

Ascending chain conditions on principal left and right ideals for semidirect products of ordered semigroups

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Abstract. An ordered semigroup S is said to satisfy the ascending chain condition for principal left ideals (respectively, principal right ideals) of S if there does not exist an infinite strictly ascending chain of principal left ideals (respectively, principal right ideals) of S . In this paper, the ascending chain condition on principal left and right ideals for semidirect products of ordered semigroups are considered. The results obtained extend the results on semigroups.

1. Preliminaries

Ascending chain conditions have been studied in both rings and semigroups. Stopar [4] investigated the ascending chain condition on principal left and right ideals for semidirect products of semigroups; the results obtained are similar to the results on rings in [3]. The purpose of this paper is to extend the results on semigroups into ordered semigroups.

An *ordered semigroup* [1] is defined to be a semigroup S together with a partial order \leq that is *compatible* with the semigroup operation, meaning that for $x, y, z \in S$,

$$x \leq y \Rightarrow zx \leq zy, \quad xz \leq yz.$$

In this paper, we write (S, \cdot, \leq) for an ordered semigroup S with a partial order \leq .

If A and B are nonempty subsets of an ordered semigroup (S, \cdot, \leq) , we write

$$AB = \{xy \mid x \in A, y \in B\},$$

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

When we deal with singleton sets it is customary to write Ax and xA as $A\{x\}$ and $\{x\}A$, respectively.

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A nonempty subset A of an ordered semigroup (S, \cdot, \leq) is called a *left* (respectively, *right*) *ideal* [2] of S if the following conditions hold:

- (i) $SA \subseteq A$ (respectively, $AS \subseteq A$);
- (ii) $A = \langle A \rangle$, that is, for $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$.

If A is both a left and a right ideal of S , then A is called a (two-sided) *ideal* of S . It is easy to see that, for a nonempty subset A of S ,

$$\langle A \rangle_l = (A \cup SA), \quad \langle A \rangle_r = (A \cup AS) \quad \text{and} \quad \langle A \rangle = (A \cup SA \cup AS \cup SAS)$$

are left ideal, right ideal and ideal of S , respectively. In particular, if $A = \{a\}$, a singleton set, then

$$\langle a \rangle_l = (a \cup Sa), \tag{1}$$

$$\langle a \rangle_r = (a \cup aS) \tag{2}$$

and

$$\langle a \rangle = (a \cup Sa \cup aS \cup SaS).$$

An ordered semigroup (S, \cdot, \leq) is said to satisfy the ascending chain condition on principal left ideals (*ACCP*L) if there does not exist an infinite strictly ascending chain of principal left ideals of S . In this case S is said to be an *ACCP*L-ordered semigroup. An *ACCP*R-ordered semigroup can be defined similarly.

Let \mathbb{N} denote the set of all positive integers. We now begin with the following lemma characterized *ACCP*R-ordered semigroups:

Lemma 1.1. *The following conditions are equivalent for an ordered semigroup (S, \cdot, \leq) :*

- (i) S is an *ACCP*R-ordered semigroup;
- (ii) For any two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ in S such that $a_n \leq a_{n+1}b_n$ for all $n \in \mathbb{N}$, then there exists $N \in \mathbb{N}$ and a sequence $(c_n)_{n \in \mathbb{N}}$ in S such that $a_{n+1} \leq a_n c_n$ for all $n \geq N$;
- (iii) For any two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ in S such that $a_n \leq a_{n+1}b_n$ for all $n \in \mathbb{N}$, then there exists $N \in \mathbb{N}$ and c_N in S such that $a_{N+1} \leq a_N c_N$.

Proof. We prove (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i). Assume that (i) holds, i.e. that S is an *ACCP*R-ordered semigroup. To show (ii), we let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be two sequences in S such that

$$a_n \leq a_{n+1}b_n \quad \text{for all } n \in \mathbb{N}.$$

This implies that

$$\langle a_1 \rangle_r \subseteq \langle a_2 \rangle_r \subseteq \langle a_3 \rangle_r \subseteq \cdots$$

of principal right ideals of S . By assumption, the chain must stabilize, say N . Then by (2), for each $n \geq N$, $a_{n+1} \leq a_n c_n$ for some $c_n \in S$ or $a_{n+1} \leq a_n$. If $a_{n+1} \leq a_n$, then $a_{n+1} \leq a_n \leq a_{n+1} b_n$, and hence for this case we can take b_n for c_n . Hence there exists a sequence $(c_n)_{n \in \mathbb{N}}$ in S such that $a_{n+1} \leq a_n c_n$ for all $n \geq N$. This proves that (ii) holds.

It is clear that (ii) implies (iii).

We show that (iii) \Rightarrow (i). Assume that (iii) holds, i.e. that for any sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ in S such that $a_n \leq a_{n+1} b_n$ for all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ and c_N in S such that $a_{N+1} \leq a_N c_N$. To show (i), we suppose that there exists an infinite strictly ascending chain

$$\langle a_1 \rangle_r \subset \langle a_2 \rangle_r \subset \langle a_3 \rangle_r \subset \dots$$

of principal right ideals of S (The notation \subset stands for proper subset of sets). Thus there exists a sequence $(b_n)_{n \in \mathbb{N}}$ in S such that $a_n \leq a_{n+1} b_n$ for all $n \in \mathbb{N}$. By assumption, there exists $N \in \mathbb{N}$ such that $a_{N+1} \leq a_N c_N$ and $c_N \in S$, and hence $\langle a_{N+1} \rangle = \langle a_N \rangle$. This is a contradiction. So S is an ACCPR-ordered semigroup. \square

Dually, using (1), we have the following lemma for ACCPL-ordered semigroup:

Lemma 1.2. *The following conditions are equivalent for an ordered semigroup (S, \cdot, \leq) :*

- (i) S is an ACCPL-ordered semigroup;
- (ii) For any two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ in S such that $a_n \leq b_n a_{n+1}$ for all $n \in \mathbb{N}$, then there exists $N \in \mathbb{N}$ and a sequence $(c_n)_{n \in \mathbb{N}}$ in S such that $a_{n+1} \leq c_n a_n$ for all $n \geq N$;
- (iii) For any two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ in S such that $a_n \leq b_n a_{n+1}$ for all $n \in \mathbb{N}$, then there exists $N \in \mathbb{N}$ and c_N in S such that $a_{N+1} \leq c_N a_N$. \square

2. On semidirect products of ordered semigroups

Let (S, \cdot, \leq_S) be an ordered semigroup. A mapping $\varphi : S \rightarrow S$ such that

- (i) $\varphi(ab) = \varphi(a)\varphi(b)$;
- (ii) $a \leq_S b \Rightarrow \varphi(a) \leq_S \varphi(b)$

for all $a, b \in S$ is called an *endomorphism* on S . Let $\text{End}(S)$ denote the set of all endomorphisms on S . Note that the identity map $id_S \in \text{End}(S)$. Under the composition of functions, it is clear that $\text{End}(S)$ is a semigroup. Moreover, if $\varphi, \psi \in \text{End}(S)$, we define

$$\varphi \preceq \psi \text{ if and only if } \varphi(x) \leq_S \psi(x) \text{ for all } x \in S.$$

It is a routine matter to check that $\text{End}(S)$ forms an ordered semigroup.

Let (S, \cdot, \leq_S) and (T, \circ, \leq_T) be two ordered semigroups. Let $\omega : T \rightarrow \text{End}(S)$ such that, for $t \in T$, ω_t denote the image of ω under t . We define a multiplication and a partial order on $S \times T$ as follows:

$$(s_1, t_1)(s_2, t_2) = (s_1 \cdot \omega_{t_1}(s_2), t_1 \circ t_2) \quad (3)$$

and

$$(s_1, t_1) \leq (s_2, t_2) \text{ if and only if } s_1 \leq_S s_2, t_1 \leq_T t_2. \quad (4)$$

It is easy to verify that $S \times T$ forms an ordered semigroup, called the *semidirect product* of S and T under ω and will be denoted by $S \times_\omega T$. Hereafter, we will skiped the operations \cdot on S and \circ on T . Then the condition (3) becomes

$$(s_1, t_1)(s_2, t_2) = (s_1 \omega_{t_1}(s_2), t_1 t_2).$$

Lemma 2.1.

- (1) *If $S \times_\omega T$ is an ACCPL-ordered semigroup and there exists $f \in T$ such that $ff = f$ and $\omega_f = id_S$, then S is an ACCPL-ordered semigroup.*
- (2) *If $S \times_\omega T$ is an ACCPR-ordered semigroup and there exists $f \in T$ such that $ff = f$ and $\omega_f = id_S$, then S is an ACCPR-ordered semigroup.*

Proof. Assume that $S \times_\omega T$ is an ACCPL-ordered semigroup and there exists $f \in T$ such that $ff = f$ and $\omega_f = id_S$.

Let $s_1, s_2 \in S$. If $s_1 \leq_S s_2$ for some $s \in S$, then, by (3) and (4), we have

$$(s_1, f) \leq (ss_2, f) = (s, f)(s_2, f),$$

and hence

$$\langle s_1 \rangle_l \subseteq \langle s_2 \rangle_l \Rightarrow \langle (s_1, f) \rangle_l \subseteq \langle (s_2, f) \rangle_l.$$

Thus S is an ACCPL-ordered semigroup.

Similarly, if $s_1 \leq_S s_2 s$ for some $s \in S$, then

$$(s_1, f) \leq (s_2 s, f) = (s_2, f)(s, f).$$

Thus

$$\langle s_1 \rangle_r \subseteq \langle s_2 \rangle_r \Rightarrow \langle (s_1, f) \rangle_r \subseteq \langle (s_2, f) \rangle_r.$$

We have S is an ACCPR-ordered semigroup. □

Lemma 2.2.

- (1) *If $S \times_\omega T$ is an ACCPL-ordered semigroup and there exists $e \in S$ such that $ee = e$, then T is an ACCPL-ordered semigroup.*

(2) If $S \times_{\omega} T$ is an ACCPR-ordered semigroup and there exists $e \in S$ such that $ee = e$, then T is an ACCPR-ordered semigroup.

Proof. Assume that $S \times_{\omega} T$ is an ACCPL-ordered semigroup and there exists $e \in S$ such that $ee = e$. To show that T is an ACCPL-ordered semigroup, we let $t_n, u_n \in T$ be such that

$$t_n \leq u_n t_{n+1}$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define

$$x_n = (\omega_{t_n}(e), t_n) \quad \text{and} \quad y_n = (\omega_{t_n}(e), u_n).$$

Thus $x_n, y_n \in S \times_{\omega} T$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} y_n x_{n+1} &= (\omega_{t_n}(e), u_n)(\omega_{t_{n+1}}(e), t_{n+1}) \\ &= (\omega_{t_n}(e)\omega_{u_n t_{n+1}}(e), u_n t_{n+1}) \\ &\geq (\omega_{t_n}(e)^2, t_n) \\ &= (\omega_{t_n}(e), t_n) \\ &= x_n \end{aligned}$$

for all $n \in \mathbb{N}$. Since $S \times_{\omega} T$ is an ACCPL-ordered semigroup, there exists $N \in \mathbb{N}$ and $(s, t) \in S \times_{\omega} T$ such that $x_{N+1} \leq (s, t)x_N$, and so $t_{N+1} \leq tt_N$. By Lemma 1.2, T is an ACCPL-ordered semigroup.

For the second case, we assume that $S \times_{\omega} T$ is an ACCPR-ordered semigroup and there exists $e \in S$ such that $ee = e$. To show that T is an ACCPR-ordered semigroup, we let $t_n, v_n \in T$ be such that

$$t_n \leq t_{n+1}v_n$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define

$$x_n = (\omega_{t_1}(e), t_n) \quad \text{and} \quad y_n = (\omega_{v_n v_{n-1} \dots v_1}(e), v_n).$$

Thus $x_n, y_n \in S \times_{\omega} T$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} x_{n+1} y_n &= (\omega_{t_1}(e), t_{n+1})(\omega_{v_n v_{n-1} \dots v_1}(e), v_n) \\ &= (\omega_{t_1}(e)\omega_{t_{n+1} v_n v_{n-1} \dots v_1}(e), t_{n+1} v_n) \\ &\geq (\omega_{t_1}(e)\omega_{t_1}(e), t_n) \\ &= (\omega_{t_1}(e), t_n) \\ &= x_n \end{aligned}$$

for all $n \in \mathbb{N}$. In the same manner as the case before and using Lemma 1.1, we have T is an ACCPR-ordered semigroup. \square

Lemma 2.3. *If S and T are ACCPL-ordered semigroups and for all $t, u \in T$ with $t \leq ut$ there exists $v \in T$ with $t \leq vt$ such that $\omega_v \omega_u = \omega_u \omega_v = id_S$, then $S \times_{\omega} T$ is an ACCPL-ordered semigroup.*

Proof. To show that $S \times_{\omega} T$ is an ACCPL-ordered semigroup, we let $s_n, r_n \in S$ and $t_n, u_n \in T$ be such that

$$(s_n, t_n) \leq (r_n, u_n)(s_{n+1}, t_{n+1})$$

for all $n \in \mathbb{N}$. That is

$$s_n \leq r_n \omega_{u_n}(s_{n+1}) \text{ and } t_n \leq u_n t_{n+1}$$

for all $n \in \mathbb{N}$. Since T is ACCPL-ordered semigroup, there exists $m \in \mathbb{N}$ such that $t_{n+1} \leq v_n t_n$ for some $v_n \in T$ for all $n \geq m$. If $n > m$, we let

$$s'_n = \omega_{u_m u_{m+1} \cdots u_{n-1}}(s_n) \in S.$$

Then

$$s'_n \leq \omega_{u_m u_{m+1} \cdots u_{n-1}}(r_n \omega_{u_n}(s_{n+1})) = \omega_{u_m u_{m+1} \cdots u_{n-1}}(r_n) s'_{n+1}.$$

Since S is ACCPL-ordered semigroup, so there exist $k > m$ and $s \in S$ such that $s'_{k+1} \leq s s'_k$. Since $t_{n+1} \leq v_n u_n t_{n+1}$ for all $n \geq m$, by assumption, there exist $w_n \in T$ such that $t_{n+1} \leq w_n t_{n+1}$ and $\omega_{w_n} \omega_{v_n u_n} = id_S$ for all $n \geq m$. Setting

$$z_n = w_n v_n$$

for all $n \geq m$, then

$$s_{k+1} = \omega_{z_k z_{k-1} \cdots z_m}(s'_{k+1}) \leq \omega_{z_k z_{k-1} \cdots z_m}(s) \omega_{z_k z_{k-1} \cdots z_m}(s'_k) = s' \omega_{z_k}(s_k)$$

where $s' = \omega_{z_k z_{k-1} \cdots z_m}(s)$. Since

$$z_k t_k = w_k v_k t_k \geq w_k t_{k+1} \geq t_{k+1},$$

it follows that $(s_{k+1}, t_{k+1}) \leq (s', z_k)(s_k, t_k)$. Hence $S \times_{\omega} T$ is an ACCPL-ordered semigroup and completes the proof. \square

Lemma 2.4. *Let S and T are ACCPR-ordered semigroups such that for all $p, r, s \in S$ and all $t, u \in T$, if $s \leq sr \omega_t(p)$ and $t \leq tu$, then $r \in (Im(\omega_t))$. (Let $Im(\omega_t)$ denote the range of ω_t). Then $S \times_{\omega} T$ is an ACCPR-ordered semigroup.*

Proof. To show that $S \times_{\omega} T$ is an ACCPR-ordered semigroup, we let $s_n, r_n \in S$ and $t_n, u_n \in T$ be such that

$$(s_n, t_n) \leq (s_{n+1}, t_{n+1})(r_n, u_n)$$

for all $n \in \mathbb{N}$. That is

$$s_n \leq s_{n+1} \omega_{t_{n+1}}(r_n) \text{ and } t_n \leq t_{n+1} u_n$$

for all $n \in \mathbb{N}$. Since S and T are ACCPR-ordered semigroups and Lemma 1.1, there exist $N \in \mathbb{N}$, $s \in S$ and $t \in T$ such that $s_{N+1} \leq s_N s$ and $t_{N+1} \leq t_N t$, and hence

$$s_N \leq s_N s \omega_{t_{N+1}}(r_N) \text{ and } t_{N+1} \leq t_{N+1} u_N t.$$

By assumption, there exists $r \in S$ such that $s \leq \omega_{t_{N+1}}(r)$. We have

$$\begin{aligned} (s_{N+1}, t_{N+1}) &\leq (s_N \omega_{t_{N+1}}(r), t_{N+1}) \\ &\leq (s_N \omega_{t_N t}(r), t_N t) \\ &= (s_N, t_N)(\omega_t(r), t). \end{aligned}$$

Therefore $S \times_\omega T$ is an ACCPR-ordered semigroup. \square

An element b of an ordered semigroup (S, \cdot, \leq) is called a *complete inverse* of $a \in S$ if $a = aba$, $b = bab$ and $ab = ba$. It is easy to see that if such b exists then it is unique.

Lemma 2.5. *Let $S \times_\omega T$ be an ACCPR-ordered semigroup. If for all $p, r, s \in S$ and $t, u \in T$, $s \leq sr\omega_t(p)$ and $t \leq tu$ implies that r is a complete inverse of $\omega_t(p)$, then $Im(\omega_t)$ is closed for complete inverse for all $t \in T$ with $t \leq tu$ for some $u \in T$.*

Proof. To show that $Im(\omega_t)$ is closed for complete inverse for all $t \in T$ with $t \leq tu$ for some $u \in T$, we let $t \in T$ such that $t \leq tu$ for some $u \in T$ and let $r \in S$ be a complete inverse of $\omega_t(p)$ for some $p \in S$. Define

$$x_n = (r^n, t) \in S \times_\omega T \quad \text{and} \quad y = (p, u) \in S \times_\omega T.$$

We have

$$\begin{aligned} x_{n+1}y &= (r^{n+1}, t)(p, u) = (r^{n+1}\omega_t(p), tu) \\ &= (r^n\omega_t(p)r, t) \\ &\geq (r^n, t) \\ &= x_n \end{aligned}$$

for all $n \in \mathbb{N}$. Since $S \times_\omega T$ is an ACCPR-ordered semigroup, there exists $N \in \mathbb{N}$, $q \in S$ and $v \in T$ such that $x_{N+1} \leq x_N(q, v)$ in $S \times_\omega T$, and so $r^{N+1} \leq r^N \omega_t(q)$. Since r is a complete inverse of $\omega_t(p)$, it follows that

$$r^N \omega_t(q) \omega_t(p) = r^{N+1} \omega_t(p) = r^N.$$

This shows that $\omega_t(q)$ is a complete inverse of $\omega_t(p)$. Hence $r = \omega_t(q)$. \square

Theorem 2.6. *Let S and T be ordered semigroups with idempotents e and f , respectively. Let $\omega : T \rightarrow End(S)$. Suppose that for all $t, u \in T$ with $t \leq ut$ there exists $v \in T$ with $t \leq vt$ such that $\omega_v \omega_u = \omega_u \omega_v = id_S$. Then $S \times_\omega T$ is an ACCPL-ordered semigroup if and only if S and T are ACCPL-ordered semigroups.*

Proof. If $ff = f$, then by assumption there exists $v \in T$ such that $f \leq vf$ and $\omega_v \omega_f = \omega_f \omega_v = id_S$. Hence ω_f is invertible. Since $\omega_f \omega_f = \omega_f$, so $\omega_f = id_S$. Using Lemma 2.1(1), Lemma 2.2(2) and Lemma 2.3, we have the theorem. \square

Theorem 2.7. *Let S and T be ordered semigroups with idempotents e and f , respectively. Let $\omega : T \rightarrow \text{End}(S)$. Suppose that $\omega_f = \text{id}_S$ and for all $p, r, s \in S$ and $t, u \in T$, $s \leq sr\omega_t(p)$ and $t \leq tu$ implies that r is a complete inverse of $\omega_t(p)$. Then $S \times_\omega T$ is an ACCPR-ordered semigroup if and only if S and T are ACCPR-ordered semigroups and $\text{Im}(\omega_t)$ is closed for complete inverse for all $t \in T$ with $t \leq tu$ for some $u \in T$.*

Proof. This is a consequence of Lemma 2.2(2), and 2.3. □

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