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On regular medial division algebras

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Abstract. We prove a Toyoda's type theorem for regular medial division n-ary groupoids and regular medial division algebras without unary operations.

1. Introduction

We recall that the algebra (Q, Σ) is said to be *medial* (*entropic*), if it satisfies the mediality hyperidentity (for hyperidentities see [13]), i.e., for any $f, g \in \Sigma$:

$$f(g(x_{11}, ..., x_{1n}), ..., g(x_{m1}, ..., x_{mn})) = g(f(x_{11}, ..., x_{m1}), ..., f(x_{1n}, ..., x_{mn})). \quad (1)$$

In particular, the n-ary groupoid $Q(f)$ is said to be *medial*, if it satisfies the identity:

$$f(f(x_{11}, ..., x_{1n}), ..., f(x_{n1}, ..., x_{nn})) = f(f(x_{11}, ..., x_{n1}), ..., f(x_{1n}, ..., x_{nn})).$$

It should be noted here that medial identity studies have been made under various names: abelian, alternation, bi-commutative, bisymmetric, entropic, surcommutative.

Medial systems were studied by many authors (Sade, Stein, Toyoda, Bruck, Belousov, Kurosh, Smith, Romanowska, Dudek, Ježek, Kepka, Movsisyan, Shcherbacov and others). Medial systems are connected with the notion of entropy in information theory [18], and have some applications in cybernetics, economics, physics and biology.

In [16], multiplicative semigroups of a field are characterized by the Cayley type theorem, using the transitive mode (i.e., an idempotent and medial algebra [17]).

Some special types of medial n-ary groupoids are described in [4] and [5]. Some aspects of binary medial algebras are considered in [3].

The n-ary groupoid $Q(f)$ is called an n-ary quasigroup or in short, an *n-quasigroup*, if in the equation $f(x_1, ..., x_n) = x_{n+1}$ any n elements of $x_1, x_2, ..., x_n, x_{n+1}$ uniquely determine the remaining one.

In [2] V.D. Belousov proved the following theorem. (This theorem follows from results of T. Evans ([7], Theorem 6.2), too.)

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Theorem 1.1. *Let $Q(f)$ be a medial n-ary quasigroup. Then there exist an abelian group $Q(+)$, its pairwise commuting automorphisms $\alpha_1, \dots, \alpha_n$, and an element a of the set Q such that*

$$f(x_1, x_2, \dots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n + a$$

for all $x_i \in Q$, $i = 1, \dots, n$. □

The classical Toyoda theorem (see [1]) follows from Theorem 1.1.

Let $G(\cdot)$ be a groupoid and $a \in G$. Denote by L_a (R_a) the map of G to G such that $L_a(x) = ax$ ($R_a(x) = xa$) for all $x \in G$.

A groupoid $G(\cdot)$ is said to be a division groupoid if L_a and R_a are surjective for every $a \in G$.

A groupoid $G(\cdot)$ is called *left regular* if $R_a = R_b$ whenever $a, b \in G$ and $ca = cb$ for some $c \in G$. *Right regular groupoids* are defined dually. A groupoid is *regular* if it is both left and right regular.

The following characterization of medial regular division binary groupoids was obtained by Kepka ([10]).

Theorem 1.2. *A groupoid $G(\cdot)$ is a regular medial division groupoid if and only if there exist an abelian group $G(+)$, two surjective endomorphisms f, g of $G(+)$, and an element $a \in G$ such that $fg = gf$ and $x \cdot y = f(x) + g(y) + a$, for all $x, y \in G$.* □

In this paper we generalized the Kepka theorem for medial regular division n-ary groupoids and medial regular division algebras without unary operations.

2. Preliminary notions and results

First we introduce some notations. The sequence $x_n, x_{n+1}, \dots, x_m$ is denoted by x_n^m or $\{x_i\}_n^m$, where n, m are natural numbers, $n \leq m$. If $n = m$, then x_n^m is an element x_n. The sequence $a, a, \dots, a$ (m times) is denoted by a^m. The operations on the set Q are denoted by A, B, C or $(a_1^n) = b$ and $[a_1^n] = b$. The nonempty set Q with an n-ary operation A is called an *n-ary groupoid* or in short, an *n-groupoid*.

Let $Q(A)$ be an m-groupoid and $A(x_1^m) = y$. If we replace $x_{k_1}, x_{k_2}, \dots, x_{k_n}$ ($n < m$) by fixed elements $a_1, a_2, \dots, a_n$ in $A(x_1^n)$, then we obtain

$$A(x_1^{k_1-1}, a_1, x_{k_1+1}^{k_2-1}, a_2, \dots, x_{k_{n-1}+1}^{k_n-1}, a_n, x_{k_n+1}^m).$$

Thus we get a new operation $B(x_1^{k_1-1}, x_{k_1+1}^{k_2-1}, \dots, x_{k_n+1}^m)$ with the arity, $m - n$. The $(m-n)$-groupoid, $Q(B)$, is called the *retract* of the m-groupoid $Q(A)$.

Let $Q(\ )$ be an n-groupoid. Denote by $\overline{a}$ the sequence $a_1^n \in Q$ and by $L_i(\overline{a})$ the map from Q to Q such that

$$L_i(\overline{a})x = (a_1 \dots a_{i-1} x a_{i+1} \dots a_n) = (a_1^{i-1} x a_{i+1}^n)$$

for all $x \in Q$. The map $L_i(\overline{a})$ is called the *i-translation* with respect to a.

An n-groupoid $Q(\ )$ is called a *division n-groupoid* if every $L_i(\overline{a})$ is a surjection for all $\overline{a} \in Q$ and all $i = 1, \dots, n$. Note that every retract of the medial division n-groupoid also is medial.

An n-groupoid $Q(\ )$ is *i-regular* if $L_i(\overline{a}) = L_i(\overline{b})$, whenever $\overline{a}, \overline{b} \in Q$ and $L_i(\overline{a})c = L_i(\overline{b})c$, for some $c \in Q$. An n-groupoid $Q(\ )$ is *regular* if it is i-regular, for every $i = 1, \dots, n$. Note that our i-regularity is different from the regularity proposed bt Sioson (see for example [6]).

It is clear that every retract of the regular n-groupoid also is regular.

The triplet $T = (\alpha, \beta, \gamma)$ of maps of $Q(\cdot)$ into itself is called an *endotopy* of $Q(\cdot)$ if the identity $\gamma(x \cdot y) = \alpha x \cdot \beta y$ is true for all $x, y \in Q$. The third component γ of this endotopy is called a *quasiendomorphism*. In the case $\alpha = \beta = \gamma$ the triplet $T = (\gamma, \gamma, \gamma)$ is called an *endomorphism*.

The following two lemmas are proved in [19].

Lemma 2.1. *Any quasiendomorphism γ of a group, $Q(+)$ has the form:*

$$\gamma = \widetilde{R}_s \gamma_0, \tag{2}$$

where γ_0 is an endomorphism of the group $Q(+)$, $\widetilde{R}_s(x) = x + s$, $s \in Q$, and, conversely, the map γ defined by (2) *is a quasiendomorphism of the group $Q(+)$.* □

Lemma 2.2. *Let γ be a quasiendomorphism of the group, $Q(+)$. Then γ is endomorphism if and only if $\gamma(0) = 0$, where 0 is the identity element of the group $Q(+)$.* □

The groupoid $Q(\cdot)$ is *homotopic* to the groupoid $Q(*)$ if there exist three maps α, β, γ of Q to Q such that $\gamma(x * y) = \alpha x \cdot \beta y$ for all $x, y \in Q$. The homotopy of the form $T = (\alpha, \beta, \varepsilon)$, where ε is the identity map, is called *principal*.

Lemma 2.3. *If the group $Q(*)$ is principally homotopic to the group $Q(\cdot)$, then $x * y = x \cdot k \cdot y$ for some $k \in Q$ and all $x, y \in Q$.*

Proof. We have $x * y = \alpha x \cdot \beta y$, where α, β are the maps of Q to Q. Putting in this equality: $y = e$ and $x = e$, where e is the identity element of the group $Q(*)$, we obtain:

$$x = \alpha x \cdot \beta e, \qquad y = \alpha e \cdot \beta y,$$

i.e.,

$$\alpha x = x(\beta e)^{-1}, \qquad \beta y = (\alpha e)^{-1} \cdot y.$$

Therefore, we get: $x * y = (x \cdot (\beta e)^{-1}) \cdot ((\alpha e)^{-1} \cdot y) = x \cdot k \cdot y.$ □

3. Main results

Theorem 3.1. *Let $Q(\ )$ be a regular medial division n-groupoid. Then there exist an abelian group $Q(+)$, its pairwise commuting surjective endomorphisms $\alpha_1, \ldots, \alpha_n$, and a fixed element $b \in Q$ such that*

$$(x_1 x_2 \ldots x_n) = (x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n + b$$

for all $x_i \in Q$, $i = 1, \ldots, n$.

Proof. The proof is by induction on n. For $n = 2$, the assumption follows from Theorem 1.2. Suppose the theorem is true for all natural numbers which are less than n. Let us write the medial identity as a matrix:

$$\begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{pmatrix}, \tag{3}$$

$$\left(\{x_{ij}\}_{j=1}^n\right) = y_i, \qquad \left(\{x_{ij}\}_{i=1}^n\right) = z_j.$$

Then, the medial identity can be represented as:

$$(y_1^n) = (z_1^n). \tag{4}$$

Consider the following matrix:

$$\begin{pmatrix} a & a & a & a & \ldots & a \\ x_1 & a & a & a & \ldots & a \\ a & x_2 & x_3 & x_4 & \ldots & x_n \\ a & a & a & a & \ldots & a \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a & a & a & a & \ldots & a \end{pmatrix}.$$

For y_i and z_j from (4), we have:

$$\begin{cases} y_i = (a^n) = b, \quad i \neq 2, 3, \\ y_2 = (x_1 a^{n-1}) = \alpha x_1, \\ y_3 = (a x_2^n), \end{cases} \qquad \begin{cases} z_1 = (a x_1 a^{n-2}) = \beta x_1, \\ z_i = (a^2 x_i a^{n-3}) = \mu x_i, \quad i \neq 1, \end{cases}$$

where α, β, μ are some surjections from Q to Q. Thus, from (4), we obtain:

$$(b, \alpha x_1, (a x_2^n), b^{n-3}) = (\beta x_1, \{\mu x_i\}_{i=2}^n).$$

Let $A(u, v) = (b, u, v, b^{n-3})$. Then $Q(A)$ is a regular, medial and division groupoid.

Let

$$B(x_2^n) = (ax_2^n). \tag{5}$$

Then B is a regular, medial and division $(n-1)$-ary operation.

By the assumption, there exist abelian groups $Q(\oplus)$ and $Q(\dot{+})$ such that:

$$A(u,v) = \gamma u \oplus \delta v \oplus d,$$

$$B(u_2^n) = \lambda_2 u_2 \dot{+} \lambda_3 u_3 \dot{+} \ldots \dot{+} \lambda_n u_n \dot{+} c,$$

where $d, c \in Q$, γ, δ are commuting surjective endomorphisms of the group $Q(\oplus)$ and λ_i $i = 2, \ldots, n$, are pairwise commuting surjective endomorphisms of the group $Q(\dot{+})$.

Thus, (5) has the form $A(\alpha x_1, B(x_2^n)) = (\beta x_1, \{\mu x_i\}_2^n)$, i.e.,

$$\gamma\alpha x_1 \oplus \delta(\lambda_2 x_2 \dot{+} \lambda_3 x_3 \dot{+} \ldots \dot{+} \lambda_n x_n \dot{+} c) \oplus d = (\beta x_1, \{\mu x_i\}_2^n). \tag{6}$$

Let h_μ be the map of Q to Q such that $\mu h_\mu = \varepsilon$ (ε is the identity map of Q to Q); then, from (6), we obtain:

$$\gamma\alpha x_1 \oplus \delta(\lambda_2 h_\mu x_2 \dot{+} \lambda_3 h_\mu x_3 \dot{+} \ldots \dot{+} \lambda_n h_\mu x_n \dot{+} c) \oplus d = (\beta x_1, x_2^n).$$

There exists an element $a_1 \in Q$ such that $\gamma\alpha a_1 \oplus d = 0_\oplus$, where $0_\oplus$ is the identity of the group $Q(\oplus)$. Hence, we get:

$$\delta(\lambda_2 h_\mu x_2 \dot{+} \ldots \dot{+} \lambda_n h_\mu x_n \dot{+} c) = (\beta a_1, x_2^n).$$

The retract $(\beta a_1, x_2^n)$ is an $(n-1)$-ary regular, medial, division groupoid; therefore, there exist: an abelian group $Q(+)$ and its commuting surjective endomorphisms φ_i $(i = 2, \ldots, n)$, such that:

$$\delta(\lambda_2 h_\mu x_2 \dot{+} \ldots \dot{+} \lambda_n h_\mu x_n \dot{+} c) = \varphi_2 x_2 + \ldots + \varphi_n x_n + l = \varphi_2 x_2 + \ldots + \varphi'_n x_n, \tag{7}$$

where $\varphi'_n x_n = \varphi_n x_n + l$ and $l \in Q$. Let us rewrite (6) in the form:

$$\gamma\alpha h_\beta x_1 \oplus \delta(\lambda_2 h_\mu x_2 \dot{+} \ldots \dot{+} \lambda_n h_\mu x_n \dot{+} c) \oplus d = (x_1^n)$$

where $\beta h_\beta = \varepsilon$. Using (7), we get $(x_1^n) = \gamma\alpha h_\beta x_1 \oplus (\varphi_2 x_2 + \ldots + \varphi'_n x_n) \oplus d$, i.e.,

$$(x_1^n) = \pi_1 x_1 \oplus (\varphi_2 x_2 + \ldots + \varphi'_n x_n), \tag{8}$$

where $\pi_1 x_1 = \gamma\alpha h_\beta x_1$. It follows from (8), that π_1 is a surjection.

Now we consider the retract, $(x_1^{n-1}a)$. By the inductive assumption, there exists an abelian group $Q(*)$ such that:

$$(x_1^{n-1}a) = \mu_1 x_1 * \mu_2 x_2 * \ldots * \mu_{n-1} x_{n-1} * h, \tag{9}$$

where μ_i $(i = 1, \dots, n-1)$ are commuting surjective endomorphisms of the group $Q(*)$ and $h \in Q$.

Substituting a for x_n in (8) and taking into account (9), we obtain:

$$\pi_1 x_1 \oplus (\varphi_2 x_2 + \dots + \varphi'_{n-1} x_{n-1}) = \mu_1 x_1 * \mu_2 x_2 * \dots * \mu'_{n-1} x_{n-1}, \tag{10}$$

where $\varphi'_{n-1} x_{n-1} = \varphi_{n-1} x_{n-1} + \varphi'_n a$, $\mu'_{n-1} x_{n-1} = \mu_{n-1} x_{n-1} * h$.

Choose the elements a_3^{n-1} such that $\varphi_3 a_3 + \dots + \varphi'_{n-1} a_{n-1} = 0$, where 0 is the identity of the group $Q(+)$; then from (5) we get:

$$\pi_1 x_1 \oplus \varphi_2 x_2 = \mu_1 x_1 * \mu'_2 x_2,$$

where $\mu'_2 x_2 = \mu_2 x_2 * \mu_3 a_3 * \dots * \mu^*_{n-1} a_{n-1}$; therefore μ'_2 is a surjection.

Hence, $x_1 * x_2 = \pi_1 h_{\mu_1} x_1 \oplus \varphi_2 h_{\mu'_2} x_2$, where $\mu_1 h_{\mu_1} = \varepsilon$ and $\mu'_2 h_{\mu'_2} = \varepsilon$. Thus, the groups $Q(*)$ and $Q(\oplus)$ are principally homotopic and, by Lemma 2.3, we get:

$$u \oplus v = u * v * l. \tag{11}$$

Now we choose the elements a_1, a_4^{n-1} such that $\pi_1 a_1 = 0_\oplus$ and $\varphi_4 a_4 + \dots + \varphi'_{n-1} a_{n-1} = 0$. Then, from (10), we obtain:

$$\varphi_2 a_2 + \varphi_3 a_3 = \mu_2 x_2 * \mu'_3 x_3,$$

where μ'_3 is a surjection. By Lemma 2.3, we have:

$$u + v = u * v * l'. \tag{12}$$

Combining (11) and (12), we obtain:

$$u \oplus v = u + \ v + \ l''. \tag{13}$$

According to (13), we get from (8):

$$(x_1^n) = \pi_1 x_1 + \varphi_2 x_2 + \dots + \varphi'_n x_n + l''' = \psi_1 x_1 + \dots + \psi_n x_n + r, \tag{14}$$

where $\psi_1, \dots, \psi_n$ are surjections.

Note that we can assume in (14) that $\psi_i 0 = 0$, $i = 1, \dots, n$.

Now, we prove that ψ_i $(i = 1, \dots, n)$ are endomorphisms of the group $Q(+)$, and $\psi_i \psi_j = \psi_j \psi_i$ for all $i, j \in \{1, \dots, n\}$. Let us consider the following retract of matrix (3).

$$\begin{array}{c c} & \begin{array}{ccccc} & j & & k & \end{array} \\ i & \left(\begin{array}{ccccc} & \vdots & & \vdots & \\ \dots & u & \dots & v & \dots \\ & \vdots & & \vdots & \end{array} \right), \end{array}$$

where $x_{ij} = u,\ x_{ik} = v$ and other elements are equal to 0. For y_i and z_i, we have:

$$\begin{cases} y_i = \psi_j u + \psi_k v + r, \\ y_s = r, \text{ if } s \neq i, \end{cases} \qquad \begin{cases} z_j = \psi_i u + r, \\ z_k = \psi_i v + r, \\ z_s = r, \text{ if } s \neq j, k. \end{cases}$$

Thus,

$$(y_1^n) = (r^{i-1}, \psi_j u + \psi_k v + r, r^{n-i}),$$

$$(z_1^n) = (r^{j-1}, \psi_i u + r, r^{k-j-1}, \psi_i v + r, r^{n-k}).$$

Hence,

$$(r^{i-1}, \psi_j u + \psi_k v + r, r^{n-i}) = (r^{j-1}, \psi_i u + r, r^{k-j-1}, \psi_i v + r, r^{n-k}).$$

From the last equality, by (14), we obtain:

$$\sum_{s=1}^{i-1} \psi_s r + \psi_i(\psi_j u + \psi_k v + r) + \sum_{s=i+1}^{n} \psi_s r + r =$$

$$\sum_{s=1}^{j-1} \psi_s r + \psi_j(\psi_i u + r) + \sum_{s=j+1}^{k-1} \psi_s r + \psi_k(\psi_i v + r) + \sum_{s=k+1}^{n} \psi_s r + r.$$

From this equality we get:

$$\psi_i(\psi_j u + \psi_k v + r) = \psi_j(\psi_i u + r) + \psi_k(\psi_i v + r) + t, \tag{15}$$

where t is some element of Q. Substituting $h_{\psi_j} u$ and $h_{\psi_k}(v - r)$ for u and v in (15), respectively, we obtain:

$$\psi_i(u + v) = \psi_j(\psi_i h_{\psi_j} u + r) + \psi_k(\psi_i h_{\psi_k}(v - r) + r) + t,$$

i.e.,

$$\psi_i(u + v) = \sigma u + \tau v,$$

where σ and τ are some maps of Q to Q. Thus, ψ_i is a quasiendomorphism of the group $Q(+)$. Since, $\psi_i 0 = 0$, it follows from Lemma 2.2 that each ψ_i is an endomorphism of the group $Q(+)$.

If we take $v = 0$ in (15) and since ψ_i is an endomorphism of the group $Q(+)$, we have:

$$\psi_i \psi_j u + \psi_i r = \psi_j \psi_i u + \psi_j r + \psi_k r + t. \tag{16}$$

Now, if we take $u = 0$ in (16), we obtain: $\psi_i r = \psi_j r + \psi_k r + t$. Substituting $\psi_j r + \psi_k r + t$ for $\psi_i r$ in (16), we get: $\psi_i \psi_j u = \psi_j \psi_i u$. To conclude the proof, it remains to note that i, j are arbitrary. □

Denote by $L_i^A(\overline{a})$ the i-translation of the algebra (Q,Σ) with respect to $\overline{a}\in Q^{|A|}$ ($|A|$ is the arity of the operation A) and the operation $A\in\Sigma$, namely:

$$L_i^A(\overline{a})x = A(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_{|A|}).$$

The algebra (Q,Σ) is a *division* (*invertible*) algebra, if every $L_i^A(\overline{a})$ is a surjection (bijection), for all $\overline{a}\in Q^{|A|}$, $A\in\Sigma$ and $i=1,\ldots,|A|$.

(Q,Σ) is *i-regular* if $L_i^A(\overline{a})=L_i^A(\overline{b})$, whenever $\overline{a},\overline{b}\in Q^{|A|}$, $A\in\Sigma$ and $L_i^A(\overline{a})c = L_i^A(\overline{b})c$, for some $c\in Q$. If (Q,Σ) is is i-regular for all $i=1,\ldots,|A|$, then it is called *regular.*

Theorem 3.2. *Let (Q,Σ) be a regular medial division algebra. Then there exists an abelian group $Q(+)$ such that every operation $A\in\Sigma$ has the representation:*

$$A(x_1,\ldots,x_{|A|}) = \varphi_1^A x_1 + \ldots + \varphi_{|A|}^A x_{|A|} + t_A, \tag{17}$$

where $\varphi_1^A,\ldots,\varphi_{|A|}^A$ are pairwise commuting surjective endomorphisms of the group $Q(+)$ and t_A is a fixed element of Q.

Proof. According to Theorem 3.1, every operation $A\in\Sigma$ ($|A|=m$) has the form:

$$A(x_1,\ldots,a_m) = \varphi_1^n x_1 +_A \ldots +_A \varphi_m^A x_m +_A t'_A, \tag{18}$$

where the abelian group $Q(+_A)$ corresponds to the operation: $A\in\Sigma$. Let us rewrite medial hyperidentity (1) (in terms of the operations, $+_A$ and $+_B$) in the following way:

$$\varphi_1^A(\varphi_1^B x_{11} +_B \ldots +_B \varphi_n^B x_{1n} +_B t_B) +_A \ldots +_A \varphi_m^A(\varphi_1^B x_{m1} +_B \ldots +_B \varphi_n^B x_{mn} +$$
$$_B t_B) +_A t_A = \varphi_1^B(\varphi_1^A x_{11} +_A \ldots +_A \varphi_m^A x_{m1} +_A t_A) +_B \ldots +_B \varphi_n^B(\varphi_1^A x_{1n} +$$
$$+_A \ldots +_A \varphi_m^A x_{mn} +_A t_A) +_B t_B.$$

If we take each of x_{ij} equal to 0_B, (where 0_B is the identity of the group $Q(+_B)$), besides x_{11} and x_{mn} in the last equality, then we obtain:

$$\varphi_1^A(\varphi_1^B x_{11} +_B t_B) +_A +\varphi_m^A(\varphi_n^B x_{mn} +_B t_B) +_A c_A$$
$$= \varphi_1^B(\varphi_1^A x_{11} +_A k_1) +_B +\varphi_n^B(\varphi_m^A x_{mn} +_A k_2) +_B c_B,$$

where c_A, c_B, k_1 and k_2 are some elements of the set Q.

From the last equality we get:

$$\alpha x_{11} +_A +\beta x_{mn} = \gamma x_{11} +_B \delta x_{mn},$$

where $\alpha,\beta,\gamma,\delta$ are surjective maps of Q to Q.

Thus, the groups $Q(+_A)$ and $Q(+_B)$ are homotopic and, by Lemma 2.3, we obtain:

$$x +_A y = x +_B y +_B k, \tag{19}$$

$$x +_B y = x +_A y +_A t, \tag{20}$$

for some $k, t \in Q$.

We fix the operation $+_B$ and further we denote it by $+$. According to (18) and (19), for the operation $A \in \Sigma$ we have:

$$A(x_1, \ldots, x_m) = \varphi_1^A x_1 +_A \ldots +_A \varphi_m^A x_m +_A t_A = \varphi_1^B x_1 +_B \ldots +_B \varphi_n^B x_n +_B u_A =$$

$$\varphi_1^A x_1 + \ldots + \varphi_m^A x_m + u_A.$$

Since the operation A is arbitrary, we have proved that every operation $A \in \Sigma$ has the form:

$$A(x_1, \ldots, x_m) = \varphi_1^A x_1 + \ldots + \varphi_m^A x_m + u_A. \tag{21}$$

Let us prove that φ_i^A $(i = 1, \ldots, m)$ are quasiendomorphisms of the group $Q(+)$. According to (19) and (20), we have:

$$\varphi_i^A(x + y) = \varphi_i^A(x +_A y +_A t) = \varphi_i^A x +_A +\varphi_i^A y +_A \varphi_i^t = \varphi_i^A x + \alpha y,$$

where α is a map of Q to Q. Thus, φ_i^A is a quasiendomorphism of the group $Q(+)$ and, by Lemma 2.1, we have:

$$\varphi_i^A = \widetilde{R}_s \gamma_i^A,$$

where $\gamma_i^A \in \operatorname{End} Q(+)$. Hence, from (21), it follows that:

$$A(x_1, \ldots, x_m) = \gamma_1^A x_1 + \ldots + \gamma_m^A x_m + d_A,$$

where $d_A \in Q$.

Similar to the proof of Theorem 3.1, we can show that the endomorphisms γ_i^A are pairwise commuting for all $i = 1, \ldots, |A|$ and $A \in \Sigma$. □

Analogously we can prove the following theorem.

Theorem 3.3. *Let (Q, Σ) be a medial invertible algebra. Then there exists an abelian group $Q(+)$ such that every operation $A \in \Sigma$ has the representation:*

$$A(x_1, \ldots, x_{|A|}) = \varphi_1^A x_1 + \ldots + \varphi_{|A|}^A x_{|A|} + t_A,$$

where $\varphi_1^A, \ldots, \varphi_{|A|}^A$ are pairwise commuting automorphisms of the group $Q(+)$ and t_A is a fixed element of Q. □

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