b = 1$, $c = 3$.

In the case (i) we obtain the quasigroup

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	6	4	5
3	3	1	4	5	6	2
4	4	5	6	3	2	1
5	5	6	2	1	3	4
6	6	4	5	2	1	3

which is not isotopic to the quasigroup obtained in Example 3.2. Indeed, as is not difficult to see, these two quasigroups have different indicators (cf. [6]). Hence they cannot be isotopic.

The quasigroup obtained in the case (ii) is isotopic to the quasigroup from the case (i). In the case (iii) we obtain a quasigroup which is not isotopic to any of the previous. So, the quasigroup $Q(\cdot)$ allows contractions to three non-isotopic quasigroups. □

Proposition 3.4. *A quasigroup isotopic to a quasigroup having a D-contraction also has a D-contraction.*

Proof. Indeed, in a quasigroup $Q(\cdot)$ possessing D-contraction there are $a, b, c \in Q$ satisfying (2). In a quasigroup $Q(\circ)$ isotopic to $Q(\cdot)$ elements $\alpha(a), \beta(b), \gamma(c)$, where $\gamma(x \cdot y) = \alpha(x) \circ \beta(y)$, satisfy (2). Thus a quasigroup $Q(\cdot)$ also has the D-contraction. □

Proposition 3.5. *If a quasigroup $Q(\cdot)$ has elements a, b, c satisfying (2), then it is isotopic to a loop $Q(*)$ with the identity $e = a \cdot b$ in which $z * z = c$ holds for all $z \in Q - \{a \cdot b, c\}$.*

Proof. Let a, b, c satisfy (2). Then $Q(*)$, where $x * y = R_b^{-1}(x) \cdot L_a^{-1}(y)$, is a loop with the identity $e = a \cdot b$. Since for $z \in Q$ there are $x, y \in Q$ such that $z = x \cdot b = a \cdot y$, so, according to (2), for $z \neq c$, $x \neq a$, $y \neq b$, we have $x \cdot y = c$. Thus $z * z = R_b(x) * L_a(y) = R_b^{-1} R_b(x) \cdot L_a^{-1} L_a(y) = x \cdot y = c$ for $z \neq c$ and $z \neq a \circ b$. In the case $z = a \cdot b$ we have $z * z = z$ since z is the identity of $Q(*)$. □

For $z = c = x \cdot b = a \cdot y$ we obtain $c * c = x \cdot y$. Hence, $c * c = c$ for $x = a$, $y = b$ (in this case c is the identity of $Q(*)$) or $c * c \neq c$.

Proposition 3.6. *If in a loop $Q(*)$ with the identity e there exists c such that $z * z = c$ for all $z \in Q - \{e, c\}$, then $Q(*)$ has a D-contraction.*

Proof. Indeed, in this loop (2) is satisfied for $a = b = e$. □

As a consequence of the above two propositions we obtain

Theorem 3.7. *A quasigroup $Q(\cdot)$ has a D-contraction if and only if it is isotopic to a loop $Q(*)$ with the identity e in which there exists an element c such that $z * z = c$ for all $x \in Q - \{e, c\}$.* □

Corollary 3.8. *A quasigroup isotopic to a Boolean group has a D-contraction.*

Proof. For all elements of a Boolean group we have $x^2 = e$, hence in such group (2) is satisfied for $a = b = c = e$. Consequently, each quasigroup isotopic to this group has a D-contraction. □

Also it is not difficult to see that the cyclic group $\mathbb{Z}_3$ has a D-contraction. For groups we have the following results.

Theorem 3.9. *A group G allows a D-contraction if and only if it has an element c such that $c^2 = e$ and $x^2 = c$ for all $x \in G - \{e, c\}$.*

Proof. Suppose that a group G allows a D-contraction. Then, by Theorem 3.7, there exist $c \in G$ such that $x^2 = c$ for all $x \in G - \{e, c\}$. If $c = e$, then obviously $c^2 = e$. If $c \neq e$, then for $x \in G - \{e, c\}$ we have $c = (x^{-1})^2 = (x^2)^{-1} = c^{-1}$. Hence $c^2 = e$.

Conversely, if in a group G there is an element c such that $x^2 = c$ for all $x \in G - \{e, c\}$, then G satisfies (2) with $a = b = e$, so it has a D-contraction. □

Corollary 3.10. *If a group G of order $n \geqslant 4$ has a D-contraction, then its exponent is a divisor of* 4.

Proof. In fact, in such group $x^4 = c^2 = e$ for each $x \in G$. □

The converse statement is not true. The exponent of the group $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ is equal to 4 but this group has no c such that $x^2 = c$ for all $x \in G - \{e, c\}$. Thus it has no D-contraction.

4. Special cases

1. Our D-contraction for $c = a \cdot b$ gives the contraction proposed by Belyavskaya (see [3] and [4]). In particular, as a simple consequence of our results we obtain the following characterizations of quasigroups allowing the contraction proposed by Belyavskaya.

Proposition 4.1. *A quasigroup $Q(\cdot)$ allows a contraction proposed by Belyavskaya if and only if it is isotopic to a loop of exponent* 2. □

2. In another special case when $a = b = c$ from (2) we obtain

$$x \cdot y = c \Longleftrightarrow x \cdot c = c \cdot y,$$

i.e., $R_c = L_c \varphi_c$ for all $x, y \in Q - \{c\}$. In this case Theorem 3.1 has the form

Theorem 4.2. *If in a quasigroup $Q(\cdot)$ there exists an element $c \in Q$ such that $R_c = L_c \varphi_c$, then $Q(\cdot)$ allows a contraction to a quasigroup $Q'(\circ)$, where $Q' = Q - \{c\}$ and*

$$x \circ y = \begin{cases} x \cdot y & \text{if} \quad x \cdot y \neq c, \\ x \cdot c & \text{if} \quad x \cdot y = c. \end{cases}$$ □

The last formula may be written in the form

$$x \circ y = \begin{cases} x \cdot y & \text{if} \quad y \neq \varphi_c(x), \\ x \cdot c & \text{if} \quad y = \varphi_c(x). \end{cases}$$

Example 4.3. Consider the quasigroup $Q(\cdot)$ defined by the table:

$\cdot$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	1	7	3	6	5	4
3	3	4	1	5	2	7	6
4	4	5	2	6	7	1	3
5	5	3	6	7	4	2	1
6	6	7	5	1	3	4	2
7	7	6	4	2	1	3	5

In this quasigroup the element $c = 4$ has the same projection onto the fourth row and the fourth column, except the case when it is located in the fourth row and in the fourth column. Hence from the multiplication table of $Q(\cdot)$ we can delete the fourth row and the fourth column. Replacing in the reduced table all $c = 4$ by the corresponding elements of the deleted column we obtain the multiplication table of a quasigroup $Q'(\circ)$, where $Q' = Q - \{4\}$ and

$\circ$	1	2	3	5	6	7
1	1	2	3	5	6	7
2	2	1	7	6	5	3
3	3	5	1	2	7	6
5	5	3	6	7	2	1
6	6	7	5	3	1	2
7	7	6	2	1	3	5

$\sim$

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	5	4	3
3	3	4	1	2	6	5
4	4	3	5	6	2	1
5	5	6	4	3	1	2
6	6	5	2	1	3	4

By the converse procedure we can reconstruct $Q(\cdot)$ from $Q'(\circ)$. □

Obviously, the reconstruction of $Q(\cdot)$ from $Q'(\circ)$ is a prolongation of $Q'(\circ)$ but it is not a prolongation by the method proposed by Belyavskaya (see the formula (6) in [7]). According to this formula in the above case should be $q = c = 4$, $a = 6$, $x_a \cdot q = q$ and $6 = x_a \circ \sigma(x_a)$. Hence $x_a = 1$ and $\sigma(1) = 6$. But using this method it should be $q \cdot \sigma(x_a) = q$, i.e., $4 \cdot 6 = 4$ which is not true.

$Q(\cdot)$ can be reconstructed from $Q'(\circ)$ as a prolongation of $Q'(\circ)$ obtained by the formula (9) in [7], where $x_1 = 1$, $x_2 = 6$, $a = 1$, $d = 6$ and $\sigma = (1.273.5.6.)$.

Example 4.4. Consider a quasigroup $Q(\cdot)$ defined by the table:

$\cdot$	1	2	3	4	5	6
1	1	2	4	3	5	6
2	4	3	6	5	2	1
3	2	6	3	4	1	5
4	5	4	1	6	3	2
5	6	1	5	2	4	3
6	3	5	2	1	6	4

In this quasigroup 3 is an idempotent and $x \circ y = 3$ only in the case when $x \circ 3 = 3 \circ y$. So, 3 can be deleted. Hence $Q' = Q - \{3\}$ and $Q'(\circ)$ has the following multiplication table:

$\circ$	1	2	4	5	6
1	1	2	4	5	6
2	4	6	5	2	1
4	5	4	6	1	2
5	6	1	2	4	5
6	2	5	1	6	4

$\sim$

$\circ$	1	2	3	4	5
1	1	2	3	4	5
2	3	5	4	2	1
3	4	3	5	1	2
4	5	1	2	3	4
5	2	4	1	5	3

In this case $Q(\cdot)$ can be reconstructed from $Q'(\circ)$ by the Belousov's method (the formula (3) in [7]), where $\overline{\sigma}(x) = x \cdot 3 = (14.26.5.)$ and $\sigma = (1456.2.)$. □

3. Cyclic group have a D-contraction only in the case when they have 2, 3 or 4 elements. But if the multiplication table of a cyclic group has a special form another type of contraction is possible. It is presented below where the cyclic group $\mathbb{Z}_6$ is contracted to the group $\mathbb{Z}_5$.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\Longrightarrow$

	0	1	2	3	4	
0	0	1	2	3	4	
1	1	2	3	4		0
2	2	3	4		0	1
3	3	4		0	1	2
4	4		0	1	2	3

$\Longrightarrow$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

In a similar way we can reduce any group $\mathbb{Z}_n$ to the group $\mathbb{Z}_{n-1}$. The converse procedure gives a prolongation of $\mathbb{Z}_{n-1}$ to $\mathbb{Z}_n$.

5. Isotopic contractions

Consider a quasigroup $Q(\cdot)$ in which for some $a, b, c \in Q$ the following identity

$$\varphi_c^{-1} L_a^{-1} R_b = (a, p), \tag{5}$$

where (a, p) is a transposition of a and $p = R_b^{-1}(c)$, is satisfied.

Let $Q(\circ)$ be a quasigroup isotopic to $Q(\cdot)$. Then

$$\gamma(x \circ y) = \alpha(x) \cdot \beta(y) \tag{6}$$

for some permutations α, β, γ of the set Q.

Since $a = \alpha(a_1)$, $b = \beta(b_1)$, $c = \gamma(c_1)$, $p = \alpha(p_1)$ for some $a_1, b_1, c_1, p_1 \in Q$, we have $\gamma(c_1) = c = p \cdot b = \alpha(p_1) \cdot \beta(b_1) = \gamma(p_1 \circ b_1)$, which gives $c_1 = p_1 \circ b_1$. So, $p \cdot b = c$ in $Q(\cdot)$ implies $p_1 \circ b_1 = c_1$ in $Q(\circ)$. Also, as it is not difficult to see, $\alpha^{-1}(a, p)\alpha = (a_1, p_1)$.

Moreover, according to [6], for all $i \in Q$ we have

$$\overset{\circ}{\varphi}_i = \beta^{-1}\varphi_{\gamma(i)}\alpha, \qquad \overset{\circ}{L}_i = \gamma^{-1}L_{\alpha(i)}\beta, \qquad \overset{\circ}{R}_i = \gamma^{-1}R_{\beta(i)}\alpha,$$

where $\overset{\circ}{\varphi}_i, \overset{\circ}{L}_i, \overset{\circ}{R}_i$ are defined on a quasigroup $Q(\circ)$. Thus

$$\overset{\circ}{\varphi}{}_{c_1}^{-1} = \alpha^{-1}\varphi_c^{-1}\beta, \qquad \overset{\circ}{L}{}_{a_1}^{-1} = \beta^{-1}L_a^{-1}\gamma, \qquad \overset{\circ}{R}_{b_1} = \gamma^{-1}R_b\alpha.$$

Consequently,

$$\overset{\circ}{\varphi}{}_{c_1}^{-1}\overset{\circ}{L}{}_{a_1}^{-1}\overset{\circ}{R}_{b_1} = \alpha^{-1}(\varphi_c^{-1}L_a^{-1}R_b)\alpha \overset{(5)}{=} \alpha^{-1}(a, p)\alpha = (a_1, p_1).$$

This proves that an isotopy saves (5). Hence this identity is universal.

Moreover, for $x \in Q - \{a, p\}$ it has the form $\varphi_c^{-1} L_a^{-1} R_b(x) = x$. Hence, any quasigroup satisfying (5) also satisfies (3). Thus, a quasigroup satisfying (5) allows a contraction.

Suppose that a quasigroup $Q(\cdot)$, $Q = \{1, 2, \ldots, n\}$, satisfies (5). Let $Q(\circ)$ be a quasigroup connected with $Q(\cdot)$ by (6), where $\gamma = (c, n)$, $\alpha = (a, n)$, $\beta = (b, n)$. Then $\overset{\circ}{\varphi}{}_n^{-1} \overset{\circ}{L}{}_n^{-1} \overset{\circ}{R}_n = (n, p_1)$, where $p_1 \circ n = n$. In the case $a \cdot b = c$ we have $p_1 \circ n = n = \gamma(c) = \gamma(a \cdot b) = \alpha(a) \cdot \beta(b) = n \circ n$, which implies $p_1 = n$. Consequently $\overset{\circ}{\varphi}{}_n^{-1} \overset{\circ}{L}{}_n^{-1} \overset{\circ}{R}_n = \varepsilon$, i.e., $\overset{\circ}{R}_n = \overset{\circ}{L}_n \overset{\circ}{\varphi}_n$. So, in the multiplication table of $Q(\circ)$ the element n has the same projection onto the last row and onto the last column. If $a \cdot b \neq c$, then $n \circ n \neq n$, and consequently, $p_1 \neq n$. In this case in the multiplication table of $Q(\circ)$ the element n has the same projection onto the last column and the last row except when it is in the p or in the last row.

Hence from the multiplication table of $Q(\circ)$ the last row and the last column can be deleted and all others elements n should be replaced by the corresponding elements of the last column (or the last row).

Example 5.1. Consider the quasigroup $Q(\cdot)$ defined in Example 4.4. From this quasigroup we can delete 3. Since $a = b = c = 3$, $a \cdot b = c$ and $\alpha = \beta = \gamma = (3, 6)$, according to the above procedure, in the multiplication table of $Q(\cdot)$ we must exchange the third row on the sixth, next we exchange the third column on the sixth column. As a result we obtain the multiplication table of $Q(\circ)$. Now in this table we replace all $c = 3$ by the elements of the new sixth column (located in the same row) and delete the last row and the last column.

$\circ$	1	2	6	4	5	3
1	1	2	6	3	5	4
2	4	3	1	5	2	6
3	3	5	4	1	6	2
4	5	4	2	6	3	1
5	6	1	3	2	4	5
6	2	6	5	4	1	3

$\Longrightarrow$

$*$	1	2	6	4	5
1	1	2	6	4	5
2	4	6	1	5	2
4	2	5	4	1	6
5	5	4	2	6	1
6	6	1	5	2	4

$\sim$

$*$	1	2	3	4	5
1	1	2	3	4	5
2	4	3	1	5	2
3	3	1	5	2	4
4	2	5	4	1	3
5	5	4	2	3	1

Obtained quasigroup $Q'(*)$ is isotopic to the quasigroup $Q'(\circ)$ constructed in Example 4.4. □

Example 5.2. The quasigroup $Q(\cdot)$ defined in Example 3.2 allows the contraction for $a = 5$, $b = 7$, $c = 3$. For this quasigroup we have $a \cdot b \neq c$ and

$$\varphi_3^{-1} L_5^{-1} R_7 = (1,3)(1462735.)(1735264.) = (5,7) = (a,p).$$

Hence this quasigroup satisfies (5). Putting $\alpha = (a, 7) = (5, 7)$, $\beta = (b, 7) = (7, 7)$, $\gamma = (c, 7) = (3, 7)$ in (6) we construct the quasigroup $Q(\circ)$ with the multiplication table

∘	1	2	3	4	5	6	7
1	1	2	7	4	5	6	3
2	2	7	1	3	4	5	6
3	7	4	2	6	3	1	5
4	4	5	6	7	2	3	1
5	3	1	4	5	6	2	7
6	6	3	5	2	1	7	4
7	5	6	3	1	7	4	2

Deleting the last row and the last column and replacing all elements 7 by their projection on the last column, we obtain the table

∗	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	6	1	3	4	5
3	5	4	2	6	3	1
4	4	5	6	1	2	3
5	3	1	4	5	6	2
6	6	3	5	2	1	4

$\sim$

∗	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	6	1	3	4	5
3	3	1	4	5	6	2
4	4	5	6	1	2	3
5	5	4	2	6	3	1
6	6	3	5	2	1	4

Such constructed quasigroup $Q'(*)$ is isotopic to the quasigroup $Q'(\circ)$ obtained in Example 3.2. □

Example 5.3. As was mentioned in Example 3.3 the quasigroup $Q(\cdot)$ defined in Example 3.2 allows also three other contractions induced by:

(*i*) $a = 3,\ b = 7,\ c = 7$,

(*ii*) $a = 6,\ b = 7,\ c = 2$,

(*iii*) $a = 1,\ b = 1,\ c = 3$.

In the case (*i*) we have $\varphi_7^{-1}L_3^{-1}R_7 = (3,1)$. So, in the above procedure we put $\alpha = (3,7)$, $\beta = (7,7)$, $\gamma = (7,7)$. Consequently,

∘	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	1	7	4	5	6
3	7	1	4	5	6	2	3
4	4	5	6	3	2	7	1
5	5	6	7	1	3	4	2
6	6	7	5	2	1	3	4
7	3	4	2	6	7	1	5

$\Longrightarrow$

∗	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	6	4	5
3	3	1	4	5	6	2
4	4	5	6	3	2	1
5	5	6	2	1	3	4
6	6	4	5	2	1	3

Obtained quasigroup $Q'(*)$ is isotopic to the quasigroup $Q'(\circ)$ constructed in Example 3.3 (*i*).

In the case (*ii*) we have $\varphi_2^{-1}L_6^{-1}R_7 = (6,5)$, which leads to the quasigroup isotopic to the quasigroup from the case (*i*). In the case (*iii*) we obtain the quasigroup which is not isotopic to any of the previous. □

In any case we can reconstruct the initial quasigroup $Q(\cdot)$ from each of its contractions. The method of reconstruction depends on the method of contraction. This reconstruction is a prolongation of the obtained contraction. In our method the map φ_c (calculated in the initial quasigroup) defines on the final quasigroup the complete or quasicomplete map. Indeed, as it is not difficult to see, if in the initial quasigroup $a \cdot b \neq c$ but $a \cdot b = p \cdot q$, where $p \cdot b = a \cdot q = c$, then φ_c defines on a contracted quasigroup the complete map. If $a \cdot b \neq p \cdot q$, then φ_c defines the quasicomplete map. In the first case we can use the method of prolongation proposed by Belyavskaya (see [4]); in the second – our method described in [7].

References

[1] **V. D. Belousov:** *Extensions of quasigroups*, (Russian), Bull. Akad. Stiinte RSS Moldoven **8** (1967), 3 – 24.

[2] **V. D. Belousov:** *Foundations of the theory of quasigroups and loops*, (Russian), "Nauka", Moscow 1967.

[3] **G. B. Belyavskaya:** *Contraction of quasigroups, I*, (Russian), Izv. Akad. Nauk Moldav. SSR, ser. Fiz.-tehn. mat. Nauk **1** (1970), 6 – 12.

[4] **G. B. Belyavskaya:** *On generalized prolongation of quasigroups*, (Russian), Mat. Issled. **5** (1970), No.2, 28 – 48.

[5] **R. H. Bruck**, *Some results in the theory of quasigroups*, Trans. Amer. Math. Soc. **55** (1944), 19 – 52.

[6] **I. I. Deriyenko**, *Indicators of quasigroups*, Quasigroups and Related Systems **19** (2011), 223 – 226;

[7] **I. I. Deriyenko and W. A. Dudek**, *On prolongations of quasigroups*, Quasigroups and Related Systems **16** (2008), 187 – 198.

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